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A STUDY OF THE EFFECT OF RELATIVITY CHANGE OF MASS OF THE ELECTRONS WITH VELOCITY ON THE $M-R$ RELATIONS FOR A COLD SPHERICAL BODY

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By introducing the basic conception of the effect of the relativity change of mass of the electrons with velocity into the elementary theory of pressure ionization, we have derived, in the present article, expressions for the mean molecular weight μ, the density ρ^{-}, for single ionization, and mass-radius relations $(M-R)$ for cold spherical bodies composed of iron, nickle and yttrium, which are in static equilibrium. Fig. 1 [(log ρ, log P) — curve] and 2 [(log ρ, log (M/R) — curve] represent outcome of our discussions. Our study reveals that as long as the electronic concentrations n^{-} in the interior regions are small, for example, $n^{-} \sim 10^{10}-10^{21}$ (cm^{-3}), one may very easily ignore the effect, but for the larger concentrations, $n^{-} \sim 10^{28}-10^{34}$ (cm^{-3}), this effect cannot be ignored. It has been further shown that the exchange potential energy contributes little to the density. In consequence, Vardya [25] previous work (non-relativistic case) has been reviewed.

1. Introduction. The theory of pressure ionization has been put forward by many eminent writers [1—9] since 1935 in the study of the internal constitution of non-relativistic polytropic stars, massive stars, white dwarfs, and planetary bodies as well as to the general relativistic problems [10—16] (a step ahead of purely relativistic phenomena) for describing equations of state of cold, catalyzed matter in different density ranges from 10^{4} to 10^{7}, 10^{7} to 10^{12}, 10^{12} to 10^{15} (g/cm^{3}) and from 10^{15} to the highest densities.

A detailed study of the internal constitution of the planets, and white dwarfs has been made by Ramsey [7], Bullen [17—19], Brown

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[20], Mestel [21–22], Inglis [23], which later on followed by Lawden's [16] and author's [14–15] general relativistic treatment of planetary structures. By using an equation of state of the kind (2) the author [24] has further discussed the behaviour of dense stellar matters and obtained quite satisfactory results, such as, ($\log \rho$, $\log M_1$) — curves, in the relativistic density ranges $10^6 \leq \rho \leq 10^{11}$).

Kothari [1–5] developed his non-relativistic theory with the help of the following virial theorem:

$$\left.\begin{aligned} 2E_{KI} + W_{EL} + W_{EX} &= 3PV, \\ \text{or} \quad 2E_{KI} + E_{GR} &= 3PV \end{aligned}\right\} \qquad (1)$$

and derived expressions for the mean molecular weight, $M-R$ relations and other results of cognate interest by making certain simplifying assumptions (E_{KI}, W_{EL}, W_{EX} and are respectively the kinetic, electrostatic, exchange and gravitational potential energies, and V and P are volume and pressure). Bhatnagar and Kothari [6] concluded on the basis of ($\log \rho$, $\log P$) — curve for Fe that their results were in good agreement with those of Kothari. The discrepancy in Kothari's theory was removed by Vardya [25] by including the exchange potential energy term. In this paper it is claimed that even Vardya's treatment does not yield satisfactory results for he has excluded the relativity corrections This aspect of the problem would be examined here more closely for cold spherical bodies at zero temperatures. Our calculations leads to the fact that for small bodies for which the electronic concentrations in the interior regions are small ($x < 1$, or, $n^- < 5.88 \cdot 10^{29}$), we can neglect the relativity corrections, but we cannot do so in the case of larger celestial bodies possessing larger concentrations n^- ($x > 1$, or, $n^- > 5.88 \cdot 10^{29}$) of orders of say, 10^{30}, 10^{33} and 10^{34} (cm^{-3}).

Our study of the plots of ($\log \rho$, $\log P$) — and ($\log \rho$, $\log (R/M)$) curves (Figs. 1 and 2) make the picture more clear. The equation of state to be followed is that of the polytropic relation [24]

$$(P_e) = P = k\rho^{4/3}/\mu^{4/3}, \qquad (2)$$

where k is the relativistic degenerate constant.

Significance of the relativistic corrections in view of the modified equation of state (2). As already shown in author's earlier work [24] (also in sect. IV below) that equation of state (2) provides a good way to study the properties of dense stellar matters in relativistic density ranges up to $\sim 10^{11}$ ($g\, cm^{-3}$). Although, for $x > 20$ the matter becomes neutron rich and in that case use of the exact equation of state, or TOV equation of general relativity of hydrostatic equilibrium, would be desirable, but even then (2) provides quite satisfactory results. This should not be much surprising in view of the following reasonings.

