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DEGENERATE NONSELFADJOINT HIGH-ORDER ORDINARY DIFFERENTIAL EQUATIONS ON AN INFINITE INTERVAL

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Abstract. The paper considers the generalized Dirichlet problem for a class of degenerate nonselfadjoint high-order ordinary differential equations on an infinite interval. The spectrum of the corresponding operator is studied, and in the special case, the domain of definition of the selfadjoint operator is described.

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1. Introduction

We consider the Dirichlet problem for degenerate ordinary differential equations of the form:

$$Lu \equiv (-1)^m(t^\alpha u^{(m)})^{(m)} + a(-1)^{m-1}(t^{\alpha-1}u^{(m)})^{(m-1)} + pt^\beta u = f(t), \tag{1.1}$$

where $t \in (1;+\infty)$, $m \in \mathbb{N}$, $\alpha \neq 1,3,\ldots,2m-1$, $\beta \leq \alpha - 2m$, a and p are real constants, and $f \in L_{2,-\beta}(1,+\infty)$.

The dependence of the setting of boundary conditions relative to t for $t = 0$ on the order of degeneration α and on the sign of number a was first noticed in the paper by M.S. Keldysh [1] for degenerating into parts of the boundary of a second-order elliptic equation. The case $m = 1$, $\beta = 0$, $0 \leq \alpha < 2$ was studied in the papers by A.A. Dezin [2] and V.V. Kornienko [3], while the case $m = 2$, $\beta = 0$, $0 \leq \alpha \leq 4$ was considered in [4] (on a finite interval). Notice that the problem (1.1) in the case where $A = 0$ has been studied in the paper by L. Tepoyan [5].

The present paper is structured as follows. We first define the weighted Sobolev spaces $\dot{W}^m_\alpha(1,+\infty)$, and discuss some properties of functions $u \in \dot{W}^m_\alpha(1,+\infty)$ and embedding theorems. Notice that weighted Sobolev spaces on infinite intervals have been studied, in particular, in the papers by L.D. Kudryavcev [6] and P.A. Zharov

[7]. Then we define the generalized solution of the Dirichlet problem for equation (1.1) and study the spectral properties of the corresponding operator. Finally, in the special case where $\beta = -2m$, we describe the domain of definition of the selfadjoint operator. .

2. Weighted Sobolev spaces $\dot{W}_\alpha^m(1,+\infty)$

Denote by $\dot{C}^m[1,+\infty)$ the set of functions from $u \in C^m[1,+\infty)$, satisfying the boundary conditions:

$$u^{(k)}(1) = u^{(k)}(+\infty) = 0, \quad k = 0,1,\dots,m-1, \tag{2.1}$$

and define the space $\dot{W}_\alpha^m(1,+\infty)$ to be the completion of $\dot{C}^m[1,+\infty)$ by the norm

$$\|u\|^2_{\dot{W}_\alpha^m(1,+\infty)} = \int_1^{+\infty} t^\alpha |u^{(m)}(t)|^2\, dt.$$

The inner product in $\dot{W}_\alpha^m(1,+\infty)$ we denote by $\{u,v\}_\alpha = (t^\alpha u^{(m)}, v^{(m)})$, where $(\cdot,\cdot)$ stands for the inner product in $L_2(1,+\infty)$. Observe that for any function $u \in \dot{W}_\alpha^m(1,+\infty)$ and any number $t_0 \in [1,+\infty)$ the boundary values $u^{(k)}(t_0)$ and $u^{(k)}(1) = 0, k = 0,1,\dots,m-1$ exist (see [8]). The proofs of the next two propositions can be found in [5].

Proposition 2.1. *For functions $u \in \dot{W}_\alpha^m(1,+\infty)$, $\alpha \neq 1,3,\dots,2m-1$, the following inequalities are satisfied:*

$$|u^{(k)}(t)|^2 \le C_1 t^{2m-2k-1-\alpha} \|u\|^2_{\dot{W}_\alpha^m(1,+\infty)}, \quad k = 0,1,\dots,m-1. \tag{2.2}$$

It follows from Proposition 2.1 that in the case $\alpha > 2m-1$ (weak degeneration), we have $u^{(j)}(+\infty) = 0$ for $j = 0,1,\dots,m-1$, while for $\alpha < 2m-1$ (strong degeneration) not all conditions $u^{(j)}(+\infty) = 0$ are "preserved". For instance, for $1 < \alpha < 3$ after completion only the condition $u^{(m-1)}(+\infty) = 0$ is "preserved", and for $\alpha < 1$ all the values $u^{(j)}(+\infty)$, $j = 0,1,\dots,m-1$, in general, can be infinite.

