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SOME UPPER BOUND ESTIMATES FOR THE MAXIMAL MODULUS OF THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. This paper deals with the problem of finding some upper bound estimates for the maximal modulus of the polar derivative of a complex polynomial on a disk under certain constraints on the zeros and on the functions involved. A variety of interesting results follow as special cases from our results.

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1. Introduction

Let $\mathbb{P}_n$ denote the space of all complex polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n and $P'(z)$ is the derivative of $P(z)$. A famous result known as Bernstein's inequality (for reference, see [3]) states that if $P \in \mathbb{P}_n$, then

$$\max_{|z|=1}\left|P'(z)\right| \le n\max_{|z|=1}\left|P(z)\right|, \tag{1.1}$$

where as concerning the maximum modulus of $P(z)$ on the circle $|z| = R \ge 1$, we have (for reference see [11]),

$$\max_{|z|=R}\left|P(z)\right| \le R^n\max_{|z|=1}\left|P(z)\right|. \tag{1.2}$$

Both the above inequalities are sharp and equality in each holds only when $P(z)$ is a constant multiple of z^n.

It was observed by Bernstein [3] that (1.1) can be deduced from (1.2), by making use of Gauss - Lucas theorem and the proof of this fact was given by Govil, Qazi and Rahman [4].

If we restrict ourselves to the class of polynomials $P \in \mathbb{P}_n$, with $P(z) \ne 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced

$$\max_{|z|=1}\left|P'(z)\right| \le \frac{n}{2}\max_{|z|=1}\left|P(z)\right|, \tag{1.3}$$

and

$$\max_{|z|=R\ge 1}\left|P(z)\right| \le \frac{R^n+1}{2}\max_{|z|=1}\left|P(z)\right|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later proved by Lax [8], where as inequality (1.4) was proved by Ankeny and Rivlin [1], for which they made use of (1.3).

Inequality (1.1) can be seen as a special case of the following inequality which is also due to Bernstein [3].

Theorem A. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \le 1$ and $f(z)$ be a polynomial of degree at most n. If $|f(z)| \le |F(z)|$ for $|z| = 1$, then for $|z| \ge 1$, we have*

$$|f'(z)| \le |F'(z)|. \tag{1.5}$$

Equality holds in (1.5) for $f(z) = e^{i\eta}F(z), \eta \in \mathbb{R}$.

Inequality (1.1) can be obtained from inequality (1.5) by taking $F(z) = Mz^n$, where $M = \max_{|z|=1}|f(z)|$. In the same way, inequality (1.2) follows from a result which is a special case of Bernstein-Walsh lemma ([10], Corollary 12.1.3).

Theorem B. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \le 1$ and $f(z)$ be a polynomial of degree at most n. If $|f(z)| \le |F(z)|$ for $|z| = 1$, then*

$$|f(z)| < |F(z)|, \quad \text{for} \quad |z| > 1,$$

unless $f(z) = e^{i\eta}F(z)$ for some $\eta \in \mathbb{R}$.

In 2011, Govil et al. [5] proved a more general result which provides a compact generalization of inequalities (1.1), (1.2), (1.3) and (1.4) and includes Theorem A and Theorem B as special cases. In fact, they proved that if $f(z)$ and $F(z)$ are as in Theorem A, then for any β with $|\beta| \le 1$ and $R \ge r \ge 1$, we have

$$|f(Rz) - \beta f(rz)| \le |F(Rz) - \beta F(rz)|, \quad \text{for} \quad |z| \ge 1. \tag{1.6}$$

Further, as a generalization of (1.6), Liman et al. [6] in the same year 2011 and under the same hypothesis as in Theorem A, proved that

$$\begin{aligned} &\Big|f(Rz) - \beta f(rz) + \gamma\Big\{\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big\}f(rz)\Big| \\ &\le \Big|F(Rz) - \beta F(rz) + \gamma\Big\{\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big\}F(rz)\Big|, \end{aligned} \tag{1.7}$$

for every $\beta, \gamma \in \mathbb{C}$ with $|\beta| \le 1, |\gamma| \le 1$ and $R > r \ge 1$.

