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On Essential Components and Critical Sets of a Graph

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Abstract

We introduce some new concepts, which generalize the concepts of critical edge and critical component and investigate their relationship with the α-coverings of a graph, perfect graphs and the Strong Perfect Graph Conjecture(SPGC). The most of obtained results could be viewed either as analogues or as generalization of already known results. In particular, the notion of essential component is introduced and SPGC is reformulated with the help of this notion. A number of open problems is also suggested.

1 Introduction

If $G = (V, E)$ is a graph, $\alpha(G)$ and $\omega(G)$ denote stability number and clique number of G respectively. An i-stable set (i-clique) means a stable set (clique) of size i. Partitioning of vertices of a graph G into i stable sets (cliques) is called i-coloring (i-covering) of G. $k(G)$ is the clique covering number of G, that is the minimum of i for which an i-covering exists.

Throughout this paper, *subgraph* means *induced subgraph*, and $\{v\}$ is replaced by v. $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G respectively, i.e. $G = (V(G), E(G))$. $\overline{G}$ denotes the complementary graph. For $V' \subseteq V(G)$ the graph induced by V' is denoted by $G(V')$, and $G \setminus V'$ will mean $G(V(G) \setminus V')$.

C_{2k+1} (an odd cycle without diagonals), where $2k + 1 \geq 5$, is called a *hole* and its complement is called an *antihole*. A graph is said to be a *Berge graph* if it does not contain holes and antiholes.

Let $\{V_1, V_2\}$ be a partitioning of vertices of a graph $G = (V(G), E(G))$, i.e. $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. A subset of edges E' of G joining the vertices of V_1 and V_2 is called an *edge-cutset* or simply a *cutset* and is denoted by $E' = (V_1, V_2)$.

We say that a cutset (V_1, V_2) separates the vertices u and v, if $u \in V_1$ and $v \in V_2$ (or vice versa). A cutset (V_1, V_2) is called *augmental*, if $\alpha(G(V_1)) + \alpha(G(V_2)) > \alpha(G(V))$, otherwise it is called *nonaugmental* (in the sequel we will replace $\alpha(G(V_1))$ by $\alpha(V_1)$).

In 1961 Claude Berge [2] introduced a concept of perfect graphs which play the central role in this paper: a graph G is called *perfect* if $\alpha(G') = k(G')$ for every subgraph $G' \subseteq G$, otherwise it is called *imperfect*. A graph is called *minimal imperfect* if it is not perfect, but all its proper subgraphs are perfect. Berge also suggested two conjectures concerning perfect graphs:

If a graph G is perfect, then its complement $\overline{G}$ is also perfect.

(Strong Perfect Graph Conjecture) *A graph G is perfect iff G does not contain holes and antiholes.*

The first conjecture is proved by Lovász [8], while the second is still open. The Strong Perfect Graph Conjecture (SPGC) is equivalent to the following: if G is a minimal imperfect graph, then it is either a hole or an antihole. If SPGC is not true, then there must exist a minimal imperfect graph distinct from C_{2k+1} and $\overline{C}_{2k+1}$. Such a graph is called a *monster* (P.Duchet).

In [9, 18] the following properties of a minimal imperfect graph G are proved:

*P*1. $|V(G)| = \alpha\omega + 1$, where $\alpha = \alpha(G), \omega = \omega(G)$.

*P*2. Every vertex of G is contained in exactly ω ω-cliques and α α-stable sets.

*P*3. For every ω-clique Q there exists a unique α-stable set S such that $Q \cap S = \emptyset$ and vice versa.

*P*4. No matter which vertex is removed, the remaining subgraph can be uniquely partitioned into α ω-cliques and ω α-stable sets.

*P*5. The incidence matrix of the α-stable sets (ω-cliques) and the vertices of G is nonsingular.

A graph satisfying properties $P1 - P5$ is called a (α, ω)-graph. (α, ω)-graph by itself is a very interesting object for investigation (see for example [4, 6]).

In order to find criteria for existence of an α-covering in a graph, in [12] a concept of critical component was introduced. Below the existing results concerning critical edges and critical components are summerized.

An edge e of a graph G is called *critical*, if $\alpha(G \setminus e) > \alpha(G)$. A chain of a graph G is called *critical* if it consists of either critical edges or a single vertex. Further we do not distinguish a critical chain from the subgraph induced by the vertices of that chain.