Let $L_{2,\beta}(1,+\infty) := \left\{f;\int_1^{+\infty} t^\beta |f(t)|^2\, dt < +\infty\right\}$. Notice that for $\beta_1 \le \beta_2$ we have the embedding $L_{2,\beta_2}(1,+\infty) \subset L_{2,\beta_1}(1,+\infty)$.

Proposition 2.2. *For $\beta \le \alpha - 2m$ the following continuous embedding holds:*

$$\dot{W}_\alpha^m(1,+\infty) \subset L_{2,\beta}(1,\infty), \tag{2.3}$$

which is compact for $\beta < \alpha - 2m$.

Notice that the embedding (2.3) is not compact for $\beta = \alpha - 2m$, and for $\beta > \alpha - 2m$ it fails. Let $d(m,\alpha) = 4^{-m}(\alpha-1)^2(\alpha-3)^2\cdots(\alpha-(2m-1))^2$. It is worth to note that in the paper [10] was considered a question concerning the number of real roots for a polynomial with constant term $d(m,\alpha)$. Using Hardy inequality (see [6]) it can be shown that (see [5])

$$\int_1^\infty t^\alpha |u^{(m)}(t)|^2\,dt \geq d(m,\alpha)\int_1^\infty t^{\alpha-2m}|u(t)|^2\,dt. \tag{2.4}$$

It is important to note that in the inequality (2.4) the number $d(m,\alpha)$ is exact. Also, as an immediate consequence of the inequality (2.4), for any $\beta \leq \alpha - 2m$ we have

$$\|u\|^2_{\dot W^m_\alpha(1,+\infty)} \geq d(m,\alpha)\|u\|^2_{L_{2,\beta}(1,+\infty)}. \tag{2.5}$$

3. Degenerate nonselfadjoint differential equations

Now we define a generalized solution of the Dirichlet problem for equation (1.1) for $\alpha \neq 0$.

Definition 3.1. *A function $u \in \dot W^m_\alpha(1,+\infty)$ is called a generalized solution of the Dirichlet problem for equation* (1.1), *if for any $v \in \dot W^m_\alpha(1,+\infty)$ the following equality is fulfilled:*

$$\{u,v\}_\alpha + a(-1)^{m-1}\left(t^{\alpha-1}u^{(m)}, v^{(m-1)}\right) + p(t^\beta u, v) = (f,v). \tag{3.1}$$

For the proof of the next theorem we refer to [9].

Theorem 3.1. *Let the following conditions be satisfied:*

$$\begin{aligned} &a(\alpha-1) > 0,\\ &\gamma = d(m,\alpha) + \frac{a}{2}(\alpha-1)d(m-1,\alpha-2) + p > 0. \end{aligned} \tag{3.2}$$

Then a generalized solution of the Dirichlet problem for equation (1.1) *exists and is unique for every $f \in L_{2,-\beta}(1,+\infty)$.*

The definition of a generalized solution usually generates some linear operator $L : L_{2,\beta}(1,+\infty) \to L_{2,-\beta}(1,+\infty)$ with dense in $L_{2,\beta}(1,+\infty)$ domain of definition $D(L) \subset \dot W^m_\alpha(1,+\infty)$ (see [5]). To obtain an operator acting in the same space, which is necessary from spectral theory viewpoint, we define the operator $\mathbb{L} = t^{-\beta}L$, $D(\mathbb{L}) = D(L)$. It is clear that the operator $\mathbb{L}$ acts in the space $L_{2,\beta}(1,+\infty)$. In [9] it was proved that under the condition (3.2) the inverse operator $\mathbb{L}^{-1}$ is bounded in $L_{2,\beta}(1,+\infty)$ for $\beta \leq \alpha - 2m$ and is a compact operator for $\beta < \alpha - 2m$.

