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ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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VIBRATIONS OF NONEQUABLE STRETCHED RECTANGULAR MEMBRANE. DIRECT AND INVERSE PROBLEMS

Ambartsumian S.A., Belubekian M.V., Movsisian L.A.

С.А. Амбарцумян, М.В. Белубекян, Л.А. Мовсисян
Колебания неравномерно растянутой мембраны. Прямая и обратная задачи

Задача решается методом разложения в ряд Фурье. Показано, что решение обратной задачи (определение функций предварительного напряжения, когда задан спектр частот для определенных граничных условий) неединственно. Обратная задача имеет единственное решение, когда заданы два спектра частот. Предлагается метод приближенного решения обратной задачи.

Ս.Ա. Համբարձումյան, Մ.Վ. Բելուբեկյան, Լ.Ա. Մովսիսյան
Ոչ համասեռ ձգված մեմբրանի տատանումները: Ուղիղ և հակադարձ խնդիրներ

The problems are solved by method of decay in Fourier series. It is shown that the solution of the inverse problem (determination of the functions of prestretch, when vibrations frequencies spectrum is given for certain boundary condition) is nonunique. The inverse problem has unique solution, when two frequencies spectrums are given. A method of the inverse problem approximate solution is suggested.

1. Let the rectangular membrane ($a \times b$) to be stretched in the direction of $0x$ axis by tension $T_1(y)$, and in the direction of $0y$ axis by tension $T_2(x)$.

The equation of membrane (drum) transverse vibration has the form

$$T_1(y)\frac{\partial^2 u}{\partial x^2} + T_2(x)\frac{\partial^2 u}{\partial y^2} = \rho\frac{\partial^2 u}{\partial t^2} \tag{1}$$

where $0 \le x \le a$, $0 \le y \le b$, $\rho = \mathrm{const}$ is the density of membrane material.

Let us consider the vibration problem with the boundary conditions

$$u = 0, \text{ when } x = 0, a \text{ and } y = 0, b \tag{2}$$

The direct problem is to find out vibrations frequencies when tensions $T_1(y)$ and $T_2(x)$ are given. The solution of the equation (1) which satisfies the boundary conditions (2), is the Fourier series

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}(t)\sin\lambda_m x \sin\mu_n y \tag{3}$$

where

$$\lambda_m = \frac{m\pi}{a}, \ \mu_n = \frac{n\pi}{b} \tag{4}$$

The validity of the following expansions is accepted

$$T_1(y) = \sum_{k=0}^{\infty} a_k \cos\mu_k y, \ T_2(x) = \sum_{k=0}^{\infty} b_k \cos\lambda_k x \tag{5}$$

Substituting (3) and (5) into the equation (1) and using

$$2\sin\mu_n y\cos\mu_k y = \sin\mu_{n-k} y + \sin\mu_{n+k} y \tag{6}$$

$$2\sin\lambda_m x\cos\lambda_k y = \sin\lambda_{m-k} y + \sin\lambda_{m+k} y \tag{7}$$

after some transformations, we shall receive the following equation

$$\sum_{m=1}^{\infty}\lambda_m^2\left[\sum_{n=1}^{\infty}A_{mn}\left(\sum_{p=1}^{n}a_{n-p}\sin\mu_p y-\sum_{p=1}^{\infty}a_{n+p}\sin\mu_p y+\sum_{p=n}^{\infty}a_{p-n}\sin\mu_p y\right)\right]\sin\lambda_m x+$$

$$+\sum_{n=1}^{\infty}\mu_n^2\left[\sum_{n=1}^{\infty}A_{nm}\left(\sum_{p=1}^{m}b_{m-p}\sin\lambda_p x-\sum_{p=1}^{\infty}b_{m+p}\sin\lambda_p x+\sum_{p=m}^{\infty}b_{p-m}\sin\lambda_p x\right)\right]\sin\mu_n y=$$

$$=-2\rho\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\ddot{A}_{mn}\sin\lambda_m x\sin\mu_n y \quad (8)$$

From (8), equalizing to zero coefficients of the members $\sin\lambda_q x\cdot\sin\mu_r y$, we shall receive the following infinite system of differential equations

$$A_{qr}+\frac{\lambda_q^2}{\rho}\left[\frac{1}{2}\sum_{n=1}^{r-1}(a_{r-n}-a_{r+n})A_{qn}+\left(a_0-\frac{a_{2r}}{2}\right)A_{qr}+\frac{1}{2}\sum_{n=r+1}^{\infty}(a_{n-r}-a_{n+r})A_{qn}\right]+$$

$$+\frac{\mu_r^2}{\rho}\left[\frac{1}{2}\sum_{m=1}^{q-1}\left(b_{q-m}-b_{m+q}\right)A_{mr}+\left(b_0-\frac{b_{2q}}{2}\right)A_{qr}+\frac{1}{2}\sum_{m=q+1}^{\infty}\left(b_{m-q}-b_{m+q}\right)A_{mr}\right]=0$$

$$q=1,2,\ldots,\ r=1,2,\ldots \quad (9)$$

Representing the solution of system (9) in the form

$$A_{qr}=A_{qr}^0\exp i(\omega t) \quad (10)$$

we shall receive infinite system of homogeneous algebraic equations with respect to A_{qr}^0 constants. Equalizing to zero the determinant of this system, we shall obtain an equation which defines the frequencies spectrum of the problem.