Definition 1 *A maximal subgraph of a graph G, the vertices of which are connected by critical chains, is called a critical component of G.*

It is clear, that if $e = (u, v)$ is a critical edge of a graph G, then for each α-covering of G (if any exists), vertices u and v are covered by the same clique of that α-covering. Hence, if a graph has an α-covering, then its critical components are complete. Converse is not true. Indeed, we can construct a counterexample by connecting all the vertices of arbitrary imperfect graph G with $K(G) > \alpha(G) + 1$ to the vertices of an empty graph with $\alpha(G) + 1$ vertices. It is obvious that obtained graph is imperfect and has no critical edges, therefore, its critical components are complete. Based on this kind of observations, in 1975, S.Markossian [12] suggested the following conjecture.

Conjecture 1 *Graph is perfect iff the critical components of all its subgraphs are complete.*

Now let us suppose, that Conjecture 1 is true. How one can use it to prove the SPGC? For this reason we need to prove this.

Conjecture 2 *Critical components of a Berge graph are complete.*

Thus, Conjectures 1 and 2 together are equivalent to the SPGC. It is proved in [13], that Conjecture 1 alone is equivalent to the SPGC. Later, András Sebö ellegantely strengthened this result in [20], by the following

Theorem 1 *The critical components of a monster are complete.*

It follows from Theorem 1 that holes and antiholes, and only they, are minimal imperfect graphs containing an incomplete critical component. It is also clear, that holes and antiholes are among minimal graphs containing an incomplete critical component, since their subgraphs are perfect. Now one can ask a question: are the holes and antiholes the only minimal graphs containing an incomplete critical component? If Conjecture 2 is true then it provides positive answer to the question above. Section 2 is devoted to Conjecture 2 and other problems concerning critical components of a graph.

As utilizing critical edges and critical components for exploring new approaches to the SPGC proved to be useful, the authors of this paper decided to go further and generalize the concepts of critical edges and components in different directions. In Sections 3 and 4 we introduce concepts of essential components, critical and α-critical sets and subgraphs, quasi-critical edges, and present many results and open problems regarding them.

2 Critical edges and critical components

In this section the subject of our interest is the class of minimal graphs containing an incomplete critical component. For convenience, let's call this class of graphs MICC (for Minimal, Incomplete Critical Component containing graphs). If Conjecture 2 is true, then MICC coincides with the class of holes and antiholes. (Indeed, by definitions, the only non-Berge graphs in MICC are holes and antiholes; from the other hand Conjecture 2 claims that there is no Berge graph in MICC.) Thus, if Conjecture 2 is proved, we will have a new definition for a Berge graph: if the critical components of all subgraphs of a graph are complete then it is a Berge graph.

Let us now suppose, that Conjecture 2 is not true. Then there exists a Berge graph in MICC. Throughout this section this graph will be denoted by N. It is obvious that existence of N is equivalent to Conjecture 2. It worths also to mention that unlike Conjecture 1, Conjecture 2 is not equivalent to the SPGC, i.e. if there is no N graph then SPGC and Conjecture 1 may not necessarily be true.

Making an attempt to prove Conjecture 2, the authors explored some properties of the graph N which seem to be very interesting because of their similarity to the ones of (α, ω)-graphs ([11, 12, 13]).

Let $P_{m+1} = \{v_1, v_2, \dots, v_m, v_{m+1}\}$ be a critical chain and S_i be the $(\alpha - 1)$-stable set such that $S_i \cup v_i$ and $S_i \cup v_{i+1}$ are $\alpha(N)$-stable sets.

Statement 1 *A non-tirvial incomplete critical component of the graph N is either a critical chain of length at least $\omega(N)$ or a critical cycle.*

Proof. Let H be an incomplete critical component of N and $P_{m+1} = \{v_1, v_2, \dots, v_m, v_{m+1}\}$ be a minimal incomplete critical chain in H. Then

$$N' = N(\{\bigcup_{i=1}^{m} S_i\} \cup \{P_{m+1}\})$$

has an incomplete critical component, hence, by minimality of N, $N' = N$. Thus,

$$V(N) = V(N') \quad m(\alpha(N) \quad 1) + m + 1 = m\alpha(N) + 1.$$

Let Q be a clique in N. Since $(v_1, v_{m+1}) \notin E(N)$, without loss of generality we may assume, that $v_{m+1} \notin Q$, which leads to

$$|Q| = \left|\bigcup_{i=1}^{m}\Big\{(S_i \cup \{v_i\}) \cap Q\Big\}\right| \leq m.$$

From the minimality of P_{m+1} we have that $\{v_1, \dots, v_m\}$ and $\{v_2, \dots, v_{m+1}\}$ are cliques in N and we conclude that $m = \omega(N)$. Thus, every critical chain of length less than $\omega(N)$ induces a complete subgraph in N.