This, in particular, implies that for $\beta < \alpha - 2m$ the spectrum of the operator $\mathbf{L}$ is discrete.

For the conjugate to (1.1) equation

$$S \equiv (-1)^m (t^\alpha v^{(m)})^{(m)} - a(-1)^{m-1}(t^{\alpha-1} v^{(m-1)})^{(m)} + pt^\beta v = g(t), \tag{3.3}$$

where $g \in L_{2,-\beta}(1,+\infty)$, a generalized solution of the Dirichlet problem is defined as follows:

Definition 3.2. *A function $v \in L_{2,\beta}(1,+\infty)$ is called a generalized solution of the Dirichlet problem for equation (3.3), if for any $u \in D(L)$ the equality $(Lu, v) = (u, g)$ is fulfilled.*

Now the existence and uniqueness of a generalized solution of the Dirichlet problem for equation (3.3) for any $g \in L_{2,-\beta}(1,+\infty)$ follows from Theorem 3.1 and boundedness of the operator L^{-1} (see [9]). As above, we define the operator $\mathbf{S} = t^{-\beta} S$, $D(\mathbf{S}) = D(S)$.

Remark 3.1. *For $\alpha < 1$ every generalized solution $v \in L_{2,\beta}(1,+\infty)$ of equation (3.3) satisfies the condition:*

$$\left(t^{\alpha-1} |v^{(m-1)}(t)|^2\right)\big|_{t=+\infty} = 0. \tag{3.4}$$

Also, notice that for a generalized solution $u \in \dot{W}^m_\alpha(1,+\infty)$ of equation (1.1) for $\alpha < 1$, it can be guaranteed only the finiteness of the left-hand side of (3.4). This is some analog of the Keldysh theorem (see [1]).

Proposition 3.1. *The spectra of operators $\mathbf{L}, \mathbf{S} : L_{2,\beta}(1,+\infty) \to L_{2,\beta}(1,+\infty)$ lie on the right half-space.*

Proof. Since $\mathbf{S} = \mathbf{L}^*$, it is enough to prove the proposition for the operator $\mathbf{L}$. Let $\mathrm{Re}\lambda \le 0$, $f_1 = t^{-\beta} f$ and $f \in L_{2,-\beta}(1,+\infty)$. Then we have $f_1 \in L_{2,\beta}(1,+\infty)$. Consider the equation $\mathbf{L}u - \lambda u = f_1$. In view of the definition of the operator $\mathbf{L}$, the last equation can be written in the form:

$$Lu - \lambda t^\beta u = f, \quad f \in L_{2,-\beta}(1,+\infty). \tag{3.5}$$

From (3.1) for $v = u$ we obtain

$$\{u,u\}_\alpha + a(-1)^{m-1}\left(t^{\alpha-1}u^{(m)}, u^{(m-1)}\right) + (p-\lambda)(t^\beta u, u) = (f, u).$$

It follows from the last equality and the proof of Theorem 3.1 (see [9]) that under the conditions (3.2) and $\mathrm{Re}\lambda \leq 0$ the equation (3.5) is uniquely solvable for any $f \in L_{2,-\beta}(1,+\infty)$, that is, the equation $\mathbb{L}u - \lambda u = f_1$ is uniquely solvable for every $f_1 \in L_{2,\beta}(1,+\infty)$. □

4. Description of the domain of definition of the degenerate selfadjoint differential equation

Consider the selfadjoint differential equation

$$(4.1) \quad Lu \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + pt^\beta u = f(t), \quad f \in L_{2,-\beta}(1,+\infty), \quad \beta \leq \alpha - 2m.$$

Define a generalized solution of the Dirichlet problem for equation (4.1) as in the Definition 1 (for $a = 0$). Now consider the special case of equation (4.1) for $p = 0$:

$$(4.2) \qquad Bu \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} = f, \quad f \in L_{2,-\beta}(1,+\infty).$$