The solutions of shortened system sequence of the equations with respect to A_{qr}^0 is used for vibrations frequencies approximate determination. Let us take the shortened system with respect to $A_{11}^0, A_{12}^0, A_{21}^0, A_{22}^0$

$$\left[\lambda_1^2\left(a_0-\frac{a_2}{2}\right)+\mu_1^2\left(b_0-\frac{b_2}{2}\right)-\rho\omega^2\right]A_{11}^0+\frac{\lambda_1^2}{2}(a_0-a_4)A_{12}^0+\frac{\mu_1^2}{2}(b_0-b_4)A_{21}^0=0$$

$$\frac{\lambda_1^2}{2}(a_1-a_3)A_{11}^0+\left[\lambda_1^2\left(a_0-\frac{a_4}{2}\right)+\mu_2^2\left(b_0-\frac{b_2}{2}\right)-\rho\omega^2\right]A_{12}^0=0 \quad (11)$$

$$\frac{\mu_1^2}{2}(b_1-b_3)A_{11}^0+\left[\lambda_2^2\left(a_0-\frac{a_2}{2}\right)+\mu_1^2\left(b_0-\frac{b_4}{2}\right)-\rho\omega^2\right]A_{21}^0=0$$

$$\frac{\mu_2^2}{2}(b_1-b_3)A_{12}^0+\frac{\lambda_2^2}{2}(a_1-a_3)A_{21}^0+\left[\lambda_2^2\left(a_0-\frac{a_4}{2}\right)+\mu_2^2\left(b_0-\frac{b_4}{2}\right)-\rho\omega^2\right]A_{22}^0=0$$

The equality to zero of determinant of the system (11) brings to the following two equations with respect to vibration frequencies

$$\Delta(a_0,a_1,a_2,a_3,a_4,b_0,b_1,b_2,b_3,b_4,\omega^2)=0 \quad (12)$$

$$\lambda_2^2\left(a_0-\frac{a_4}{2}\right)+\mu_2^2\left(b_0-\frac{b_4}{2}\right)-\rho\omega^2=0 \quad (13)$$

The equation (12) is cubic relative to ω^2 and, therefore determines three vibration frequencies

$$\omega_1^2=\omega_{11}^2,\omega_2^2=\omega_{12}^2,\omega_3^2=\omega_{21}^2 \quad (14)$$

The equation (13) determines the forth frequency of vibrations ω_{22}^2.

The inverse problem for systems (12), (13) is the determination of the coefficients a_k and b_k when four vibrations frequencies are known. Consequently the inverse problem for shortened system (11) is brought to the solution of four equations system

$$\Delta(a_0,\dots a_4,b_0,\dots b_4,\omega_i^2)=0,\quad i=1,2,3$$
$$\lambda_2^2\left(a_0-\frac{a_4}{2}\right)+\mu_2^2\left(b_0-\frac{b_4}{2}\right)-\rho\omega_{22}^2=0 \tag{15}$$

It is evident. that the system (15) with respect to ten unknown coefficients a_k, b_k has not unique solution.

Thus. in general, the inverse problem-determination of the tension functions $T_1(y), T_2(x)$ (or coefficients of their expansion a_k, b_k) with known frequencies spectrum of the problem under boundary conditions (2) – has not unique solution. This result is analogous to the result for the inverse problem of the nonhomogeneous string vibrations [1].

From (15) it follows also. that the inverse problem can have unique solution in particular cases. If $a_k=b_k$ and a_0 is known. the system of equations (15) has unique solution. Such case will take place for quadratic membrane $(a=b)$ with the same distribution of the tension function in directions of x and y.

In the case $a>>b$, it may be accepted approximately. that tension function in the direction of x is constant $(a_0\neq 0, a_k=0)$. Then the coefficients b_k are determined by unique way. when a_0 and b_0 are known.

It should be noted also that unique solution exists in the case. when a_0 and b_0 are known and $a_{2k-1}=0$, $b_{2k-1}=0$.