Let $e = (u, v)$ be a critical edge in H which is not in P_{m+1}, but has a common vertex with it. Clearly, P_{m+1} cannot contain both vertices u and v simultaneously, otherwise the v_1 and $v_{\omega(N)+1}$ would be connected by a critical chain of length less than $\omega(N)$ and would be adjacent. Suppose $v = v_i$, where $1 \leq i \leq \omega(N) + 1$. If $i \neq 1, \omega(N) + 1$, then the vertices $v_2, \dots, v_{\omega(N)}$ are connected with the vertex u by critical chains of length less than $\omega(N)$, hence, at least one of the sets $\{u, v_1, \dots, v_{\omega(N)}\}$ and $\{u, v_2, \dots, v_{\omega(N)+1}\}$ induce a clique of size $\omega(N) + 1$ (here we assumes that $\omega(N) > 2$, otherwise N contains a hole: see Corollary 9, Section 4). Obtained contradiction proves that either $v = v_1$ or $v = v_{m+1}$, which ends the proof of the statement.

Corollary 1 $|V(N)| \leq \omega(N)\alpha(N) + 1$.

Corollary 2 *Critical components of the graph N containing no more than $\omega(N)$ vertices are complete.*

Our analysis involves a class of graphs $C^{\omega}_{\alpha\omega+1}$ known as webbs. $G = C^{\omega}_{\alpha\omega+1}$is a graph with $|V(G)| = \alpha\omega + 1$ vertices indexed $v_1, v_2, \dots, v_{\alpha\omega+1}$ in such a way that v_i is adjacent to v_j iff $(|i - j| < \omega)(\text{mod } \omega)$. It is easy to see that $\alpha(G) = \alpha$ and $\omega(G) = \omega$. One can observe many properties of webbs that are similar to the ones of holes and antiholes. For example, $P_{m+1} = \{v_1, \dots, v_{m+1}\}$, where $m = \alpha\omega$, is a hamiltonian critical cycle of $C^{\omega}_{\alpha\omega+1}$. Another one is the following: if S_i is the $(\alpha - 1)$-stable set such that $S_i \cup v_i$ and $S_i \cup v_{i+1}$ are α-stable sets, then

*P*6. for each i, $1 \leq i \leq m$, the following equality holds:

$$P_{m+1} \cap S_i = \{v_j / j < i, (j - i = 0)(\text{mod } \omega)\} \cup \{v_j / j > i + 1, (j - i = 1)(\text{mod } \omega)\}.$$

Statement 2 *If P_{m+1} is an incomplete critical component of the graph N, then property P6 is satisfied for P_{m+1}.*

Proof. Let $k < i \leq k + w$, $\omega = \omega(N)$ and Q_l denote the ω-clique $\{v_l, \dots, v_{l+\omega-1}\}$ $(1 \leq l \leq m - \omega + 2)$. We know that

$$V(N) = \left\{\bigcup_{j=k+1}^{k+\omega} S_j\right\} \cup Q_{k+1} \cup V_{k+\omega+1}.$$

Therefore, $|S_i \cap Q_l| = 1$ if $i + 1 < l$. Obviously, $S_i \cap Q_{i+2} = v_{i+\omega+1}$, since $Q_{i+1} = \{v_{i+1}, \dots, v_{i+\omega}\}$ is a clique. Continuing in the same manner we can show that $S_i \cap Q_{i+(l-1)\omega+2} = v_{i+l\omega+1}$. Thus,

$$v_{i+2}, \dots, v_{m+1} \quad S_i = \ v_j / j > i + 1, (j \quad i = 1)(\text{mod } \omega) \ .$$

Using similar speculations and going in the "opposit" direction from v_i, we can show that

$$\{v_1, \ldots, v_{i-1}\} \cap S_i = \{v_j / j < i, (i - j = 0)(\mathrm{mod}\ \omega)\}.$$

Statement is proved.

Corollary 3 *If $(j - i = 1)(mod\ \omega)$ and $j > i + 1$, then v_i and v_j are nonadjacent.*

Corollary 4 *If $(i - j = 0)(mod\ \omega)$ and v_i and v_j are adjacent, then the vertices v_i and v_j are the endpoints of P_{m+1}.*

Statement 3 *If P_{m+1} is an incomplete critical component of the graph N, $m = k\omega$ and $e = (v_1, v_{m+1}) \in E(N)$, then $k = \alpha(N)$ and e is a critical edge, i.e. $P_{m+1} \cup e$ is a Hamiltonian critical cycle.*

Proof. Since v_1 and v_{m+1} are adjacent, then $\alpha(P_{m+1}) \leq k$. From the other hand, by Statement 2 $|(P_{m+1} \cap S_1) \cup v_1| = k$, hence, $\alpha(P_{m+1}) = k$. All the edges (v_i, v_{i+1}) remain critical in P_{m+1}, therefore, from minimality of N follows that $P_{m+1} = N$. Based on Statement 2, it is also easy to verify, that $\{v_1, v_{\omega+1}, \ldots, v_{\alpha\omega+1}\}$ is a $(\alpha(N)+1)$-stable set in graph $P_{m+1} \backslash e$. End of proof.

