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On a Class of Number-Theoretic Predicates Represented by Means of General Form Recursive Equations

A. R. Aslyan

Institute for Informatics and Automation Problems
of NAS of RA and YSU

Abstract

This paper is an attempt to tackle the problem of determination of class of number-theoretic predicates represented by means of general form recursive equations. By the representation of number-theoretic predicate by means of general form recursive equation the author suggests the existence of system of recursive equations ,which has unique solution, in the class of number-theoretic functions one of which is the characteristic function of the predicate given. The precise notion of representation of number-theoretic predicate by recursive equations will be given bellow. In Marandjian [6] it is shown the existence of recursive equation which represents the set K. Here it is shown that arbitrary arithmetical predicate $P(x_1, ..., x_k)$ representable by the system of recursive equations.

1 Introduction

The aim of this article is to prove that arbitrary arithmetical predicate is representable by general form recursive equations. The general form recursive equations has been introduced in Marandjian [6]. To be more precise in our considerations we will define the language *Rec* of recursive terms. Each recursive term of language *Rec* will denote a partial recursive functional. The definition we will now give for the language *Rec* is not symbol by symbol as in Marandjian [6], but in fact the set of functionals denoted by the terms of the language *Rec* coincide with that of 1 **Definition** in Marandjian [6].

Definition 1.1 *For arbitrary* $i, i_1, ..., i_n, n, q, k, 0 < j \leq n$:

1. $S(x_i) \in Rec$.
2. $C_q^n(x_{i_1}, ..., x_{i_n}) \in Rec$.
3. $P_j^n(x_{i_1}, ..., x_{i_n}) \in Rec$.
4. $f_i^n(x_{i_1}, ..., x_{i_n}) \in Rec$.
5. *If* $g_j(y_{i_1}, ..., y_{i_m})$; $f(x_{k_1}, ..., x_{k_n}) \in Rec$ *for* $0 < j \leq n$ *then* $Sub(f(x_{k_1}, ..., x_{k_n}); g_1(y_{i_1}, ..., y_{i_m}), ..., g_n(y_{i_1}, ..., y_{i_m})) \in Rec$.

6. *If* $f(x_{i_1},...,x_{i_n}), g(x_{i_1},...,x_{i_{n+2}}) \in Rec$ *and* $i_1,...,i_n \neq j$ *then*
$Pr[j; f(x_{i_1},...,x_{i_n}); (x_{i_1},...,x_{i_{n+2}})] \in Rec$.

7. *If* $f(x_{i_1},...,x_{i_n}) \in Rec$ *for* $0 < k \leq n$ *then* $\mu x_{i_k}[f(x_{i_1},...,x_{i_n})] \in Rec$.

Definition 1.2 *Let a finite set* $F_0,...,F_n,G_0,...,G_n$ *of recursive terms be given, then the system of equations of the form (1) is called general form recursive equations.*

$$\begin{array}{rcl} F_0[f_0,...,f_k](x_1,...,x_m) & \simeq & G_0[f_0,...,f_k](x_1,...,x_m) \\ & ... & \\ F_n[f_0,...,f_k](x_1,...,x_m) & \simeq & G_n[f_0,...,f_k](x_1,...,x_m) \end{array} \qquad (1)$$

Definition 1.3 *We will say that the solution of the system of general form recursive equation (1) with respect to the function symbol* $f_j, 0 \leq j \leq k$ *represents a number-theoretic predicate* $P(x_1,...,x_k)$ *if the system of recursive equations (1) has a unique solution in the class of number-theoretic functions, (which is a system of functions* $\mathsf{f}_0,...,\mathsf{f}_k$*) and the function* $\mathsf{f}_j(x_1,...,x_q)$ *from the list of functions* $\mathsf{f}_0,...,\mathsf{f}_k$ *is the characteristic function of predicate* $P(x_1,...,x_q)$.

2 Main Result

Theorem 2.4 *For arbitrary arithmetical predicate*
$P(x_1,...,x_k) \in \sum_n^0 (\prod_n^0)$ $n = 0,1,...,k,...$ *there exist a system of recursive equations*

$$\begin{array}{rcl} F_0[f_0](x_1,...,x_{k+n}) & \simeq & 0 \\ F_1[f_0,f_1](x_1,...,x_{k+n-1};y) & \simeq & 0 \\ ... & & \\ F_n[f_{n-1},f_n](x_1,...,x_k;y) & \simeq & 0 \end{array} \qquad (2)$$

such that, the solution of it with respect to the function symbol f_n *from (2) represents the predicate* $P(x_1,...,x_k)$.