2. Let us consider the vibrations problem of nonuniformly stretched rectangular membrane with following boundary conditions

$$u=0,\ \text{when}\ x=0,a;\quad \frac{\partial u}{\partial y}=0,\ \text{when}\ y=0,b \tag{16}$$

The solution of equation (1). which satisfies the boundary condition (16). is represented in the form

$$u=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}B_{mn}(t)\sin\lambda_m x\cos\mu_n y \tag{17}$$

Substituting (17) and (5) into equation(1)m using the formulae (7) and

$$2\cos\mu_n y\cos\mu_k y=\cos\mu_{n-k}y+\cos\mu_{n+k}y \tag{18}$$

after some transformations and equalization to zero of the coefficients of the members $\sin\lambda_q x\cos\mu_r y$. we shall receive the following system of infinite equations

$$B_{qr}+\frac{\lambda_q^2}{\rho}\left[\frac{1}{2}\sum_{n=0}^{r-1}(a_{r-n}-a_{r+n})B_{qn}+\left(a_0-\frac{a_{2r}}{2}\right)B_{qr}+\frac{1}{2}\sum_{n=r+1}^{\infty}(a_{n-r}-a_{n+r})B_{qn}\right]+$$
$$+\frac{\mu_r^2}{\rho}\left[\frac{1}{2}\sum_{m=1}^{q-1}\left(b_{q-m}-b_{m+q}\right)B_{mr}+\left(b_0-\frac{b_{2q}}{2}\right)B_{qr}+\frac{1}{2}\sum_{m=q+1}^{\infty}\left(b_{m-q}-b_{m+q}\right)B_{mr}\right]=0$$
$$q=1,2,\dots,\ r=0,1,2,\dots \tag{19}$$

Instead of the system (19) let us consider the following shortened system of equations

$$B_{10}+\frac{\lambda_1^2}{2\rho}a_0B_{10}=0$$
$$B_{11}+\frac{\lambda_1^2}{\rho}\left[\left(a_0-\frac{a_2}{2}\right)B_{11}+\frac{1}{2}(a_1-a_3)B_{12}\right]+\frac{\mu_1^2}{\rho}\left[\left(b_0-\frac{b_2}{2}\right)B_{11}+\frac{1}{2}(b_1-b_3)B_{21}\right]=0$$
$$B_{20}+\frac{\lambda_2^2}{\rho}\left(a_0-\frac{a_4}{2}\right)B_{20}=0 \tag{20}$$

$$B_{21}+\frac{\lambda_2^2}{\rho}\left[\left(a_0-\frac{a_4}{2}\right)B_{21}+\frac{1}{2}(a_1-a_3)B_{22}\right]+\frac{\mu_1^2}{\rho}\left[\frac{1}{2}(b_1-b_3)B_{11}+\left(b_0-\frac{b_4}{2}\right)B_{21}\right]=0$$

$$B_{12}+\frac{\lambda_1^2}{\rho}\left[\frac{1}{2}(a_1-a_3)B_{11}+\left(a_0-\frac{a_4}{2}\right)B_{12}\right]+\frac{\mu_2^2}{\rho}\left[\left(b_0-\frac{b_2}{2}\right)B_{12}+\frac{1}{2}(b_1-b_3)B_{22}\right]=0$$

$$B_{22}+\frac{\lambda_2^2}{\rho}\left[\frac{1}{2}(a_1-a_3)B_{21}+\left(a_0-\frac{a_4}{2}\right)B_{22}\right]+\frac{\mu_2^2}{\rho}\left[\frac{1}{2}(b_1-b_3)B_{12}+\left(b_0-\frac{b_4}{2}\right)B_{22}\right]=0$$

Representing the solution of the system (20) in the form

$$B_{qr}=B_{qr}^0\exp i(\theta t) \tag{21}$$

we shall obtain a system of algebraic equations with respect to B_{qr}^0. Equalizing to zero the determinant of this system we shall receive

$$\frac{\lambda_1^2}{2}a_0-\rho\theta^2=0 \tag{22}$$

$$\frac{\lambda_1^2}{2}(a_0-\frac{a_4}{2})-\rho\theta^2=0 \tag{23}$$

$$\delta(a_0,\dots,a_4,b_0,\dots,b_4,\theta^2)=0 \tag{24}$$

The equation (24) has fourth degree with respect to θ^2. Consequently the equations (22)-(24) determine six vibrations frequencies. Let us signify the solution of the equation (22) by θ_{10}^2, (23) by θ_{20}^2 and (24) by

$$\theta_1=\theta_{11},\ \theta_2=\theta_{21},\ \theta_3=\theta_{12},\ \theta_4=\theta_{22} \tag{25}$$

In the inverse problem we shall receive the following six equations for the determination of the coefficients $a_0,\dots,a_4,b_0,\dots,b_4$ if indicated six vibrations frequencies are known

$$\frac{\lambda_1^2}{2}a_0-\rho\theta_{10}^2=0,\ \ \frac{\lambda_2^2}{2}(a_0-\frac{a_4}{2})-\rho\theta_{20}^2=0 \tag{26}$$

$$\delta(a_0,\dots,a_4,b_0,\dots,b_4,\theta_i^2)=0\qquad i=1,2,3,4$$

The system of the equations together with the system (15) make up ten equations with ten unknowns $a_0,\dots,a_4,b_0,\dots,b_4$. Therefore, if the first four frequencies of the problem with boundary conditions (2) and first six frequencies of the problem with boundary conditions (16) are known, then the coefficients $a_0,\dots,a_4,b_0,\dots,b_4$ are determined by unique way. Hence it is clear that the inverse problem has unique solution when the two frequencies spectrums corresponding to the membrane vibrations problem under different boundary conditions are given.

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Институт механики НАН Армении

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