Mathematical Problems of Computer Science 17, 1997, 111–119

General Algorithm of the 2-D Discrete Hartley Transformation

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Abstract

Efficient algorithms for computing of the two-dimensional discrete Hartley transformation (2-D DHT) that based on construction of special coverings of the square lattice is given. The concepts of starts of the transform and image are presented and comparative estimates, describing the efficiency of the proposed algorithm with respect to the known ones, are given.

1 Introduction

The discrete Hartley transformation plays an important role in digital signal and image processing. This transformation was created as an alternative form to the complex discrete Fourier transformation (DFT), to elimate the necessity of complex operation fulfillment [1, 2]. So, given a sequence f of input data, the DFT output is formed by rules:

$$f \to (T_1(f), T_2(f)) \to T_1(f) + \iota T_2(f), \tag{1}$$

where as the DHT output is formed by rules:

$$f \to (T_1(f), T_2(f)) \to T_1(f) + T_2(f), \tag{2}$$

Here $T_1(f)$ and $T_2(f)$ denote the real transformations which kernels defined by the cosine and sine functions, respectively. So, for a 1-D sequence $f = \{f_n;\ n = 0 \div N-1\}$, these transformations are

$$\begin{aligned} T_1 &: f \to \{\sum_{n=0}^{N-1} f_n \cos(\frac{2\pi}{N} pn); p = 0 \div N-1\}, \\ T_2 &: f \to \{\sum_{n=0}^{N-1} f_n \sin(\frac{2\pi}{N} pn); p = 0 \div N-1\}, \end{aligned} \tag{3}$$

where N is the transformation length. Transition from the one-dimensional case to a multidimensional case leads to essential difficulty of calculation of the DHT. So, in the two-dimensional case, product np in (3) are the scalar product $np = n_1p_1 + n_2p_2$, for pairs $n = (n_1, n_2)$, $p = (p_1, p_2)$. Therefore the 2-D DHT is not separable and cannot be calculated using the standard row-column method. As well known many attempts have been made to overcome this problem [3, 4, 5]. But this problem is very better solved when using the concept of coverings such that open 2-D DHT presented further on. Coverings of that kind for the first time was effectively used for computing of the 2-D DFT and known as coverings opening the 2-D DFT [6, 7]. In this paper we shall be proving that coverings of that kind may be effectively used for construction fast algorithms of the $N \times N$-point 2-D DHT, where N is an arbitrary natural number.

2 The covering opening 2-D DHT

Let X be the square lattice of dimension N by N, for N a certain natural number, i.e.

$$X = X_{N,N} = \{(p_1, p_2); p_1 = 0 \div N-1, p_2 = 0 \div N-1\}. \tag{4}$$

Let $\mathcal{H}_{N,N}$ be the $N \times N$-point 2-D DHT which image $\mathcal{H}_{N,N} \circ f$ of an $N \times N$ sequence $f = \{f_{n_1,n_2}\}$ is given by the following:

$$H_{p_1,p_2} = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} f_{n_1,n_2} Cas(n_1 p_1 + n_2 p_2), \qquad (p_1, p_2) \in X, \tag{5}$$

where the kernel of the transformation is the real periodic function

$$Cas(x) = Cas_N(x) = cas(2\pi x/N) = \cos(2\pi x/N) + \sin(2\pi x/N). \tag{6}$$

The 2-D DHT is a real-to-real and non-separable transformation. It is the two-dimensional analog of the one-dimensional N-point orthogonal discrete Hartley transformation $\mathcal{H}_N$ determined for a 1-D sequence $f = \{f_n\}$ by the relation:

$$H_p = (\mathcal{H}_N \circ f)_p = \sum_{n=0}^{N-1} f_n Cas(np), \qquad p = 0 \div N-1. \tag{7}$$

An interesting feature of the DHT is that the inversion formula for the DHT coincides with the initial formula (accurate to the factor $1/N$), i.e.

$$f_n = 1/N \sum_{p=0}^{N-1} H_p Cas(np), \qquad n = 0 \div N-1. \tag{8}$$

Therefore, we can write $\{1\text{–D DHT}\}^{-1} = 1/N\{1\text{–D DHT}\}$. Similarly, for 2-D DHT the following $\{2\text{–D DHT}\}^{-1} = 1/N^2\{2\text{–D DHT}\}$ takes place.

Let $\sigma = (T)$ be a certain family of subsets $T \subseteq X$. We shall assume that the sets $T \in \sigma$ are numbered in a way, i.e. $\sigma = (T_i)_{i \in I}$ with a certain set I of indices.