It follows from Statement 3, that in the graph N every critical cycle is Hamiltonian. Thus, we proved

Theorem 2 *A non-trivial incomplete critical component of the graph N is either a critical chain of length at least $\omega(N)$ or a Hamiltonian critical cycle.*

If N has a Hamiltonian critical cycle, it is called *N-webb*, because it contains a webb $C^{\omega(N)}_{\alpha(N)\omega(N)+1}$ as a spanning subgraph. Chvátal [5] has proved that a monster M cannot contain a spanning subgraph that is a webb with the stability number $\alpha(M)$. Since we believe that no N graph exists, we should also think that the statement similar to the Chvátal's theorem is true for the N graph. Unfortunately, we could not prove this fact so we must put it as a conjecture, which is weaker than Conjecture 2.

Conjecture 3 *There is no N-webb.*

Statement 4 *If P_{m+1} is an incomplete critical chain of N intersecting with a $\omega(N)$-clique Q but not contaning it, then the intersectioin of P_{m+1} and Q consists of either one or two end-subchains of P_{m+1} (end-subchain means a subchain containing an endpoint of the chain).*

Proof. It is enough to prove the statement only for $m = \omega(N)$. Let $Q \cap P_{m+1} = \{v_{i_1}, \ldots, v_{i_l}\} = Q'$ and $v_{m+1} \notin Q'$ (the latter is a valid assumption since, by the conditions of the Statement and by Statement 3, Q cannot contain both of v_1 and v_{m+1}). Then $Q \cap S_{i_j} = \emptyset, j = 1, \ldots, l$. If $v_i \in Q'$ and $i > 1$ then $v_{i-1} \in Q'$, otherwise $Q \cap S_{i-1} = \emptyset$, and Q has no vertices from at least $l + 1$ S_i-s, which contradicts to the fact, that Q is a $\omega(N)$-clique.

Statement 5 *If an incomplete critical component H of N contains a $\omega(N)$-clique Q, then Q is a critical subchain of H.*

Proof. Let $P_{\omega(N)+1}$ be a critical chain from H such that $v_1 \notin Q$ and $v_2 \in Q$. Existence of such a $P_{\omega(N)+1}$ is garanteed by Statement 4. Now, from the same statement it follows that $Q = \{v_2, \ldots, v_{\omega(N)+1}\}$.

Definition 2 *Graph is called cuttable, if it contains a nonaugmental cutset, otherwise it's called uncuttable.*

Statement 6 *N is an uncuttable graph.*

Proof. Statement is trivial for N-webbs.

Clearly, if the cutset has a critical edge, then it is augmental. Hence, if the cutset (V_1, V_2) of N is nonaugmental, then an incomplete critical chain P_{m+1} of N is completely contained in one of the sets V_1 or V_1. Let's say, $P_{m+1} \subset V_1$. For any pair S, S' of $\alpha(N)$-stable set of N $|V_1 \cap S| = |V_1 \cap S'| = \alpha(N(V_1))$. Hence, the graph $N(V_1)$ contains an incomplete critical component which contradicts to the definition of N.

Corollary 5 *For any clique Q of N there exists an $\alpha(N)$-stable set not intersecting with Q.*

One can see that (α, ω)-graphs satisfy to all the properties proved for the N graph above. It will be interesting to check validity of the other properties of (α, ω)-graphs for N:

- For any critical edge there are exactly $\omega - 1$ ω-cliques containing it.
- For any $v \in V(N)$, $N \setminus v$ has a ω-coloration.
- For any $v \in V(N)$, $N \setminus v$ has a α-covering.

In the next section we will see (Statement 11), that only the last property is enough to show that N is a (α, ω)-graph.

At the end of this section we formulate two conjectures concerning critical edges and monster. These conjectures are weaker than SPGC and Conjecture 1, but seem to be very difficult to prove as well.

Conjecture 4 *Monster has a critical edge.*

Conjecture 5 *Monster with a removed critical edge is perfect.*

Within the context of last two conjectures, it also would be interesting to investigate the monsters with minimal number of edges since for such a monster M, one of the following properties holds:

1. M has a critical edge, removing of which makes the graph perfect.
2. For any edge $e \in E(M)$, the graph $M \setminus e$ contains a hole or antihole.

It must be mentioned that there are Berge graphs and perfect graphs satisfying 2. For example the graph in Fig.1 is a perfect Berge graph satisfying 2.

