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A NOTE ON RECURSIVE INTERPOLATION FOR THE LIPSCHITZ CLASS

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Abstract. This note is framed in the field of complex analysis and deals with some types of interpolating sequences for Lipschitz functions in the unit disk. We introduce recursion between each point of a sequence and the next. We also add interpolation by the derivative, linking its values to those that the function takes. On the supposition that the sequences are quite contractive and lie in a Stolz angle, we relate the interpolating ones for each type to the uniformly separated sequences.

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1. Introduction

We denote by $\mathbb{D}$ the disk in the complex plane $\mathbb{C}$ and by *Lip* the Lipschitz class, that is, the space of all analytic functions f on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and such that

$$M_f = \sup_{z\neq w} \frac{|f(z)-f(w)|}{|z-w|} < \infty.$$

It is well-known that $f \in Lip$ if and only if $f' \in H^\infty$ (the space of bounded analytic functions on $\mathbb{D}$). We put $\Lambda = (\lambda_n)$ for bounded sequences of complex numbers and l^∞ for their space ($\|\Lambda\|_\infty = \sup_n |\lambda_n|$). We denote by $Z = (z_n)$ any sequence in $\mathbb{D}$ satisfying the Blaschke condition $\sum_n (1-|z_n|) < \infty$. We write $\tau(z,w) = \dfrac{z-w}{1-\overline{z}w}$, so that $|\tau(z,w)|$ is the pseudo-hyperbolic distance between z and w. We put B for the Blaschke product in $\mathbb{D}$ with zeros at Z, that is,

$$B(z) = \prod_n \frac{\overline{z}_n}{z_n}\, \tau(z_n, z),$$

and $B_{i_1,\ldots,i_m}$ for the Blaschke product with zeros at $Z \setminus \{z_{i_1}, \ldots, z_{i_m}\}$.

We recall that a sequence Z is called *k-contractive* if there is a constant $0<k<1$ such that

$$|z_{m+1} - z_m| \le k\,|z_m - z_{m-1}|, \quad m \ge 2.$$

We also recall that a sequence Z is called *uniformly separated* (we will abbreviate by writing u.s.) if

$$|B_m(z_m)| \ge \delta > 0, \quad m \in \mathbb{N}.$$

Carleson's theorem ([1]) ensures that the u.s. sequences are the interpolating ones for H^∞ (it means that given any $\Lambda \in l^\infty$, there is $f \in H^\infty$ such that $f(z_n) = \lambda_n$ for all n).

First, we bring up two types of interpolating sequences for *Lip*.

Definition 1.1. *Z is called an interpolating sequence for Lip if given any sequence (ω_n) satisfying*

$$\sup_{i \neq j} \frac{|\omega_i - \omega_j|}{|z_i - z_j|} < \infty, \tag{1.1}$$

there exists $f \in Lip$ such that $f(z_n) = \omega_n$ for all n.

Definition 1.2. *Z is called a double interpolating sequence for Lip if given any sequences (ω_n) satisfying* (1.1) *and $(\lambda_n) \in l^\infty$, there exists $f \in Lip$ such that $f(z_n) = \omega_n$ and $f'(z_n) = \lambda_n$ for all n.*

Both types are characterized in the following two Theorems, but only when the sequence Z is in a *Stolz angle*, that is, when for some $\zeta \in \partial\mathbb{D}$ and $1 < \mu < \infty$,

$$|z_n - \zeta| < \mu\,(1 - |z_n|), \quad n \in \mathbb{N}.$$

For example, the radial sequence $(1-2^{-n})$ satisfies the Blaschke condition, is $(1/2)$-contractive and lies in a Stolz angle ($\zeta = 1$).

Theorem 1.1. ([2], [3]). *A sequence Z in a Stolz angle is interpolating for Lip if and only if Z is the union of two u.s. sequences.*

Theorem 1.2. ([3]). *A sequence Z in a Stolz angle is double interpolating for Lip if and only if Z is u.s.*

The interpolation by Lipschitz functions for a closed set in $\overline{\mathbb{D}}$ has also been studied (see [4]).

Our purpose is to introduce some new types of interpolating sequences for *Lip*. For that, we modify the above Definitions for the case that a recursive relationship of the interpolating function in two consecutive points of the sequence is required. On the other hand, we impose a rather natural ligature between the interpolating function and its derivative, also adding a recursive relationship for the derivative. We are interested in knowing if doing this, we have to restrict ourselves to some sort of sequences to obtain the same results as if recursion is not considered. Recursive interpolating sequences for the space H^∞ have already been addressed in [5], and in this note, we check the effect of introducing recursion in a space of functions that are regular up to the boundary of the disk.