Definition 1 A family $\sigma = (T)$ is called a covering of set X, if the set-theoretic union of subsets $T \in \sigma$ coincides with the set X, i.e. $\cup T = X$.

A covering σ of the set X we shall denote by $\sigma = \sigma(X)$. A covering $\sigma(X)$ is called *irreducible* if any family of subsets, obtained from σ by throwing away any element $T \in \sigma$, will not be a covering of X. A covering $\sigma(X)$ such which is a family of non-empty and disjoint parts of the set X is called *a partition* of the set X.

Give now the fundamental concept of coverings opening the 2-D DHT. Suppose $\sigma = (T)$ is a certain covering of the square lattice X. We shall call $card\,\sigma$ the cardinality of σ.

If the 2-D DHT is splitted on $card\,\sigma$ one-dimensional $card\,T$-point orthogonal transformations $\mathcal{A}(T)$ (which are only depended on the sets $T \in \sigma$ and are not necessary Hartley type transformations), then we shall say that the 2-D DHT *is opened by the covering* σ, or, that the same, σ is *a covering opening* the 2-D DHT.

Let us give more strict definition. We shall call $g_{|T} = \{g_n; n \in T\}$ a restriction of a sequence g on the set T. Let f be an $N \times N$ sequence.

Definition 2 $\mathcal{H}_{N,N}$ is called *opened by a covering* σ, if for every set $T \in \sigma$ there exists a 1-D orthogonal transformation $\mathcal{A} = \mathcal{A}(T)$ such that

$$(\mathcal{H}_{N,N} \circ f)_{|T} = \mathcal{A} \circ f_T \tag{9}$$

for a certain 1-D sequence f_T.

The totality of the one-dimensional transformations $\{\mathcal{A}(T); T \in \sigma\}$ is called *a set of elements of the* $\mathcal{H}_{N,N}$ by the covering σ and is denoted by $\mathcal{R}(\mathcal{H}_{N,N};\sigma)$.

The set of irreducible coverings opening the 2-D DHT is non-empty.

3 General algorithm for 2-D DHT

Let $\sigma = (T)$ be a covering of the square lattice $X_{N,N}$, which opens the 2-D DHT. The general algorithm for computing the 2-D DHT of an $N \times N$ sequence f is splitted onto the following steps:

Step (1) Construction of the set of elements $\mathcal{R}(\mathcal{H}_{N,N};\sigma)$;
Step (2) Construction of the 1-D sequences f_T, for all $T \in \sigma$;
Step (3) Calculation of the 1-D transforms $A(T) \circ f_T$, for all $T \in \sigma$.

Theorem 3.1 *Assertion 3.1* To compute the $N \times N$-point 2-D DHT, it is enough to fulfill $card\,\sigma$ 1-D transformations of the corresponding set $\mathcal{R}(\mathcal{H}_{N,N};\sigma)$.

Next we construct an irreducible covering of a special type and consider the general algorithm application for this case.

4 Property of the 2-D DHT

Let us consider a representation of the 2-D DHT such which is analogous to the vectorial representation of the 2-D DFT[6, 7]. To this end, given a sample (p_1, p_2) and all $t = 0 \div N-1$, define the following sets and quantities:

$$V_{p_1,p_2,t} = \{(n_1, n_2);\ n_1p_1 + n_2p_2 = t \bmod N\} \tag{10}$$

$$f_{p_1,p_2,t} = \sum_{V_{p_1,p_2,t}} f_{n_1,n_2}. \tag{11}$$

Owing to the periodicity of the function Cas, we obtain the following

Property.Property of the 2-D DHT. 1 *Let* $(p_1, p_2) \in X_{N,N}$ *and let* k *be an arbitrary integer, then*

$$H_{\overline{kp_1},\overline{kp_2}} = \sum_{t=0}^{N-1} f_{p_1,p_2,t} Cas(kt), \tag{12}$$

where $\bar{l}$ *denotes* $l(\mathrm{mod}\, N)$, *for* l *an integer.*

Proof. *Really, for* $(p_1, p_2) \in X$ *fixed, the following empty intersections* $V_{p_1,p_2,t_1} \cap V_{p_1,p_2,t_2} = \emptyset$ *take place, for all* $t_1 \neq t_2 \in [0, N-1]$. *Consequently the family* $\{V_{p_1,p_2,t};\ t = 0 \div N-1\}$ *is the partition of the square lattice* X. *Therefore, by simple calculations we are obtaining:*

$$\sum_{t=0}^{N-1} f_{p_1,p_2,t} Cas(kt) = \sum_{t=0}^{N-1} \Big(\sum_{V_{p_1,p_2,t}} f_{n_1,n_2} \Big) Cas(kt) =$$

$$= \sum_{t=0}^{N-1} \sum_{V_{p_1,p_2,t}} f_{n_1,n_2} Cas[k(n_1p_1 + n_2p_2)] =$$

$$= \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} f_{n_1,n_2} Cas[n_1(kp_1) + n_2(kp_2)] = H_{kp_1,kp_2} = H_{\overline{kp_1},\overline{kp_2}}.$$