3 Essential edges and essential components.

Definition 3 *Edge $e = (v, u)$ of a graph G is called essential, if each cutset of G separating vertices u and v is augmental. If u and v are not adjacent and each cutset separating them is augmental, then the co-edge $\{u, v\}$ is called essential.*

Critical edge is essential, but the converse is not true. For example, if a 5-hole (C_5) contains one diagonal, then the diagonal is an essential edge but not critical. Moreover, graph in Fig.2 contains no critical edges, but it does have an essential edge – (5,6).

Definition 4 *A chain of a graph G is called essential if it consists of either essential edges or a single vertex. A maximal subgraph of a graph G, the vertices of which are connected by essential chains, is called an essential component of G.*

Unlike critical component which may contain non-critical edges, all the edges of essential components are essential. A critical component is contained in one essential component, which may consist of many critical components.

If a graph has a α-covering, then for any α-covering the vertices of an essential edge are covered by the same clique. Therefore, for any α-covering the vertices of an essential component are also covered by the same clique. Hence, if a graph has a α-covering, then its essential components are complete. The converse is not true: the graph in Fig.3 has 4 complete essential components, that are the edges $(1,4); (2,5); (3,6)$ and the vertex 7, but it is imperfect.

There are even graphs, which contain no essential edge and no α-covering. As an example one can consider the Rosenfeld graph, which consists of two 5-holes and a 5-stable set, all the vertices of which are adjacent with the vertices of the 5-holes.

Nevertheless, the following Statement, which sounds very similar to the Conjecture 1, is true.

Statement 7 *Graph is perfect iff the essential components of all its subgraphs are complete.*

Proof. The "only if" part is trivial. We will prove the "if" part by induction. Let us assume that the statement is true for the graphs which have less than n vertices, and let G be a n-vertex graph. If G is complete, then the statement is true. If G is not complete, then it has more than one essential components, which means that there is a nonaugmental cutset (V_1, V_2). It is easy to see, that an essential components of $G(V_i)$ is an essential component for G as well. Hence, by the induction assumption, each of $G(V_1)$ and $G(V_2)$ has an α-covering. The union of these two coverings forms an α-covering for G.

Corollary 6 *If for each subgraph of a Berge graph the critical and essential components coincide, then it is a perfect graph.*

Proof. If the statement is not true then, there exists a monster M with coinciding critical and essential components. From Statement 7 we have that M contains incomplete essential component which contradicts to Theorem 1.

Perfectness of the graphs, each hole of which contains at least two diagonals, was independently proved by Markossian and Karapetian [14] and Meyniel [17] (a graph satisfying this property is called a Meyniel graph). It is proved in [15], that a Meyniel graph, with an edge removed, is perfect as well. This result has been strengthened by A.Hertz in [16] who showed that a Meyniel graph without the edges of any its subgraph is perfect. Hence, critical and essential edges of a Meyniel graph coincide.

From Statement 7 and the fact that SPGC and Conjecture 1 are equivalent, we have also, that SPGC is equivalent to the following

Conjecture 6 *If the critical components of every subgraph of G are complete, then the essential components of every subgraph of G are also complete.*

If a graph has an α-covering, then it has no essential co-edges. The converse is not true. See a counter-example in Fig.2. If a graph has an incomplete essential component, then it has an essential co-edge. It would be interesting to check, whether this condition is also necessary for existence of an essential co-edge:

Conjecture 7 *Essential components of a graph are complete iff it has no essential co-edge.*

Below we present two simple statements in this regard.

Statement 8 *If G has no essential co-edges, then for any subset of vertices V', such that*

$$\alpha$$

lpha(V')=$\alpha(V)$, subgraph $G(V')$ also has no essential co-edges.

Proof. It is easy to check, that if deleting of a set of vertices doesn't change stability number of a graph, then obtained graph can not contain new essential edges or co-edges.

Corollary 7 *If the essential components of G are complete, then they are also complete for any subgraph G' of G with the same stability number $\alpha(G') = \alpha(G)$.*

Let us now consider the relation between essential edges and "cuttability" of a graph. It is clear, that all the edges and co-edges of an uncuttable graph are essential. We will call a graph *α-critical* (see Section 4 and [12]), if it contains only critical edges. An α-critical graph is uncuttable. A minimal imperfect graph G is also uncuttable. Indeed, if it contained a nonaugmental cutset (V_1, V_2), then the combination of α-coverings of $G(V_1)$ and $G(V_2)$ would be an α-coverings for G which contradicts to definition of G. Moreover:

Statement 9 *A (α, w)-graph is uncuttable.*

Proof. Let's suppose that there is a cuttable (α, ω)-graph G and let A be the incidence matrix of α-stable sets and vertices of G. It is well-known, that A is a non-singular matrix of size $n = \alpha\omega + 1$.