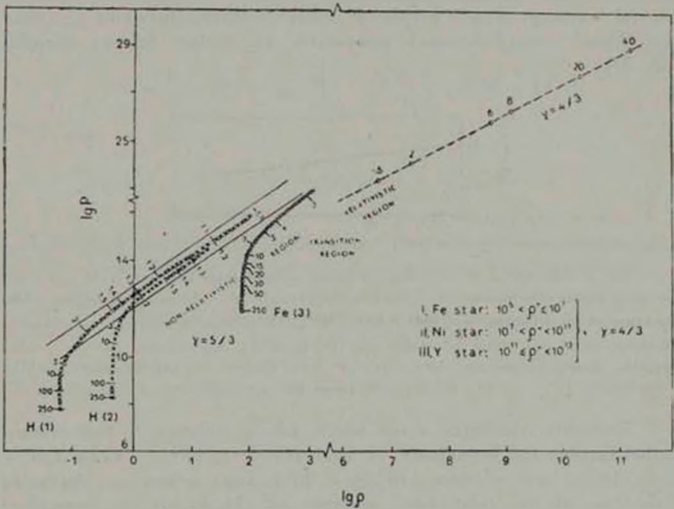

Fig. 1. The ($\log \rho$, $\log P$) — *curves: the relativity corrections on pressure-density relation for a cold spherical body.*

Cases beyond $\rho \sim 10^{11}$ need special treatment of general relativity; $\rho \sim 10^{15}$ corresponds to the neutron rich matter ($\gamma = 5/3$). ○○○ — Vardya's results, - - - - Author's results.

Van Albada [27—28] and Harrison, Wakano and Wheeler [29] have stated that no nuclear physics is required for the treatment of the present regime, but one has to apply the most elementary principles of statistical mechanics to obtain the equation of state. Landau [30] pointed out to the known circumstances that with increasing mass the material is more strongly compacted. Ultimately, the Fermi energy rises to relativistic level. It makes no difference in the pressure, whether the Fermi gas consists of electrons (densities up to 10^8 $g\ cm^{-3}$), or neutrons (ρ up to 10^{15} $g\ cm^{-3}$). For nuclear, and supernuclear densities, they accepted, for the sake of simplicity, the concept of an ideal cold degenerate gas of electrons, protons and neutrons in equilibrium, but with negligible particle-particle interactions. Schatzman [31] has also discussed the problem of mass limit of a star composed of a degenerate electron gas by using simply Newtonian equation of hydrostatic equilibrium (an excellent approximation for densities 10^{10} gcm^{-3}). There are several instances in our picture of stellar evolution in which the electron Fermi energy becomes comparable to nuclear β-decay energies [32—37].

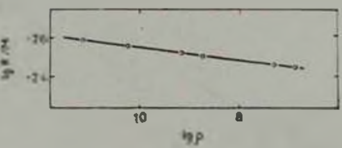

Fig. 2. Connection (i. e., overall a linear relationship) between R/M and ρ for the relativistic equilibrium configuration (spherical) of cold matter obeying γ-law equation of state of the kind [24] $P = k\rho^{4/3}$. Quite satisfactory results are obtained for densities of the orders of $\sim 10^{4}$ — $\sim 10^{13}$ (gcm^{-3}), as marked by — — — The probable density ranges for formations of iron, nickle, and yttrium stars are [11], respectively, $10^5 < \rho \lesssim 10^7$, $10^7 < \rho < 10^{11}$ and $10^{11} \lesssim \rho < 10^{13}$.

Therefore the three cases which are of interest to this investigation involve the formations of iron ($10^5 < \rho \lesssim 10^7$) —, nickle ($10^7 < \rho < 10^{11}$) — and yttrium ($10^{11} \lesssim \rho < 10^{13}$) stars where we have to make use of the relativistic analogue of (1), as has also been done elsewhere [5, 24, 35, 36, 38—44]. Specifically in our model assumed, for simplicity (and because we find no decisive indication favouring any simpler alternative procedure) the matter at very high density $\sim 10^{11}$ gcm^{-3}, can be treated for the purpose of calculations of ($\log \rho$, $\log P$) — and ($\log \rho$, $\log R/M$) relations, as if it were an ideal mixture of three Fermi gases (electrons, protons and neutrons) in statistical equilibrium. This would avoid the mathematical complications due to the assumptions of different equations of state when considering the electron capture at high densities.