The above 2-D DHT property leads to the following result. Given a sample (p_1, p_2) of X, the cyclic group T in X with this generatrix is written in the form

$$T = T_{p_1,p_2} = \{(\overline{kp_1}, \overline{kp_2});\ k = 0 \div N-1\}. \tag{13}$$

Theorem 4.2 *Assertion 4.1* Suppose σ is an irreducible covering of the square lattice X such that is composed of the groups (13). Then σ opens the 2-D DHT.

Thus, for a two-dimensional sequence f given, the 2-D DHT transformation on the each group T_{p_1,p_2} is determined by the corresponding 1-D sequence of N,

$$f_T = \{f_{p_1,p_2,0}, \ldots, f_{p_1,p_2,N-1}\}. \tag{14}$$

5 Construction of the irreducible covering

Before passing on to the common case of N, we consider separable the cases when N is a prime number, when N is a power of 2 and when N is a power of an odd prime number.

Theorem 5.3 Given a prime number N, totality of groups

$$\sigma_{N,N} = ((T_{1,p_2})_{p_2=0 \div N-1}, T_{0,1}) \tag{15}$$

is the irreducible covering of the square lattice $X_{N,N}$.

For instance, for $N = 3$, we have the following irreducible covering $\sigma = (T_{1,0}, T_{1,1}, T_{1,2}, T_{0,1})$ of X as shown in Fig. 1.

Owing to Theorem 5.3 if σ is an irreducible covering of the $X_{N,N}$, then $card\, \sigma_{N,N} = N+1$. The proof of this fact becomes simple enough if we use some known facts from group theory connected with the concept of the group orbit [8].

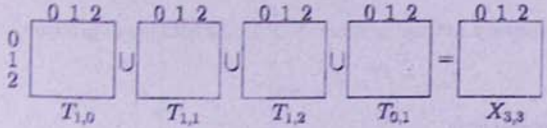


Figure 1: Reargement of samples of the groups of $\sigma_{3,3}$

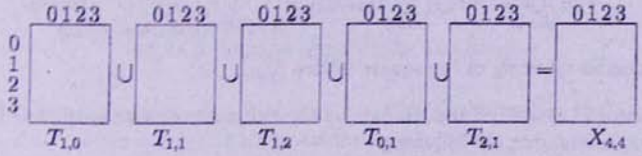


Figure 2: Reargement of samples of the groups of $\sigma_{4,4}$

Really, for N a prime, the set $Y = X_{N,N} \setminus \{(0,0)\}$ is the multiplicative group in the arithmetic modulo N. The set $G = \{1,2,3,\ldots,N-1\}$ is just such a group. Therefore, each set $T_{p_1,p_2} \setminus \{(0,0)\}$ can be considered as the orbit of the element (p_1,p_2) relative to the action group G onto Y. Inasmuch as any two orbits either intersect or coincide, then Y is the union of paired disjoint orbits, i.e.

$$Y = \sum_{(p_1,p_2)\in J} (T_{p_1,p_2} \setminus \{(0,0)\}), \tag{16}$$

for a certain set of indices $J \subseteq Y$.

Further, since $card\, Y = N^2 - 1$, and $card\,(T_{p_1,p_2} \setminus \{(0,0)\}) = N - 1$ for all points $(p_1,p_2) \in Y$, we obtain from the expansion (16), that $N^2 - 1 = (N-1)(card\, J)$. Hence $card\sigma = card\, J = N + 1$.

Theorem 5.4 Given a power of two $N = 2^r$, $(r > 1)$, totality of groups

$$\sigma_{N,N} = \left(\left(T_{1,p_2}\right)_{p_2=0\div N-1}, \left(T_{2p_1,1}\right)_{p_1=0\div N/2-1}\right) \tag{17}$$

is the irreducible covering of the square lattice $X_{N,N}$.