As G is cuttable, it must contain a nonaugmental cutset (V_1, V_2). Consider the following equation:

$$Ax = ke \quad (1)$$

where $k = \alpha(V_1)$ and $e = (1, \ldots, 1)$ is a n-vector.

Obviously, $y = (y_1, y_2 \ldots y_n)$, $y_i = k/\alpha$ is a solution of 1. But it is also easy to see that the characteristic vector of V_1 is also a solution of 1, which contradicts to the fact that A is non-singular.

Let's give a combinatorial proof of Statement 9.

If (V_1, V_2) is nonaugmental, then V_1 has only $k, 1 \leq k < \alpha$, vertices from each α-stable set. The number of α-stable sets is equal to n. Taking into account that every vertex is contained in exactly α α-stable sets, we have that cardinality of V_1 is equal to

$$|V_1| = \frac{nk}{\alpha} = wk + \frac{k}{\alpha}$$

which is not an integer. A contraduction.

If a graph has a unique α-covering, it doesn't mean yet that each clique of this covering is an essential component of the graph. However, this is true for a (α, ω)-graph with a removed vertex.

Statement 10 *If G is an (α, w)-graph, then for each vertex $v \in V(G)$, essential components of the subgraph $G \setminus v$ are the cliques of the unique α-covering of $G \setminus v$.*

Proof. Let's consider an equation:

$$Bx = ke \tag{2}$$

where k a positive integers, $1 \leq k < \alpha$, $e = \{1, \ldots, 1\}$ and B is the incidence matrix of the α-stable sets and vertices of $G \setminus v$. The dimension of B is $(n - \alpha, n - 1)$, where $n = \alpha\omega + 1$ is the number of vertices of G. As the rows of B are linearly independent, the set of solutions of 2 is a plain of dimension $\alpha - 1$. Let $\{Q_1, \cdots, Q_\alpha\}$ be the α-covering of $G \setminus v$ and z_i be the characteristic vector of the clique Q_i. Obviously, vectors $z_1, \ldots, z_\alpha$ are linearly independant and each kz_i is a solution of 2.

Now let z be the characteristic vector of some nonaugmental cutset (V_1, V_2) such that $\alpha(V_1) = k$. Since $kz_1, \ldots, kz_\alpha$ is an affinian basis for the solutions of 2, z can be represented as a linear combination of vectors z_i: $z = \lambda_1 z_1 + \ldots + \lambda_\alpha z_\alpha$. We know, that z is a 0-1-vector, and different vectors z_i and z_j do not have two 1-s at the same coordinate, which leads us to the conclusion, that $\lambda_i = 0, 1$, hence, V_1 either contains a Q_i completely or is disjoint from it. Statement is proved.

We don't know any combinatorial proof of Statement 10.

Statement 11 *If a graph G is uncuttable and for all $v \in V(G)$, the subgraphs $G \setminus v$ have an α-covering, then G is a (α, ω)-graph.*

Proof. Since G is uncuttable, for any clique of G there exists an α-stable set not intersecting with this clique, otherwise, G would have a nonaugmental cutset. This fact together with the condition that for every $v \in G$, $G \setminus v$ has an α-covering, are enough [7], for G to be an $(\alpha, w) - graph$.

The following conjecture sounds similar to Statement 11 but seems to be more difficult:

Conjecture 8 *If a graph G is uncuttable and essential components of the graphs $G \setminus v$ are complete for all $v \in V(G)$, then G is an (α, ω) graph.*

4 Critical sets and α-critical subgraphs

Another natural generalization of a critical edge is the following:

Definition 5 *A subset of edges E' of a graph G is called critical, if it is a minimal set of edges deleting of which makes the stability number grow, i.e. $\alpha(G \setminus E') > \alpha(G)$ and $\alpha(G \setminus E'') = \alpha(G)$ for each $E'' \subset E'$. An edge is called quasi-critical, if there is a critical set containing it.*

It is easy to see, that if E' is a critical set, then the subgraph $G(E')$ is an induce subgraph; i.e. $G' = G(E')$ is a minimal induced subgraph with the property $\alpha(G \setminus G') > \alpha(G)$ iff E' is a critical set. Such minimal subgraphs is called *critical subgraphs*.

How do the essential and quasi-critical edges relate to each other? An essential edges may not be quasi-critical. For instance, all the edges of the graph in Fig.4 are critical except $(2, 4)$, which is essential but not quasi-critical.

From the other hand, there are quasi-critical edges that are not essential. Moreover, it is possible that all the edges of a critical set are not essential. For example, even cycle without diagonals has no essential edges, but as any non-empty graph, it has critical sets.