Proof. We carry out the proof of this theorem by induction on n from the definition of arithmetical hierarchy.

Basis : Let $n = 0$ i.e. $P(x_1,...,x_k) \in \sum_0^0$, then the predicate $P(x_1,...,x_k)$ is recursive i.e. the characteristic function

$$\chi_P(x_1,...,x_k) = \begin{cases} 0 & if \quad x_1,...,x_k \in P; \\ 1 & if \quad x_1,...,x_k \notin P \end{cases}$$

is recursive.

Let us consider the equation $|\chi_P(x_1,...,x_k) - f_0(x_1,...,x_k)| \simeq 0$. It is easy to see that it has a unique solution which is an everywhere defined function equal to $\chi_P(x_1,...,x_k)$.

Induction step: Let $P(x_1,...,x_k) \in \sum_{n+1}^0$, then for some $Q(x_1,...,x_{k+1}) \in \prod_n^0$ we have $P(x_1,...,x_k) \Leftrightarrow \exists x_{k+1} Q(x_1,...,x_{k+1})$ and by induction hypothesis there exists a system of recursive equations

$$\begin{array}{rcl} F_0[f_0](x_1,...,x_{k+n+1}) & \simeq & 0 \\ F_1[f_0,f_1](x_1,...,x_{k+n};y) & \simeq & 0 \\ ... & & \\ F_n[f_{n-1},f_n](x_1,...,x_{k+1};y) & & 0 \end{array}$$

such that the solution of it with respect to the function symbol f_n represents the predicate $Q(x_1, ..., x_{k+1})$.

Consider the system of equations

$$\begin{array}{ll} F_0[f_0](x_1,...,x_{k+n+1}) & \simeq 0 \\ F_1[f_0,f_1](x_1,...,x_{k+n};y) & \simeq 0 \\ ... & \\ F_n[f_{n-1},f_n](x_1,...,x_{k+1};y) & \simeq 0 \\ F_{n+1}[f_n,f_{n+1}](x_1,...,x_k;y) & \simeq 0 \end{array} \quad (3)$$

where $F_{n+1}[f_n,f_{n+1}](x_1,...,x_k;y) \equiv_{Df}$

$$\equiv_{Df} \begin{cases} \overline{sg}(\mu t[f_n(x_1,...,x_k,t)=0]+1) & if \quad f_{n+1}(x_1,...,x_k)=0; \\ 0 & if \quad f_{n+1}(x_1,...,x_k)=1 \ \& \\ & \& \ f_n(x_1,...,x_k,y)=1; \\ undefined & otherwise. \end{cases}$$

Now we will prove that the system of recursive equations(3) has a unique solution $\mathsf{f}_0, ..., \mathsf{f}_{n+1}$ where f_{n+1} is the characteristic function of predicate $P(x_1, ..., x_k)$.

Let $x_1, ..., x_k, y \in N$ be arbitrary non-negative integers.

Case 1: $\mathsf{f}_{n+1}(x_1, ..., x_k)$ is the characteristic function of predicate $P(x_1, ..., x_k)$.

1.1: Let $P(x_1, ..., x_k)$ then $\exists x_{k+1} Q(x_1, ..., x_{k+1}) \Leftrightarrow \exists x_{k+1}[\mathsf{f}_n(x_1, ..., x_{k+1}) = 0]$ and $\mathsf{f}_{n+1}(x_1, ..., x_k) = 0$. From the construction of F_{n+1} it is evident that $F_{n+1}[\mathsf{f}_n, \mathsf{f}_{n+1}](x_1, ..., x_k; y) = 0$.

1.2: Let $\overline{P}(x_1, ..., x_k)$ then $\forall x_{k+1} \overline{Q}(x_1, ..., x_{k+1}) \Leftrightarrow \forall x_{k+1}[\mathsf{f}_n(x_1, ..., x_k, x_{k+1}) = 1]$ and $\mathsf{f}_{n+1}(x_1, ..., x_k) = 1$, hence for arbitrary y we have $F_{n+1}[\mathsf{f}_n, \mathsf{f}_{n+1}](x_1, ..., x_k; y) = 0$.

Case 2: $\mathsf{f}_{n+1}(x_1, ..., x_k)$ is not the characteristic function of predicate $P(x_1, ..., x_k)$.

2.1: Let $\exists x_1^*, ..., x_k^* (P(x_1^*, ..., x_k^*) \& \mathsf{f}_{n+1}(x_1, ..., x_k) = 1)$ then we have $F_{n+1}[\mathsf{f}_n, \mathsf{f}_{n+1}](x_1, ..., x_k; y) \uparrow$ by construction of F_{n+1}.