For $f \in \mathbb{P}_n$, the polar derivative $D_\alpha f(z)$ of $f(z)$ with respect to the point α is defined as

$$D_\alpha f(z) := nf(z) + (\alpha - z)f'(z).$$

Note that $D_\alpha f(z)$ is a polynomial of degree at most $n-1$. This is the so-called polar derivative of $f(z)$ with respect to α (see [9]). It generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha\to\infty}\Big\{\frac{D_\alpha f(z)}{\alpha}\Big\} := f'(z),$$

uniformly with respect to z for $|z| \leq R, R > 0$.
Recently, Liman et al. [7] besides proving some other results also proved the following generalization of (1.6) to the polar derivative $D_\alpha f(z)$ of a polynomial $f(z)$ with respect to $\alpha, |\alpha| \geq 1$.

Theorem C. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that $|f(z)| \leq |F(z)|$ for $|z| = 1$. If $\alpha, \beta, \gamma \in \mathbb{C}$ be such that $|\alpha| \geq 1, |\beta| \leq 1$ and $|\lambda| < 1$, then for $R > r \geq 1$ and $|z| \geq 1$, we have*

$$\begin{aligned} &\Big|z\Big[(n-m)\Big\{f(Rz) - \beta f(rz)\Big\} + D_\alpha f(Rz) - \beta D_\alpha f(rz)\Big] \\ &\qquad + \frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz) - \beta f(rz)\Big\}\Big| \\ &\leq \Big|z\Big\{D_\alpha F(Rz) - \beta D_\alpha F(rz)\Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz) - \beta F(rz)\Big\}\Big|. \end{aligned} \tag{1.8}$$

Equality holds in (1.8) for $f(z) = e^{i\eta}F(z), \eta \in \mathbb{R}$.

While making an attempt towards the generalization of the above inequalities, the authors found that there is a room for the generalization of (1.6) to the polar derivative of a polynomial which in turn induces inequalities towards more generalized form. The essence in the papers by Liman et al. [7] and Govil et al. [5] is the origin of thought for the new inequalities presented in this paper.

2. Main results

The main aim of this paper is to obtain some more general results for the maximal modulus of the polar derivative of a polynomial under certain constraints on $|z|$ and on the functions considered. We first prove the following generalization of inequalities (1.6) and (1.7) and of Theorem C.

Theorem 2.1. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\big|f(z)\big| \leq |F(z)|, \quad \text{for} \quad |z| = 1.$$

If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ be such that $|\alpha| \geq 1, |\beta| \leq 1, |\gamma| \leq 1$ and $|\lambda| < 1$, then for $R > r \geq 1$ and $|z| \geq 1$, we have

$$\begin{aligned} &\Big|z\Big[(n-m)\Big\{f(Rz) + \psi f(rz)\Big\} + D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big] \\ &\qquad + \frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz) + \psi f(rz)\Big\}\Big| \\ &\leq \Big|z\Big\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz) + \psi F(rz)\Big\}\Big|, \end{aligned} \tag{2.1}$$

where

$$\psi = \psi(R, r, \beta, \gamma) = \gamma\Big\{\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big\} - \beta.$$

The result is sharp and equality in (2.1) *holds for* $f(z) = e^{i\eta}F(z), \eta \in \mathbb{R}$.

The following result immediately follows from Theorem 2.1.