So, for $N = 4$, we have the following irreducible covering $\sigma = (T_{1,0}, T_{1,1}, T_{1,2}, T_{1,3}, T_{0,1}, T_{2,1})$ of $X_{3,3}$ as shown in Fig. 2.

Theorem 5.5 Given a power of an odd prime $N = L^r\,(r > 1)$, the totality of groups

$$\sigma_{N,N} = \left(\left(T_{1,p_2}\right)_{p_2=0\div N-1}, \left(T_{Lp_1,1}\right)_{p_1=0\div N/L-1}\right) \tag{18}$$

is the irreducible covering of the square lattice $X_{N,N}$.

In the general case, the construction of the irreducible covering σ is implemented in the following way. Define first the set $B_N = \{n \in [0, N-1];\, g.c.d.(n,N) > 1\}$ and let $\beta(p)$ be the function equal to the number of the elements $s \in B_N$ such which are co prime with p and $ps < N$. Further denote by $\phi(N)$ Euler's function, i.e. the number of the positive integers which are smaller than N and co prime with N.

Theorem 5.6 Given an arbitrary natural number $N > 1$, the totality of groups

$$\sigma_{N,N} = \left(T_{p_1',p_2'}\right)_{p_1,p_2 \in J} \tag{19}$$

with the following set of generatrices

$$J = \bigcup_{p_2=0}^{N-1} (1, p_2) \bigcup \left(\bigcup_{p \in B_N} (p_1, 1) \right) \bigcup \left(\bigcup_{p_1,p_2 \in B_{N\, g.c.d.(p_1,p_2)=1,\, p_1,p_2 \leq N}} (p_1, p_2) \right) \tag{20}$$

is the irreducible covering of the square lattice $X_{N,N}$.

Thus, from Theorem 5.6 and the general algorithm for computing the 2-D DHT proposed above, we are obtaining the following

Theorem 5.7 Let $N > 1$ be an arbitrary number, then:
(a) the irreducible covering σ of the square lattice $X_{N,N}$ has the cardinality

$$card\, \sigma_{N,N} = 2N - \phi(N) + \sum_{p \in B_N} \beta(p); \tag{21}$$

(b) to compute the 2-D DHT, it is sufficiently to fulfill $2N - \phi(N) + \sum\{\beta(p);\, p \in B_N\}$ N-point 1-D DHTs.

Corollary 1 *The cardinality of the irreducible covering $\sigma_{N,N}$ of the square lattice and number of N-point 1-D DHTs necessary for computing of the $N \times N$-point 2-D DHT equals to:*
(a) $N+1$, for N a prime;
(b) $3N/2$, for N a power of 2;
(c) $(L+1)L^{r-1}$, for $N = L^r$ a power of a prime number, and $r \geq 1$;
(d) $L_1L_2 + L_1 + L_2 + 1$, for $N = L_1L_2$, where $L_1 \neq L_2 > 1$ are prime numbers.

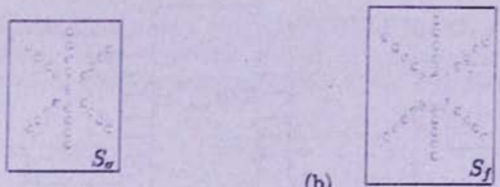
It is easy to verify, from definition of Euler's function, that $\phi(N) \geq \sum\{\beta(p);\, p \in B_N\}$, so that $card\, \sigma \leq 2N$. Herewith, the equality in this unequality takes place only when $L_1 = 2$ and $L_2 = 3$ (or when $L_1 = 3$ and $L_2 = 2$), for cases under consideration.
We shall call M_N the number of real multiplications necessary for computing of the N-point 1-D DHT.

Corollary 2 *Given an arbitrary natural number $N > 1$, for computing of the $N \times N$-point 2-D DHT, it is enough to fulfill $M_{N,N} = (card\, \sigma_{N,N})M_N$ real multiplications.*

Give now some comparative values of valuations $M_{N,N}$ with known ones.
Let N be a prime number, then $M_{N,N} = (N+1)M_N$ Compare this estimation with the known estimation $M^1_{N,N} = N^2 + 2N - 3$ which was obtained by Boussakta and Holt by using an index mapping scheme for computing the $N \times N$-point 2-D DHT[9]. For that, using the estimation $M_N = N - 1$ introduced with the Fermat number transformation [10], we have obtained

$$M_{N,N} = (N+1)(N-1) = N^2 - 1 < M^1_{N,N}. \tag{22}$$

What is more, as for the one-dimensional case, for computing of the $N \times N$-point 2-D DHT, it is enough to fulfill no more than 1 multiplication per point, whereas in the scheme proposed by Boussakta and Holt operations more that 1 multiplication are used.