Every edge (u, v) of an (α, ω)-graph is quasi-critical. Indeed, there is unique pair of disjoint α-stable set S and ω-clique Q such, that $u \in Q$ and $v \in S$. Now one can check that the set of edges going from u to S is critical.

Statement 12 *If the essential components of a graph are complete, then after deleting of any non-quasi-critical or non-essential edge they remain complete.*

Due to simplesity we omit the proof.

Conjecture 9 *If the essential components are complete, then essential edges are quasi-critical.*

T is called a *transversal* of a class of sets $C = \{A_j\}$, if

1. $T \cap A_j \neq \emptyset$, for any j.

2. For any $a_j \in T$ there exists such an A_{i_j} that $A_{i_j} \cap T = a_j$.

Maximum number of pairwise disjoint sets from the class C is called *independence* number of C and denoted by $\alpha(C)$. $\tau(C)$ denotes the *transversal* number of C, that is the size of the minimum transversal of C.

It is clear, that $\alpha(C) \leq \tau(C)$.

The set $T \subset V(G)$ is called $\alpha -$ *transversal* for a graph G, if T is a transversal for the class of all α-stable sets of G.

Theorem 3 *For each quasi-critical edge of a graph G which is not an α-transversal, there exists an odd cycle containing it.*

Proof. Let E' be a critical set, $e = (u, v) \in E'$, and S be an α-stable set of G disjoint from u, v. Further, let S' be the $(\alpha+1)$-stable set in $G \setminus E'$, and R be a subgraph of $G \setminus E'$ induced by the set $S \cup S'$. We want to show, that the vertices u and v are in the same connected component of R. As S' has more vertices than S, R must contain a connected component H, which has more vertices from S', than from S. Let $V' = (S' \quad V(H)) \quad (S \quad V(H))$ and

$R' = G(V')$. It's clear, that $|V'| = \alpha + 1$, which means that $E(R')$ is a critical set. From the other hand, since all edges of R' belong to $G(S')$ and $E' = E(G(S'))$, then $E(R') = E'$, which implies $u, v \in V(H)$.

Now, if P is a chain in H connecting u and v. Then $P \cup (u, v)$ is an odd cycle in G.

Theorem 3 generalizes some of the results from [1, 3, 12].

Corollary 8 *[12] Let E' be a critical set, and S' be an α-stable set disjoint from $V(E')$. Then there exists an edge in E', which is contained in a hole or a triangle.*

Corollary 9 *[3, 12] If two critical edges are incident, then there exists a hole or a triangle containing them.*

Corollary 10 *[1] Critical edges of a bipartite graph have no common vertices.*

Statement 13 *In a bipartite graph an edge is quasi-critical iff it is an α-transversal.*

Proof. Necessity immediately follows from Theorem 3.

Sufficiency. Let $e = (u, v)$ be an α-transversal and S be an α-stable set containing v. Let $v_1, v_2, ..., v_k, v$ be all the vertices of S adjacent to u. Then the set of edges $U = \{(u, v_1), (u, v_2), \ldots, (u, v_k), (u, v)\}$ contains a critical set U' (possibly $U' = U$). As e is an α-transversal, it follows that $(u, v) \in U'$.

Even if all quasi-critical edges are α-transversals, it is not sufficient for a graph to be bipartite. For example, all the essential edges of the graph in Fig.5, are quasi-critical, while the graph is not bipartite. But the following is true:

Statement 14 *A graph is bipartite iff for its any subgraph a quasi-critical edge is an α-transversal.*

Proof. The "only if" is the content of Statement 13. The "if" part follows from the fact that in holes and triangles every edge is quasi-critical but not an α-transversal.

Let $\{E_i\}$ be a class of all critical sets of the graph G. A transversal T of this class is called an *α-critical set* . The spanning subgraph $H = (V, T)$ of G is called a *α-critical subgraph* , Obviously, $\alpha(H) = \alpha(G)$ and for any $e \in H, \alpha(H \setminus e) > \alpha(H)$. If a graph has an α-covering, then the subset of all the edges of its cliques is a transversal for the class of critical sets of the graph. If a transversal T of that class is not an α-covering, then by Corollary 9 the graph contains a hole.

Definition 6 *The subset of edges E' of a graph G is called independent, if for any edge $e \in E'$ there exists a critical set E_e such that $E_e \cap E' = e$.*

Clearly, if T is a transversal, then it is independent. Not all independent sets could be extended to a transversal. For example, for a four-cycle without diagonals, two incident edges are independent, but there is not a transversal containing them.

Definition 7 *A subset of edges E' of a graph G is called strongly independent, if for each edge $e \in E'$ there exists a critical set E_e such that for each critical set E_j*
1. $E' \cap E_e = \{e\}$ 2. $E_j \not\subset \{\bigcup_{e\in E'} E_e\} \setminus E'$.