Let $\mathbb{B} = t^{-\beta}B$, $D(\mathbb{B}) = D(B)$. In paper [5], it was proved the unique solvability of the Dirichlet problem for equation (4.2) for every $f \in L_{2,-\beta}(1,+\infty)$, as well as, the positiveness and self-adjointness of the operator $\mathbb{B} : L_{2,\beta}(1,+\infty) \to L_{2,\beta}(1,+\infty)$, and the boundedness of the inverse operator $\mathbb{B}^{-1} : L_{2,\beta}(1,+\infty) \to L_{2,\beta}(1,+\infty)$ for $\beta \leq \alpha - 2m$ and its compactness for $\beta < \alpha - 2m$. Thus, for $\beta < \alpha - 2m$ the operator $\mathbb{B}^{-1}$ is compact and selfadjoint. Therefore the spectrum of the operator $\mathbb{B}$ for $\beta < \alpha - 2m$ is discrete and the system of eigenfunctions is complete in $L_{2,\beta}(1,+\infty)$ (see [11]). Also, notice that the spectrum of the operator $\mathbb{B}$ for $\beta = \alpha - 2m$ is purely continuous and coincides with the ray (see [5]):

$$\sigma(\mathbb{B}) = \sigma_c(\mathbb{B}) = [d(m,\alpha); +\infty).$$

Now we give the description of the domain of definition of the operator L for $\beta = -2m$, that is, consider the equation

$$(4.3) \qquad Lu \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + pt^{-2m} u = f, \quad f \in L_{2,2m}(1,+\infty),$$

$\alpha \neq 1, 3, \ldots, 2m-1, \alpha \geq 0$. Observe that $D(B) = D(L)$, hence it is enough to describe $D(B)$.

Theorem 4.1. *The domain of definition of the operator B consists of functions $u \in \dot{W}^m_\alpha(1,+\infty)$, for which the value $u^{(m-1)}(+\infty)$ is finite for $\frac{1}{2} < \alpha < 1$, and for*

$2m-2k-2<\alpha<2m-2k-1$, $k=0,1,\dots,m-2$, the values $u^{(k)}(+\infty)$ also are finite.

Proof. Let $m>2$. To find a general solution of the equation

$$(-1)^m(t^\alpha u^{(m)})^{(m)}=f(t)$$

observe first that

$$t^\alpha u^{(m)}(t)=\frac{1}{(m-1)!}\int_t^{+\infty}(\tau-t)^{m-1}f(\tau)\,d\tau+c_0+c_1t+\ldots+c_{m-1}t^{m-1}.$$

Let $\frac{1}{2}<\alpha<1$. It follows from $u\in\dot W_\alpha^m(1,+\infty)$ that $t^{\frac{\alpha}{2}}u^{(m)}\in L_2(1,+\infty)$. It is easy to check that $c_0=c_1=\ldots=c_{m-1}=0$, because the functions $t^{-\frac{\alpha}{2}},t^{1-\frac{\alpha}{2}},\dots,t^{m-1-\frac{\alpha}{2}}$ do not belong to the space $\dot W_\alpha^m(1,+\infty)$. Thus, we have

$$u^{(m)}(t)=\frac{t^{-\alpha}}{(m-1)!}\int_t^{+\infty}(\tau-t)^{m-1}f(\tau)\,d\tau. \tag{4.4}$$

Let $G(t)=\int_t^{+\infty}(\tau-t)^{m-1}f(\tau)\,d\tau$. Applying Cauchy-Schwarz inequality we get

$$|G(t)|^2\le\int_t^{+\infty}(\tau-t)^{2m-2}\tau^{-2m}\,d\tau\cdot\|f\|^2_{L_{2,2m}(1,+\infty)},$$

implying that

$$|G(t)|\le t^{-\frac{1}{2}}\|f\|_{L_{2,2m}(1,+\infty)}. \tag{4.5}$$

Now from (4.4) we obtain

$$u^{(m-1)}(t)=c-\frac{1}{(m-1)!}\int_t^{+\infty}\tau^{-\alpha}G(\tau)\,d\tau.$$

Therefore

$$\left|\int_t^{+\infty}\tau^{-\alpha}G(\tau)\,d\tau\right|\le\int_t^{+\infty}\tau^{-\frac{1}{2}-\alpha}\,d\tau\cdot\|f\|_{L_{2,2m}(1,+\infty)}\le Ct^{\frac{1}{2}-\alpha}\|f\|_{L_{2,2m}(1,+\infty)},$$