(a) (b)

Figure 3: Stars of the covering and image

Pass on now to the case $N = 2^r, r > 1$. From Corollary 2 it directly follows that the following valuation of the number of multiplications takes place

$$M_{N,N} = 3N/2M_N \leq 3N^2/4(r-3) + 3N, \tag{23}$$

for the $N \times N$-point 2-D DHT. Here we use the well-known fact that for computing of the N-point 1-D DHT, it is enough to execute $M_N = N/2(r-3) + 2$ multiplications [11]. Comparing this estimation with the number of multiplications $M^2_{N,N} = 2N^2(r-2) + 4N$ obtained in the Bracewell algorithm [3]., we have $M^2_{N,N}/M_{N,N} \approx 4/3$. In other words, the number of multiplications is reduced by 4/3 times.

Similarly one can show the advantage (by multiplications) of the proposed algorithm in comparison with the known algorithms of computation of the $N \times N$-point 2-D DHT, for any integer $N > 1$.

6 Stars of transform and image

For visual representation, we shall imagine the covering σ in the form of star, on each branch of which the samples of the certain cyclic group $T \in \sigma$ are disposed as it is shown in Fig. 3(a). All elements of the group $T_{p_1,p_2} \in \sigma$ are ordered one after another in accordance with the parameter k determing them, $(\overline{kp}, \overline{ks})$, when k increases from 0 to $N-1$. We will refer the number (p_1, p_2) to the branch with elements of the group T_{p_1,p_2}. The zero point $(0,0)$ is the generic point of the groups $T \in \sigma$, from which we count the elements of each group T along the corresponding branch of the star. Also, we can consider that the points of the groups $T \in \sigma$ are disposed on the concentric circles of radii $r = k$, $k = 1, 2, \ldots, N-1$, described from the center $(0,0)$ on the plane R^2. We shall call *the star of covering* σ this kind of star for the covering σ and denote it by S_σ.

Taking the star S_σ instead of the rectangular fundamental period X_{N_1,N_2}, which the image f is written on, we pass on now to the consideration of f to the star S_f which is like to the star S_σ. Namely, we can represent S_f as the star S_σ each branch (include the original point $(0,0)$) of which is moved aside along its radius, as it is shown in Fig. 3(b).

That is, S_f is the start without the center and has $(card\,\sigma - 1)$ points more than the star S_σ. We shall call S_f the *star of image* f corresponding to the covering σ, or, briefly the star of image. So, we wish present the given image f as the spatial figure with the base on the plane, being the star S_f. And, on each branch of S_f, the image f is described by the corresponding 1-D signal f_T, $T \in \sigma$. Just in that the whole sense of the Step (2) of the general algorithm. The presentation of the image N N in the form of the star S_f is fulfilled

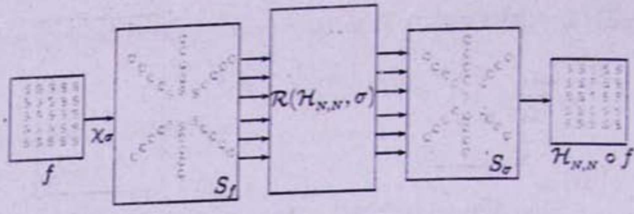


Figure 4: 2-D Hartley transform algorithm

by means of the transformation

$$\chi_\sigma : f \to \{f_{p_1,p_2,t};\ (p_1,p_2) \in J, t = 0 \div N-1\}. \tag{24}$$

associated with the covering σ of $X_{N,N}$. This transformation is called *vectorial*[6].

Thus, to calculate the 2-D transform of the given image $(N \times N)$, it is necessary to do the following:

(1) to represent image f in the form of the spatial figure with base S_f on the plane, by means of the transform (24);

(2) to fulfill the 1-D transforms over the corresponding signals f_T, lying on the branches of the image star S_f, and write the obtained values on the knots of the same branches of the transform star S_σ;

(3) to rewrite the information of the star S_σ on the square lattice $X_{N,N}$ in the following way: k-th knot of the (p_1,p_2)-th branch of the star corresponds to the knot $\overline{(kp_1}, \overline{kp_2)}$ of X.

Setting stars of image and transform, we can represent the general algorithm in the form of the block-cheme given in Fig. 4

Thus the above shows that the 2-D discrete Hartley transformation of dimension $N \times N$, where N is an arbitrary natural number, can be effectively computed by means of the stars were presented of the given image and covering which opens the transformation.

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