It is easy to see, that a set can be extended to a transversal iff it is strongly independent.

Conjecture 10 *If the essential components of a graph are complete, then the set of edges of any essential component is independent (strongly independent).*

One of the main goals of our investigation is to find criteria for a graph to have an α-covering. But "unfortunately", completeness of essential components, non-existence of essential co-edges, strongly independence of the essential edges all together are not enough for existence of an α-covering (e.g. see the graph in Fig.3).

If $\{F_j\}$ is the class of all α-critical sets of the graph G, then each element of $\{F_j\}$ is a transversal for the class of critical sets $\{E_i\}$ of G. It is easy to see, that each element of $\{E_i\}$ also is a transversal for $\{F_j\}$. This classes are uniquely determine each other, as one is the set of all transversals of the other. We have

$$\alpha(\{E_i\}) \leq \tau(\{E_i\}) \tag{3}$$

$$\alpha(\{F_j\}) \leq \tau(\{F_j\}) \tag{4}$$

If G has a critical edge e, then e belongs to each α-critical set F_j, therefore $\alpha(\{F_j\}) = \tau(\{F_j\}) = 1$. If G is an α-critical graph, then in 3 and 4 we have equalities. But there exist graphs, for which we have strict inequalities in both 3 and 4, e.g. for $K_6 \setminus e$ (K_n is the complete graph having n vertices) we have:

$$\alpha(\{E_i\}) = 5, \tau(\{E_i\}) = 6$$

$$\alpha(\{F_j\}) = 1, \tau(\{F_j\}) = 2$$

It can be shown, that the strict inequality holds for each $K_n \setminus e$, $n \geq 6$.

Statement 15 *If G is bipartite, then $\alpha(\{E_i\}) = \tau(\{E_i\})$.*

Proof. It follows from König's Theorem, that for a bipartite graph each α-critical set is a maximum matching having $n - \alpha(G)$ edges (where $n = |V(G)|$), i.e. $\tau(\{E_i\}) = n - \alpha(G)$. Conversely, there exist $n - \alpha(G)$ disjoint critical sets in G. Indeed, let S be an α-stable set of G. Then the set of all edges joining a vertex $v \notin S$ with the vertices of S contains a critical set. These critical sets are disjoint and their number is $n - \alpha(G)$.

The similar statement for (4) is not true, but it is not difficult to show, that for r-regular bipartite graphs it is true. It would be interesting to find other classes of graphs, for which the inequalities (3) and (4) hold as inequality.

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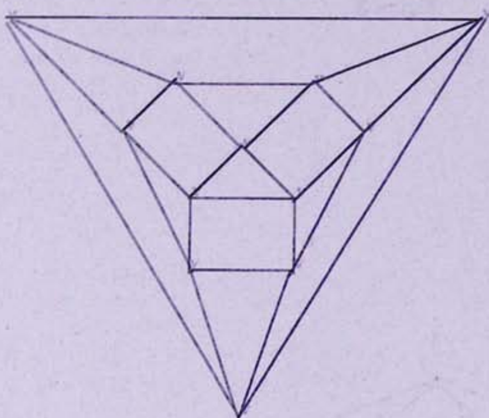

Figure 1

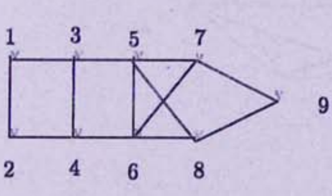

Figure 2

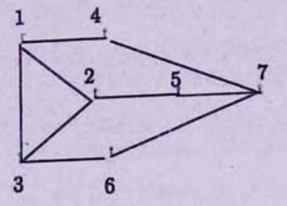

Figure 3

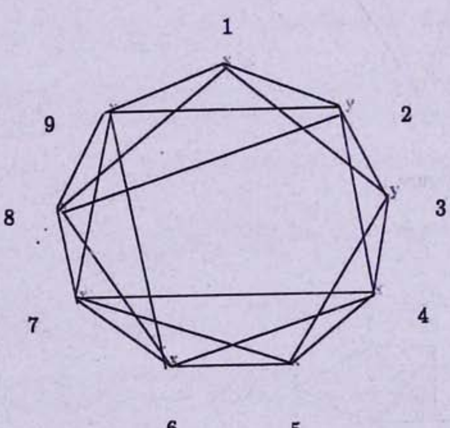

Figure 4

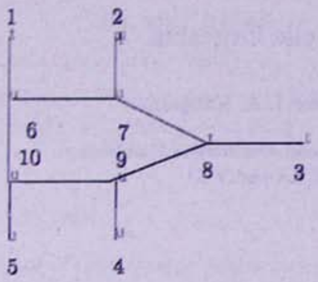

Figure 5