implying that for $\frac{1}{2}<\alpha<1$ the value $u^{(m-1)}(+\infty)$ is finite. Now let $2<\alpha<3$. It follows from $u\in\dot W_\alpha^m(1,+\infty)$ that $c_1=\ldots=c_{m-1}=0$. An integration yields

$$u^{(m-1)}(t)=C_0t^{1-\alpha}-\frac{1}{(m-1)!}\int_t^{+\infty}\tau^{-\alpha}G(\tau)\,d\tau,$$

because for $2<\alpha<3$ we have $u^{(m-1)}(+\infty)=0$. Therefore

$$u^{(m-2)}(t)=C_1+\tilde C_0t^{2-\alpha}+\frac{1}{(m-1)!}\int_t^{+\infty}\int_\tau^{+\infty}\eta^{-\alpha}G(\eta)\,d\eta\,d\tau. \tag{4.6}$$

Using (4.5) we can estimate the integral in (4.6) to obtain

$$\left|\int_t^{+\infty}\int_\tau^{+\infty}\eta^{-\alpha}G(\eta)\,d\eta\,d\tau\right|\le C_2t^{\frac{3}{2}-\alpha}\|f\|_{L_{2,2m}(1,+\infty)}.$$

Similarly, for $2m-2<\alpha<2m-1$ we obtain

$$u(t)=\frac{(-1)^m}{((m-1)!)^2}\int_t^{+\infty}(\tau-t)^{m-1}\tau^{-\alpha}G(\tau)\,d\tau+C+C_0t^{m-\alpha}+\ldots+C_{m-2}t^{2m-2-\alpha}.$$

The integral on the right-hand side of the last relation can be estimated as follows

$$\left|\int_t^{+\infty}(\tau-t)^{m-1}\tau^{-\alpha}G(\tau)\,d\tau\right|\leq ct^{m-\alpha+\frac{1}{2}}\|f\|_{L_{2,2m}(1,+\infty)}.$$

Here it is important to note that for $m>2$ we have $m-\alpha+\frac{1}{2}<\alpha<2m-2-\alpha$. In the case $m\leq 2$ the proof is evident. Notice that the conditions of Theorem 4.1 are exact, in the sense that their violation, generally, can cause the nonexistence of the values $u^{(k)}(+\infty)$, $k=1,\ldots,m-1$. Also, note that the values $u^{(k)}(+\infty)$, $k=0,1,\ldots,m-1$ cannot be given arbitrarily, they are defined by the right-hand side of the equation (4.3) (see [2], [4]). □

Список литературы

[1] M. V. Keldysh, "On some cases of degenerating of elliptic type equations on the boundary of domain", Dokl. AN SSSR, **23**, no. 2, 181 – 184 (1951).

[2] A. A. Dezin, "Degenerate operator equations", Mat. Sb., **115(157)**, no. 3(7), 323 – 336 (1981).

[3] V. V. Kornienko, "On the spectrum of degenerate operator equations", Mat. zametki, **68**, no. 5, 677 – 691 (2000).

[4] L. P. Tepoyan, "Degenerate fourth-order differential-operator equations", Differ. Uravn., **23**, no. 8, 1366 – 1376 (1987).

[5] L. P. Tepoyan, "Degenerate differential-operator equations on infinite intervals", Journal of Mathematical Sciences, **189**, no. 1, 164 – 172 (2013).

[6] L. D. Kudryavtzev, "On equivalent norms in the weight spaces", Tr. Mat. Inst. Steklova, **170**, 161 – 190 (1984).

[7] P. A. Djarov, "Compactness of embeddings in some spaces with power weight", Izv. Vyssh. Ucheb. Zav. Ser. Mat., no. 8, 82 – 85 (1988).

[8] V. I. Burenkov, Sobolev Spaces on Domains, Teubner (1999).

[9] S. Zschorn, "Nonselfadjoint degenerate differential operator equations of higher order on infinite interval", Proceedings of the YSU, Physical and Mathematical Sciences, no. 2, 39 – 45 (2014).

[10] S. A. Osipova, L. P. Tepoyan, "On Euler type equation", Proceedings of the YSU, Physical and Mathematical Sciences, no. 1, 16 – 19 (2009).

[11] M. A. Naimark, Linear differential operators, Moscow, Nauka (1969).

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