The above discussions, however, justifies our present approach for obtaining good results in the analysis of relativistic dense stellar matter up to $\rho \sim 10^{11}$ gcm^{-3}, even without an explicit (mathematically) use of the concept of electron capture. Unfortunately, we have not succeeded here in formulating straightforwardly a tractable theory leading to the study of the behaviour of dense matters beyond $\rho \sim 10^{11}$ gcm^{-3}, in which case certain modifications, as mentioned above, are to be made. This could be another aspect of the problem for future study.

2. *Basic equations.* The expression for the total kinetic energy of the electrons, in a volume V of the star at zero temperature, for small $x\,(P_F < m_0 c)$ is given by

$$(E_{KI})_{x<1} = \frac{8\pi V m_0^4 c^5}{h^3}\left[-\frac{x^3}{3} + f(x)\right] =$$
$$= n V \left(\frac{3}{40}\right)\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2 n^{2/3}}{m_0}\left[1 - \frac{5h^2}{112}\left(\frac{3}{\pi}\right)^{2/3} n^{2/3}\right], \tag{3}$$

where

$$[f(x)]_{x<1} = \int_0^x (1+y^2)^{1/2}\, y^2 dy = \frac{x^3}{3} + \frac{x^5}{5} - \frac{1}{56}x^7 + \cdots,$$

symbols have their usual significance, Neglecting the approximation, used in $f(x)$ by Vardya, we obtain the complete expression for E_{KI}, when x is large

$$(E_{KI})_{x>1} = \frac{3}{8}\left(\frac{3}{\pi}\right)^{1/3} V h c n^{4/3}. \tag{4}$$

The foregoing expression can be re-expressed as

$$(E_{KI})_{x>1} = \frac{3}{8} ch \left(\frac{3}{\pi}\,\frac{Z\rho}{\mu_0\mu_1 A m_H}\right)^{1/3} Z\frac{\rho V}{A m_H}, \tag{5}$$

where Z, ρ, A, and m_H represent respectively the atomic number, the density, the atomic weight, and the mass of a hydrogen atom. For the material which is r-fold ionized, we have

$$n = \frac{\rho}{A m_H} r, \tag{6}$$

so that the measure of the degree of ionization μ is given by

$$\mu = \frac{A}{n}. \tag{7}$$

μ ranges from A to $A/Z\,(=\mu_0 = \mu)$ respectively for single and complete ionizations. Expressions for the electrostatic and exchange potential energies respectively are:

$$W_{EL} = -\frac{9}{10} Z^2 e^2 \left(\frac{\rho}{\mu_1 A m_H}\right)^{1/3} \frac{\rho V}{A m_H}, \tag{8}$$

and

$$W_{EX} = -3e^2\left(\frac{9}{32\pi^2\mu_1}\,\frac{Z\rho}{A m_H}\right)^{1/3} \frac{\rho V}{A m_H}. \tag{9}$$

3. The desired expressions. Making use of equations (1), (2), (5), (8) and (9), we obtain

$$\mu = \frac{\mu_0(\gamma_1\gamma_2\alpha^3/8)^{1/4}}{[1-\gamma_2^{1/3}(\xi Z^2)^{1/3}\epsilon_Z]^{3/4}}, \tag{10}$$

where

$$\xi = [2.3^{2/3}e^2\lambda^{1/3}/5ch]^3; \quad \alpha = 2.98454, \tag{11}$$

and

$$\epsilon_Z = 1 + 5(12\lambda^2 Z^3)^{-1/3}. \tag{12}$$

On setting $\gamma_1 = \gamma_2 = 1$, the equation (10) can be cast into the form

$$\mu = \frac{\mu_0(\alpha^3/8)^{1/4}}{[1-(\xi Z^2)^{1/3}\epsilon_Z]^{3/4}}. \tag{13}$$

For matter in a singly ionised state ($\mu = A$), we have from (2) and (13),

$$\rho = \rho^* = \left(\frac{P}{R}\right)^{3/4} A,$$

and

$$\epsilon_Z = \epsilon_Z^* = \frac{1-\dfrac{3}{2Z^{4/3}}}{(\xi Z^2)^{1/3}} \tag{14}$$

respectively. With the help of equations (2) and (13), we obtain

$$\rho = \left(\frac{P}{R}\right)^{3/4} \frac{\mu_0(\alpha^3/8)^{1/4}}{[1-(\xi Z^2)^{1/3}\epsilon_Z]^{3/4}}. \tag{15}$$