Corollary 2.1. *If* $f \in \mathbb{P}_n$, *and* $f(z)$ *does not vanish in* $|z| < 1$, *then for every* $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ *such that* $|\alpha| \geq 1, |\beta| \leq 1, |\gamma| \leq 1$ *and* $|\lambda| < 1$, *we have for* $R > r \geq 1$ *and* $|z| \geq 1$,

$$\begin{aligned} &\Big|z\Big\{D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big\} + \frac{n\lambda}{2}(|\alpha| - 1)\Big\{f(Rz) + \psi f(rz)\Big\}\Big| \\ &\leq \Big|z\Big\{D_\alpha Q(Rz) + \psi D_\alpha Q(rz)\Big\} + \frac{n\lambda}{2}(|\alpha| - 1)\Big\{Q(Rz) + \psi Q(rz)\Big\}\Big|, \end{aligned} \tag{2.2}$$

where $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$.

Equality holds in (2.2) for $f(z) = e^{i\eta}Q(z), \eta \in \mathbb{R}$. Taking $\lambda = 0$ in Corollary 2.1, we get the following result.

Corollary 2.2. *If* $f \in \mathbb{P}_n$, *and* $f(z) \neq 0$ *in* $|z| < 1$, *then for every* $|\alpha| \geq 1, |\beta| \leq 1, |\gamma| \leq 1, R > r \geq 1$ *and* $|z| \geq 1$,

$$\begin{aligned} &\Big|D_\alpha f(Rz) - \beta D_\alpha f(rz) + \gamma\Big(\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big)D_\alpha f(rz)\Big| \\ &\leq \Big|D_\alpha Q(Rz) - \beta D_\alpha Q(rz) + \gamma\Big(\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\Big)D_\alpha Q(rz)\Big|, \end{aligned} \tag{2.3}$$

where $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$.

Inequality (2.3) should be compared with a result of Liman, Mohapatra and Shah ([6], Lemma 2.3), where $f(z)$ is replaced by $D_\alpha f(z), |\alpha| \geq 1$.

Taking $r = 1$ in Corollary 2.2, we get the following generalization of a result due to Aziz and Rather [2].

Corollary 2.3. *If* $f \in \mathbb{P}_n$, *and* $f(z)$ *does not vanish in* $|z| < 1$, *then for every* $\alpha, \beta, \gamma \in \mathbb{C}$ *with* $|\alpha| \geq 1, |\beta| \leq 1$ *and* $R > 1$,

$$\begin{aligned} &\Big|D_\alpha f(Rz) - \beta D_\alpha f(z) + \gamma\Big(\Big(\frac{R+1}{2}\Big)^n - |\beta|\Big)D_\alpha f(z)\Big| \\ &\leq \Big|D_\alpha Q(Rz) - \beta D_\alpha Q(z) + \gamma\Big(\Big(\frac{R+1}{2}\Big)^n - |\beta|\Big)D_\alpha Q(z)\Big|, \quad \textit{for} \quad |z| \geq 1, \end{aligned}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

If we take $\beta = 0$ in Theorem 2.1, we get the following.

Corollary 2.4. *Let $F \in \mathbb{P}_n$, having all zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\big|f(z)\big| \leq \big|F(z)\big|, \quad \text{for} \quad |z| = 1.$$

If $\alpha, \gamma, \lambda \in \mathbb{C}$ be such that $|\alpha| \geq 1, |\gamma| \leq 1$ and $|\lambda| < 1$, then for $R > r \geq 1$ and $|z| \geq 1$, we have

$$\Big|z\Big[(n-m)\Big\{f(Rz) + \gamma\Big(\frac{R+1}{r+1}\Big)^n f(rz)\Big\} + D_\alpha f(Rz) + \gamma\Big(\frac{R+1}{r+1}\Big)^n D_\alpha f(rz)\Big] + \frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz) + \gamma\Big(\frac{R+1}{r+1}\Big)^n f(rz)\Big\}\Big|$$

$$\leq \Big|z\Big\{D_\alpha F(Rz) + \gamma\Big(\frac{R+1}{r+1}\Big)^n D_\alpha F(rz)\Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz) + \Big(\frac{R+1}{r+1}\Big)^n F(rz)\Big\}\Big|. \tag{2.4}$$

Equality holds in (2.4) for $f(z) = e^{i\eta}F(z), \eta \in \mathbb{R}$.

