

An Achievable Rates-reliabilities-distortions Dependence for Source Coding with Three Descriptions

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Abstract

The matter of present paper is the source coding with three descriptions over three data streams at the rates R_1, R_2, R_3 . To each of the three decoders F_r , $r = \overline{1, 3}$, the data of r encoders are available, based on that the source messages must be recovered within the given distortion threshold Δ_r and error probability with exponent E_r . An inner bound for the region of $(E_1, E_2, E_3, \Delta_1, \Delta_2, \Delta_3)$ -achievable rates triplets $\mathcal{R}(E, \Delta)$ is found. With $E_r \rightarrow 0$, $r = \overline{1, 3}$, we receive the inner bound of rates-distortions region $\mathcal{R}(\Delta)$.

1 Introduction

In the string of works [1]-[7] the multiple description problem is sequently contributed. However, as yet the whole rates-distortions achievable region for a simple two-description problem of El Gamal and Cover [2] remains uncertain. In [2] the existence of an achievable region is described. A direct generalization of that problem and a special case of Zhang and Berger [6] 3-diversity problem corresponds to the communication system in Fig. 1. Our goal is determination of an (E, Δ) -achievable rates region for that configuration.

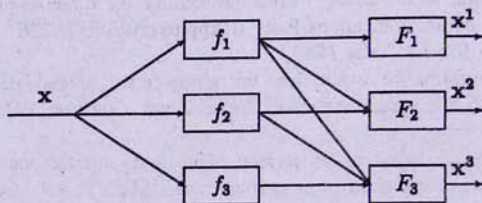


Fig. 1. Three-descriptions configuration with diversity decodings.

The difficulty of the problem in general forced researchers to introduce some restrictions—so called "without excess rate", "excess rate", "without excess marginal rate" cases [3], [5], [6], [7]. We do not put such assumptions for the system exhibited in Fig. 1.

The system under the consideration has three encoders, which generate three descriptions of n -sequence messages of the source. The first decoder receives only one of the descriptions, the second - two descriptions, and the third decoder is aware of all three descriptions. Every decoder must recover the vector of source messages within the given distortion threshold and with error probability smaller than permits the given exponent.

Discrete stationary source $\{X\}$ without memory is defined by finite alphabet \mathcal{X} and generic probability $P^* = \{P^*(x), x \in \mathcal{X}\}$. Hence, the probability of a source message vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{X}^n$ of length n is $P^{*n}(\mathbf{x}) = \prod_{i=1}^n P^*(x_i)$.

Let $\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3$ be finite sets, serving as reconstruction alphabets at each of the decoders, respectively. The encoding and decoding mappings are

$$f_r : \mathcal{X}^n \rightarrow \{1, 2, \dots, L_r(n)\}, \quad r = \overline{1, 3},$$

$$F_1 : \{1, 2, \dots, L_1(n)\} \rightarrow \mathcal{X}^{1n},$$

$$F_2 : \{1, 2, \dots, L_1(n)\} \times \{1, 2, \dots, L_2(n)\} \rightarrow \mathcal{X}^{2n},$$

$$F_3 : \{1, 2, \dots, L_1(n)\} \times \{1, 2, \dots, L_2(n)\} \times \{1, 2, \dots, L_3(n)\} \rightarrow \mathcal{X}^{3n},$$

The single letter distortion measures are defined by $d_r : \mathcal{X} \times \mathcal{X}^r \rightarrow [0; \infty)$, and averaged over length n

$$d_r(\mathbf{x}, \mathbf{x}^r) = n^{-1} \sum_{j=1}^n d_r(x_j, x_j^r), \quad r = \overline{1, 3}.$$

The probabilities of exceeding given distortion levels Δ_r are

$$e_1 = \Pr\{d_1(\mathbf{x}, F_1(f_1(\mathbf{x}))) \geq \Delta_1\},$$

$$e_2 = \Pr\{d_2(\mathbf{x}, F_2(f_1(\mathbf{x}), f_2(\mathbf{x}))) \geq \Delta_2\},$$

$$e_3 = \Pr\{d_3(\mathbf{x}, F_3(f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))) \geq \Delta_3\}.$$