2.2: Let $\exists x_1^*, ..., x_k^* \left(\overline{P}(x_1^*, ..., x_k^*) \& \mathsf{f}_{n+1}(x_1, ..., x_k) = 0\right)$ then $\forall x_{k+1}[\mathsf{f}_n(x_1, ..., x_k, x_{k+1}) = 1]$ hence $F_{n+1}[\mathsf{f}_n, \mathsf{f}_{n+1}](x_1, ..., x_k; y) \uparrow$.

On the other hand, for arbitrary predicate $P(x_1, ..., x_k) \in \prod_{n+1}^0$ we have $\overline{P}(x_1, ..., x_k) \in \sum_{n+1}^0$ hence there exists a system of recursive equations (3) such that the solution of it with respect to the function symbol f_{n+1}represents $\overline{P}(x_1, ..., x_k)$. Replacing all the occurrences of term $f_{n+1}(U_1, ..., U_k)$ in the definition of functional $F_{n+1}[f_n, f_{n+1}](x_1, ..., x_k; y)$ with $\overline{sg}(f_{n+1}(U_1, ..., U_k))$

$F_{n+1}^*[f_n, f_{n+1}](x_1, ..., x_k; y) \equiv_{Df}$

$$\equiv_{Df} \begin{cases} \overline{sg}(\mu t[f_n(x_1,...,x_k,t)=0]+1) & if \quad \overline{sg}(f_{n+1}(x_1,...,x_k))=0; \\ 0 & if \quad \overline{sg}(f_{n+1}(x_1,...,x_k))=1 \ \& \\ & \& \ f_n(x_1,...,x_k,y)=1; \\ undefined & otherwise. \end{cases}$$

we will get a system of equations

$$\begin{array}{rcl} F_0\left[f_0\right](x_1,...,x_{k+1+n}) & \simeq & 0 \\ F_1\left[f_0, f_1\right](x_1,...,x_{k+n};y) & \simeq & 0 \\ ... & & \\ F_n\left[f_{n-1}, f_n\right](x_1,...,x_{k+1};y) & \simeq & 0 \\ F^*_{n+1}\left[f_n, f_{n+1}\right](x_1,...,x_k;y) & \simeq & 0 \end{array} \tag{4}$$

where the solution of it with respect to the function symbol f_{n+1} from (4) represents the predicate $P(x_1,...,x_k)$.

□

It is easy to see that for arbitrary system of recursive equations (1) there exist recursive terms
$F^*[f_0,...,f_k](x_0,x_1,...,x_m), G^*[f_0,...,f_k](x_0,x_1,...,x_m)$ such that the system of recursive equations (1) has a solution if and only if the equation

$$F^*[f_0,...,f_k](x_0,x_1,...,x_m) \simeq G^*[f_0,...,f_k](x_0,x_1,...,x_m) \tag{5}$$

has one and the solutions of both are coincide.

One may construct terms F^*, G^* using definition by cases

$$F^*\left[\bar{f}\right](x_0,\bar{x}) \equiv_{Df} \begin{cases} F_0\left[\bar{f}\right](\bar{x}) & if \quad x_0 = 0; \\ & ... \\ F_n\left[\bar{f}\right](\bar{x}) & if \quad x_0 = n; \\ 0 & \quad otherwise. \end{cases}$$

$$G^*\left[\bar{f}\right](x_0,\bar{x}) \equiv_{Df} \begin{cases} G_0\left[\bar{f}\right](\bar{x}) & if \quad x_0 = 0; \\ & ... \\ G_n\left[\bar{f}\right](\bar{x}) & if \quad x_0 = n; \\ 0 & \quad otherwise. \end{cases}$$

Hence we conclude.

Corollary 2.5 *For arbitrary arithmetical predicate* $P(x_1,...,x_k) \in \Sigma^0_n\left(\Pi^0_n\right)$ $n = 0,1,...,k,...$ *there exists a recursive equation*

$$F^*[f_0,...,f_n](x_0,x_1,...,x_{k+n},y) \simeq 0$$

such that the solution of it with respect to the function symbol f_n *represents the predicate* $P(x_1,...,x_k)$.

The theorem 2.4 states the representability of arithmetical predicates. On the other hand it is easy to see that arbitrary predicate represented by means of general form recursive equations (1) is hyperarithmetical. Here only the representability of arithmetical predicates is proved, whereas the same problem for hyperarithmetical predicates arises. The later is considered in Asljan [1].

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