Specifically, we introduce the following sequences.

Definition 1.3. *We say that Z is a recursive interpolating sequence for Lip if given any $\alpha \in \mathbb{C}$ and $\Lambda = (\lambda_n) \in l^\infty$, there exists $f \in Lip$ such that $f(z_1) = \alpha$ and recursively, for each $n \in \mathbb{N}$,*

$$\frac{f(z_{n+1}) - f(z_n)}{z_{n+1} - z_n} = \lambda_n. \tag{1.2}$$

Note that all quotients in (1.2) are bounded, because $f \in Lip$.

Definition 1.4. *If we include $f'(z_n) = \lambda_n$ in Definition 1.3, we say that Z is a double and recursive interpolating sequence for Lip.*

In this Definition, the requirement for the derivative is added to relate its value in a point to a difference quotient of the function in that point. Finally, taking into account that if $g \in H^\infty$, then

$$|g(z) - g(w)| \leq c\,|\tau(z, w)|$$

for a constant $c > 0$, we can state:

Definition 1.5. *We say that Z is an interpolating sequence in a general sense for Lip if given any α, β, $\eta \in \mathbb{C}$, there exists $f \in Lip$ such that $f(z_1) = \alpha$, $f'(z_1) = \beta$ and, recursively, for each $n \in \mathbb{N}$,*

$$\begin{cases} f'(z_n) = \dfrac{f(z_{n+1}) - f(z_n)}{z_{n+1} - z_n} \\ f'(z_{n+1}) = f'(z_n) + \eta\,\tau(z_n, z_{n+1}) \end{cases} \tag{1.3}$$

Note that these two equalities can be interpreted as a certain system of recurrence equations. The next section is devoted to examining these types of interpolating sequences.

2. Statement and proof of results

We will use the following two Lemmas.

Lemma 2.1. ([3]). *If a function $f \in Lip$ vanishes on a sequence Z, then for each $m \in \mathbb{N}$,*

$$|f(z)| \leq M_f\,|z - z_m|\,|B_m(z)|.$$

Lemma 2.2. *If Z is a k-contractive sequence and $k \leq k_0 < 1/2$, then given any integer $p \geq 2$,*

$$|z_{m+1} - z_m| \leq K_0\,|z_{m+p} - z_m|, \quad m \in \mathbb{N},$$

where $K_0 = (1 - k_0)/(1 - 2k_0 + k_0^p)$.

Proof. By the triangle inequality and since Z is k-contractive,

$$\begin{aligned}|z_{m+1}-z_m| &\le |z_{m+p}-z_m| + |z_{m+p}-z_{m+p-1}| + \cdots + |z_{m+2}-z_{m+1}|\\ &\le |z_{m+p}-z_m| + (k^{p-1}+\cdots+k)|z_{m+1}-z_m|.\end{aligned}$$

Since $k \le k_0 < 1/2$, then $k^{p-1}+\cdots+k \le k_0^{p-1}+\cdots+k_0 < 1$, and

$$|z_{m+1}-z_m| \le \frac{1}{1-(k_0^{p-1}+\cdots+k_0)}|z_{m+p}-z_m| = K_0\,|z_{m+p}-z_m|.$$

The proof is complete. □

Our results are the following ones.

Theorem 2.1. *Let Z be a sequence in a Stolz angle and k-contractive for some $k \le k_0 < 1/2$. Then, Z is recursive interpolating for Lip if and only if Z is the union of two u.s. sequences.*

Proof. Suppose that Z is recursive interpolating for *Lip*. Take $\alpha = 0$ and for a fixed $m \in \mathbb{N}$, let Λ be defined by: $\lambda_m = \dfrac{z_{m+2}-z_{m+1}}{z_{m+1}-z_m}$, $\lambda_{m+1} = -1$ and $\lambda_n = 0$, otherwise. Because Z is k-contractive, we have $|\lambda_m| \le k$ and then, $\|\Lambda\|_\infty = 1$. Since the operator given by the quotient on the left in (1.2) is linear and surjective, by the open mapping theorem there is a function $f_m \in Lip$ and a constant $c > 0$ such that $M_{f_m} \le c\,\|\Lambda\|_\infty = c$. We have $f_m(z_m) = z_{m+1}-z_m$ and $f_m(z_n) = 0$, if $n \ne m$. Applying Lemma 2.1 to $Z \setminus \{z_m\}$,

$$|f_m(z)| \le c\,|z-z_{m+1}|\,|B_{m,m+1}(z)|,$$

and evaluating at z_m,

$$|z_{m+1}-z_m| = |f_m(z_m)| \le c\,|z_m-z_{m+1}|\,|B_{m,m+1}(z_m)|,$$

that is,

$$|B_{m,m+1}(z_m)| \ge c. \tag{2.1}$$

This condition (2.1) implies that Z is the union of two u.s. sequences (see [6], p. 1202).