In view of the pressure equation (2), the structure of the configuration will be governed by the Lane-Emden function θ_3; the $R-M$ relation for this configuration being [45]

$$R = \frac{QM}{M_\odot}\left(\frac{\mu}{\rho_c^{1/4}}\right)^{4/3}, \tag{16}$$

where $Q = 3.0782 \cdot 10^{16}$, ρ_c is the central density and $M_\odot$ the mass of the sun. Eliminating μ between (13) and the foregoing relation (16), we obtain

$$R = \frac{QM}{2M_\odot}\alpha\frac{\mu_0^{4/3}}{\rho_c^{1/3}[1-(\xi Z^2)^{1/3}\epsilon_Z]}, \tag{17}$$

or, from (14), for a singly ionised state,

$$R = \frac{QM}{M_s} \frac{\rho_0^{4/3}}{\rho_e^{1/3}} Z^{4/3}, \quad \text{or,} \quad R = \frac{QM}{2M_s}\, 2 \frac{\rho_0^{4/3}}{\rho_e^{1/3}\left|1-\left(1-\frac{2}{2Z^{4/3}}\right)\frac{\xi}{\xi_Z}\right|}, \tag{18}$$

which is the required $M-R$ relation for a cold body. From equations (17) and (18), we can easily conclude that when $M \to 0$, $R \to 0$, as M increases initially until, after attaining a certain maximum value R_{max} for $M = M_{max}$, it decreases to the stage where $M \to \infty$, $R \to 0$.

4. Numerical results and consequent discussions. As pointed out above, we have shown that as long as n^* has values of the order of 10^{19}–10^{27}, the relativity corrections can be neglected, but for larger values of n^*, such as, $n^* = 10^{30}$–10^{34}, it begins to play an important role in the results. To illustrate the significant role of the relativistic effect, we have obtained ($\log \rho$, $\log P$) relation (shown by the dotted straight line, Fig. 1) for solid spherical bodies composed of iron, nickle, and yttrium, for a few values of the electronic concentration $n^* = 1.9833 \cdot 10^{30}$, $4.7008 \cdot 10^{30}$, $1.2692 \cdot 10^{32}$, $3.0082 \cdot 10^{32}$, $4.7011 \cdot 10^{33}$, and $3.7610 \cdot 10^{34}$, corresponding to $x = 1.5$, 2.0, 6, 8, 20 and 40. Consequent effect of the relativistic correction on $M-R$ relation has been shown in Fig. 2.

The ($\log \rho$, $\log P$) — curves present a comparative study of our results (in the relativistic density ranges up to $\sim 10^{11}$ gcm^{-3}) with those of Vardya's. This brings out many useful physical consequences regarding the non-relativistic and relativistic character of a degenerate gas, as given below.

The dotted straight line exhibits the results of our investigation for I — iron, II — nickle, and III — yttrium (with exchange potential). The three curves marked with crosses (hydrogen (1) and hydrogen (2), with and without exchange potential), and circles (iron (3) — with exchange potential) sketched on our chosen scale, were obtained by Vardya. A single dotted straight line for iron, nickle, and yttrium has been drawn by the author by taking into account the potential energy term as it would not show appreciable differences in the values of ρ and P for different values of n^*, if the same is neglected. Hence, in this case, it will be almost insignificant to draw other curves. We further note in Figs. 1 and 2 that the plots of $\log P$ vs $\log \rho$ and $\log (R/M)$ vs $\log \rho$ show linear relationships up to a wide range of densities $\sim 10^{11}$ (gcm^{-3}).

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ИЗУЧЕНИЕ ЭФФЕКТА РЕЛЯТИВИСТСКОГО ИЗМЕНЕНИЯ МАССЫ ЭЛЕКТРОНОВ СО СКОРОСТЬЮ В СООТНОШЕНИЯХ *M—R* ДЛЯ ХОЛОДНОГО СФЕРИЧЕСКОГО ТЕЛА

Ж. П. ШАРМА

Введя основную концепцию эффекта релятивистского изменения массы электронов со скоростью в элементарную теорию ионизации давлением, в данной работе получены выражения для среднего молекулярного веса и плотности ρ в случае единичной ионизации, а также для соотношения масса — радиус ($M—R$) для холодных сферических тел, состоящих из железа, никеля и иттрия, находящихся в статическом равновесии.

Результаты нашего обсуждения представлены на рис. 1 ($\log \rho$, $\log P$) и рис. 2 (кривая $\log \rho$, $\log (M/R)$). Наше рассмотрение показывает, что в том случае, когда электронные концентрации n во внутренних областях малы, например, порядка $n = 10^{1}$[illegible] — 10^{27} (см[illegible]), то указанным эффектом можно легко пренебречь. Для больших же концентраций, $n^{*} = 10^{3}$[illegible] — 10^{34} см[illegible], этот эффект нельзя не учитывать.

Далее показано, что плотность слабо зависит от потенциальной энергии обмена.

В заключение обсуждена ранняя работа Варданa [25] (нерелятивистский случай).

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