If denote

$$\mathcal{A}_1 = \{\mathbf{x} : F_1(f_1(\mathbf{x})) = \mathbf{x}^1, d_1(\mathbf{x}, \mathbf{x}^1) \leq \Delta_1\},$$

$$\mathcal{A}_2 = \{\mathbf{x} : F_2(f_1(\mathbf{x}), f_2(\mathbf{x})) = \mathbf{x}^2, d_2(\mathbf{x}, \mathbf{x}^2) \leq \Delta_2\},$$

$$\mathcal{A}_3 = \{\mathbf{x} : F_3(f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x})) = \mathbf{x}^3, d_3(\mathbf{x}, \mathbf{x}^3) \leq \Delta_3\},$$

then

$$e_r = 1 - P^{*n}(\mathcal{A}_r), \quad r = \overline{1, 3}.$$

For fixed $E_r > 0$, $\Delta_r \geq 0$ a triplet (R_1, R_2, R_3) , $R_r \geq 0$, is a triplet of (E, Δ) -achievable rates $(E = (E_1, E_2, E_3), \Delta = (\Delta_1, \Delta_2, \Delta_3))$, if for every $\epsilon > 0$ and $n \geq n_0(\epsilon, E, \Delta)$ there exists a $(f_1, f_2, f_3, F_1, F_2, F_3) = (f, F)$ code, such that

$$e_r \leq \exp\{-nE_r\},$$

$$n^{-1} \log L_r(n) \leq R_r + \epsilon, \quad r = \overline{1, 3}.$$

Let $\mathcal{R}(\Delta)$ be the rate-distortion function for Fig. 1. Denote the set of all triplets of (E, Δ) -achievable rates $\mathcal{R}(E, \Delta)$ and call it "rates-reliabilities-distortions region" of our configuration.

When $E_r \rightarrow 0$, the region $\mathcal{R}(E, \Delta)$ becomes the rates-distortions region $\mathcal{R}(\Delta)$.

2 The Result

Let $P = \{P(x), x \in \mathcal{X}\}$ be a probability distribution (PD) on \mathcal{X} and

$$Q = \{Q(x^1, x^2, x^3 | x), x \in \mathcal{X}, x^r \in \mathcal{X}^r, r = \overline{1, 3}\}$$

be a conditional PD on Cartesian product $\mathcal{X}^1 \times \mathcal{X}^2 \times \mathcal{X}^3$.

Introduce the following sets

$$\alpha(E_r) = \{P : D(P \| P^*) \leq E_r\}, r = \overline{1, 3},$$

where $D(P \| P^*)$ is divergence, which are the collections of the higher probability messages types of the source for each E_r . For simplicity and originality we assume that $E_1 \geq E_2 \geq E_3$, such that due to the convexity of set $\alpha(E)$ for any E , $\alpha(E_1) \supseteq \alpha(E_2) \supseteq \alpha(E_3)$. In the other cases the exhibition of an achievable rates region is similar to the region given below.

Let $\Phi(P)$ is a function that to each P gives a conditional PD Q , such that if $D(P \| P^*) \leq E_r$, then

$$E_{P, \Phi(P)} d_r(X, X^r) = \sum_{x, x^1, x^2, x^3} P(x) Q(x^1, x^2, x^3 | x) d_r(x, x^r) \leq \Delta_r, r = \overline{1, 3}. \quad (1)$$

We denote the collection of all such Φ by $\mathcal{M}(E, \Delta)$. For any $\Phi \in \mathcal{M}(E, \Delta)$ define

$$\widetilde{\mathcal{R}}(E, \Delta, \Phi) = \{(R_1, R_2, R_3) :$$

$$R_1 \geq \max_{P \in \alpha(E_1)} I_{P, \Phi(P)}(X \wedge X^1), \quad (2)$$

$$R_1 + R_2 \geq \max_{P \in \alpha(E_2)} I_{P, \Phi(P)}(X \wedge X^1, X^2),$$

$$R_1 + R_2 + R_3 \geq \max_{P \in \alpha(E_3)} I_{P, \Phi(P)}(X \wedge X^1, X^2, X^3)\}.$$

Let

$$\widetilde{\mathcal{R}}(E, \Delta) = \bigcup_{\Phi \in \mathcal{M}(E, \Delta)} \widetilde{\mathcal{R}}(E, \Delta, \Phi). \quad (3)$$