Reciprocally, to meet the requirement in (1.2) we look for f verifying $f(z_n) = \gamma_n$, where $\gamma_1 = \alpha$ and for each $n \ge 2$,

$$\gamma_n = \alpha + \lambda_1(z_2-z_1) + \cdots + \lambda_{n-1}(z_n-z_{n-1}).$$

Suppose $i > j$. Taking into account that Z is k-contractive,

$$\begin{aligned}|\gamma_i-\gamma_j| &= |\lambda_j(z_{j+1}-z_j) + \cdots + \lambda_{i-1}(z_i-z_{i-1})|\\ &\le \|\Lambda\|_\infty\,(|z_{j+1}-z_j| + \cdots + |z_i-z_{i-1}|)\\ &\le \|\Lambda\|_\infty\,(1+k+\cdots+k^{i-j-1})\,|z_{j+1}-z_j|.\end{aligned}$$

If $i > j+1$, then by Lemma 2.2,

$$|\gamma_i - \gamma_j| \le \|\Lambda\|_\infty \frac{1-k_0^{i-j}}{1-2k_0+k_0^{i-j}} |z_i - z_j|.$$

The existence of the desired interpolating function f follows from Theorem 1.1. □

Theorem 2.2. *Let Z be a sequence in a Stolz angle and k-contractive for some $k \le k_0 < 1/2$. Then, Z is double and recursive interpolating for Lip if and only if Z is u.s.*

Proof. The necessity for the sequence Z to be u.s. is a consequence of the requirement that the function f' in H^∞ must interpolate the sequence Λ in l^∞ (Carleson's theorem). As for sufficiency, take $(\lambda_n) \in l^\infty$. By Theorem 2.1, there is $g \in Lip$ verifying (1.2). It is proved in [3] that if Z in a Stolz angle is u.s., then given any sequence $(\alpha_n) \in l^\infty$, there is a function $h \in Lip$ such that $h(z_n) = 0$ and $h'(z_n) = \alpha_n$ for all n. Taking $\alpha_n = \lambda_n - g'(z_n)$, it follows that the function $f = g + h$ performs (1.2) and $f'(z_n) = \lambda_n$ for all n. □

Theorem 2.3. *If Z is a sequence in a Stolz angle and k-contractive for some $k \le k_0 < 1/2$, it verifies the condition*

$$\sum_n |\tau(z_n, z_{n+1})| < \infty \tag{2.2}$$

and is u.s., then Z is interpolating in a general sense for Lip.

Proof. Equivalently, instead of looking for a function $f \in Lip$ that verifies (1.3), we look for it so that $f(z_n) = \gamma_n$ and $f'(z_n) = \gamma'_n$, where

$$\gamma_1 = \alpha, \quad \gamma_2 = \alpha + \beta(z_2 - z_1),$$

$$\gamma_n = \alpha + \beta(z_n - z_1) + \eta \sum_{l=3}^{n} \left(\sum_{m=1}^{l-2} \tau(z_m, z_{m+1}) \right) (z_l - z_{l-1}), \quad n \ge 3;$$

$$\gamma'_1 = \beta, \quad \gamma'_n = \beta + \eta \sum_{l=1}^{n-1} \tau(z_l, z_{l+1}), \quad n \ge 2.$$

Suppose $i > j$. By (2.2), there is a constant $c > 0$ such that

$$\begin{aligned} |\gamma_i - \gamma_j| &\le |\beta|\,|z_i - z_j| + \eta \sum_{l=j+1}^{i} \left(\sum_{m=1}^{l-2} |\tau(z_m, z_{m+1})| \right) |z_l - z_{l-1}| \\ &\le |\beta|\,|z_i - z_j| + c\,\eta \sum_{l=j+1}^{i} |z_l - z_{l-1}| \end{aligned}$$

As in the proof of Theorem 2.1, if $i > j+1$, then

$$|\gamma_i - \gamma_j| \le \left(|\beta| + c\,\eta \frac{1-k_0^{i-j}}{1-2k_0+k_0^{i-j}} \right) |z_i - z_j|.$$

On the other hand, $(\gamma'_n) \in l^\infty$ by (2.2). So, the existence of the interpolating function f follows now from Theorem 1.2. $\square$

The sequence $(1 - 2^{-n})$ is also an example for the condition (2.2).

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