Our aim is to prove the following

Theorem 1: For every $E_r > 0, \Delta_r \geq 0, (r = \overline{1, 3})$,

$$\widetilde{\mathcal{R}}(E, \Delta) \subset \mathcal{R}(E, \Delta). \quad (4)$$

Theorem 1 has the following important consequence. With $E_r \rightarrow 0, r = \overline{1, 3}$,

$$\widetilde{\mathcal{R}}(E, \Delta) \rightarrow \widetilde{\mathcal{R}}(\Delta),$$

because all expressions are continuous with respect to E_r . We obtain

Theorem 2: For every $\Delta_r \geq 0$ ($r = \overline{1,3}$),

$$\begin{aligned}\mathcal{R}(\Delta) \supset \widetilde{\mathcal{R}}(\Delta) &= \bigcup_{\Phi \in \mathcal{M}(\Delta)} \{(R_1, R_2, R_3) : \\ R_1 &\geq I_{P^*, \Phi(P^*)}(X \wedge X^1), \\ R_1 + R_2 &\geq I_{P^*, \Phi(P^*)}(X \wedge X^1, X^2), \\ R_1 + R_2 + R_3 &\geq I_{P^*, \Phi(P^*)}(X \wedge X^1, X^2, X^3)\}.\end{aligned}\quad (5)$$

Corollary: For the case when the third decoder is absent from Theorem 1 we receive the result of [8], and with $E_r \rightarrow 0$ ($r = 1, 2$) the result of Gray and Wyner [1].

3 Proof of Theorem 1

The proof of Theorem 1 is based on the following lemma about a covering of a type P from [8], [9].

Lemma: For any $\epsilon > 0$, types P, Q , for sufficiently large n a covering $\{T_{P,Q}(X | x_j^r), j = \overline{1, J(P, Q)}\}$ for $T_P(X)$ exists with

$$J(P, Q) = \exp\{n(I_{P,Q}(X \wedge X^r) + \epsilon)\}, \quad r = \overline{1, 3}.$$

Moreover, every $T_{P,Q}(X | x_j^r)$ may be covered by

$$\{T_{P,Q}(X | x_j^r, x_k^s), \quad k = \overline{1, K(P, Q)}\},$$

where $K(P, Q) = \exp\{n(I_{P,Q}(X \wedge X^s | X^r) + \epsilon)\}$, $s \neq r$, $s = \overline{1, 3}$. Analogically, $\{T_{P,Q}(X | x_j^r, x_k^s, x_m^t), m = \overline{1, M(P, Q)}\}$ covers a $T_{P,Q}(X | x_j^r, x_k^s)$ with

$$M(P, Q) = \exp\{n(I_{P,Q}(X \wedge X^t | X^r, X^s) + \epsilon)\}, \quad t \neq s \neq r, \quad t = \overline{1, 3}.$$

In the first part of the proof of the lemma the method of random selection [8], [9] is employed. The second part is a consequence of the first part.

Before describing the encoding and decoding schemes, note that \mathcal{X}^n is a union of vectors with disjoint types. For every P let us choose a conditional type $Q = \Phi(P)$, such that $\Phi \in \mathcal{M}(E, \Delta)$, i.e. (1). According to the lemma we construct a covering for $T_P(X)$ with disjoint elements

$$C_{j_1}(P, Q) = T_{P, \Phi(P)}(X | x_{j_1}^1) \setminus \bigcup_{j'_1 < j_1} T_{P, \Phi(P)}(X | x_{j'_1}^1), \quad j_1 = \overline{1, J_1(P, Q)},$$

$$J_1(P, Q) = \exp\{n(I_{P,Q}(X \wedge X^1) + \epsilon)\}.$$

It is clear that

$$T_P(X) = \bigcup_{j_1=1}^{J_1(P, Q)} C_{j_1}(P, Q).$$

For each $P \in \alpha(E_1) \cap \alpha(E_2)$, let a covering for $C_{j_1}(P, Q)$ be

$$C_{j_1, j_2}(P, Q) = C_{j_1}(P, Q) \cap \{T_{P, \Phi(P)}(X | x_{j_1}^1, x_{j_2}^2) \setminus \bigcup_{j'_2 < j_2} T_{P, \Phi(P)}(X | x_{j_1}^1, x_{j'_2}^2)\},$$

$$j_2 = \overline{1, J_2(P, Q)} \text{ and } J_2(P, Q) = \exp\{n(I_{P,Q}(X \wedge X^2 | X^1) + \epsilon)\}.$$

Similarly for each $P \in \alpha(E_1) \cap \alpha(E_2) \cap \alpha(E_3)$ a $C_{j_1, j_2, j_3}(P, Q)$ we cover by

$$C_{j_1, j_2, j_3}(P, Q) = C_{j_1, j_2}(P, Q) \cap \{T_{P, \Phi(P)}(X | x_{j_1}^1, x_{j_2}^2, x_{j_3}^3) \setminus$$

$$\bigcup_{j'_3 < j_3} T_{P, \Phi(P)}(X | x_{j_1}^1, x_{j_2}^2, x_{j'_3}^3)\}, \quad j_3 = \overline{1, J_3(P, Q)}$$

and consequently $J_3(P, Q) = \exp\{n(I_{P,Q}(X \wedge X^3 | X^1, X^2) + \epsilon)\}.$

Note that the encoding of the messages with $P \notin \alpha(E_r)$, $r = \overline{1, 3}$ can be omitted, because

$$\begin{aligned} P^{*n} \left(\bigcup_{P \notin \alpha(E_r + \epsilon)} T_P(X) \right) &\leq \sum_{P \notin \alpha(E_r + \epsilon)} P^{*n}(T_P(X)) \leq \\ &\leq (n+1)^{|X|} \exp\{-n(\min_{P \notin \alpha(E_r + \epsilon)} D(P \| P^*))\} \leq \\ &\leq \exp\{-n(E_r + |X| \log(n+1) + \epsilon)\} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Now define the encoding and decoding rules.

Encoding: All messages from $C_{j_1}(P, Q)$ f_1 encodes by j_1 , when $P \in \alpha(E_1)$. Since the first encoder helps the other decoders and the second encoder helps only to the third decoder, therefore

$$f_2(x) = j_2, \quad x \in C_{j_1, j_2}(P, Q), \quad P \in \alpha(E_1) \cap \alpha(E_2),$$

$$f_3(x) = j_3, \quad x \in C_{j_1, j_2, j_3}(P, Q), \quad P \in \alpha(E_1) \cap \alpha(E_2) \cap \alpha(E_3).$$

Decoding: F_r are converse mappings of corresponding f_r , i.e.

$$F_1(j_1) = x_{j_1}^1, \quad F_2(j_1, j_2) = x_{j_2}^2, \quad F_3(j_1, j_2, j_3) = x_{j_3}^3.$$

Distortion: Show that the distortion constraints are satisfied for these encoding and reconstruction rules. If $x \in C_{j_1, j_2, j_3}(P, Q)$ for any j_1, j_2, j_3 , then from (1)

$$\begin{aligned} d_1(x, F_1(f_1(x))) &= n^{-1} \sum_{i=1}^n d_1(x_i, x_{j_1, i}^1) = \\ &= n^{-1} \sum_{x, x^1} n(x, x^1 | x, x_{j_1}^1) d_1(x, x^1) = \\ &= \sum_{x, x^1, x^2, x^3} P(x) Q(x^1, x^2, x^3 | x) d_1(x, x^1) \leq \Delta_1. \end{aligned}$$

Similarly,

$$d_2(x, F_2(f_1(x), f_2(x))) \leq \Delta_2,$$

$$d_3(x, F_3(f_1(x), f_2(x), f_3(x))) \leq \Delta_3.$$

The number of used symbols in encoding are estimated as (for fixed P)

$$L_1(n) = \exp\{n(I_{P,Q}(X \wedge X^1) + \epsilon)\},$$

$$L_1(n)L_2(n) = \exp\{n(I_{P,Q}(X \wedge X^1) + I_{P,Q}(X \wedge X^2 | X^1) + 2\epsilon)\},$$

$$L_1(n)L_2(n)L_3(n) = \exp\{n(I_{P,Q}(X \wedge X^1) + I_{P,Q}(X \wedge X^2 | X^1) + I_{P,Q}(X \wedge X^3 | X^1, X^2) + 3\epsilon)\}.$$

The worst types among the $P \in \alpha(E_r + \epsilon)$, $r = \overline{1,3}$, determine corresponding bounds of rates in above references. It means to put

$$L_1(n) \geq \exp\{n(\max_{P \in \alpha(E_1 + \epsilon)} I_{P,Q}(X \wedge X^1) + \epsilon)\},$$

$$L_1(n)L_2(n) \geq \exp\{n(\max_{P \in \alpha(E_2 + \epsilon)} I_{P,Q}(X \wedge X^1, X^2) + 2\epsilon)\}, \quad (6)$$

$$L_1(n)L_2(n)L_3(n) \geq \exp\{n(\max_{P \in \alpha(E_3 + \epsilon)} I_{P,Q}(X \wedge X^1, X^2, X^3) + 3\epsilon)\}$$

and all demands on distortion and reliability will be satisfied. Taking into account the arbitrariness of ϵ and the function $\Phi(P)$, and the continuity of information theoretic expressions in right sides of (6), we have (2), and therefore (3).

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