Remark 1.1. For $\gamma = 0$, Corollary 2.4 reduces to Theorem C.

Theorem 2.2. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$ and $f(z)$ be a polynomial of degree $m(\leq n)$ such that*

$$\big|f(z)\big| \leq |F(z)|, \quad \text{for} \quad |z| = 1.$$

If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ be such that $|\alpha| \geq 1, |\beta| \leq 1$ and $|\gamma| \leq 1$, then for $R > r \geq 1$ and $|z| \geq 1$, we have

$$\Big|z\Big[(n-m)\Big\{f(Rz) + \psi f(rz)\Big\} + D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big]\Big| + \frac{n}{2}(|\alpha|-1)\big|F(Rz) + \psi F(rz)\big|$$

$$\leq \Big|z\Big\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\Big\}\Big| + \frac{n}{2}(|\alpha|-1)\big|f(Rz) + \psi f(rz)\big|, \tag{2.5}$$

where ψ is defined in Theorem 2.1.

Equality holds in (2.5) for $f(z) = e^{i\eta}F(z), \eta \in \mathbb{R}$.

From Theorem 2.2, we have the following:

Corollary 2.5. *If $f \in \mathbb{P}_n$, and $f(z)$ does not vanish in $|z| < 1$, then for every $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ with $|\alpha| \geq 1, |\beta| \leq 1, |\gamma| \leq 1$, we have for $R > r \geq 1$, and $|z| \geq 1$,*

$$\Big|z\Big\{D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big\}\Big| + \frac{n}{2}(|\alpha|-1)\big|Q(Rz) + \psi Q(rz)\big|$$

$$\leq \Big|z\Big\{D_\alpha Q(Rz) + \psi D_\alpha Q(rz)\Big\}\Big| + \frac{n}{2}(|\alpha|-1)\big|f(Rz) + \psi f(rz)\big|,$$

where $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$. and ψ is defined in Theorem 2.1.

Remark 1.2. For $\gamma = 0$, Corollary 2.5 reduces to a result of Liman et al. [7].

3. Lemmas

We need the following lemmas to prove our theorems. The first lemma is due to Liman, Mohapatra and Shah [6].

Lemma 3.1. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$, then for every $R > r \geq 1$,*

$$\big|f(Rz)\big| > \Big(\frac{R+1}{r+1}\Big)^n \big|f(rz)\big|, \quad \textit{for} \quad |z| = 1.$$

Lemma 3.2. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$,*

$$2\big|zD_\alpha f(z)\big| \geq n(|\alpha| - 1)\big|f(z)\big|, \quad \textit{for} \quad |z| = 1.$$

The above lemma is due to Shah [12].

Lemma 3.3. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leq k$, then for $|\alpha| \geq k$, the polar derivative*

$$D_\alpha f(z) := nf(z) + (\alpha - z)f'(z),$$

of $f(z)$ at the point α also has all its zeros in $|z| \leq k$.

The above lemma is due to Laguerre ([9], p.49).

4. Proofs of theorems

Proof of Theorem 2.1. If $F(z)$ has a zero on $|z| = 1$, then the result is obvious, so we assume that $F(z)$ has no zeros on $|z| = 1$. Since $|f(z)| \leq |F(z)|$ for $|z| = 1$, therefore, for every $\delta \in \mathbb{C}$ with $|\delta| > 1$, we have $|f(z)| < |\delta F(z)|$, for $|z| = 1$. Also all the zeros of $F(z)$ lie in $|z| < 1$, it follows by Rouche's theorem that all the zeros of $g(z) = f(z) - \delta F(z)$ lie in $|z| < 1$. Now by Lemma 3.1, we have in particular

$$|g(rz)| < |g(Rz)|, \quad \text{for} \quad |z| = 1 \quad \text{and} \quad R > r \geq 1.$$

Since $g(Rz)$ has all its zeros in $|z| \leq \frac{1}{R} < 1$, a direct application of Rouche's theorem shows that the polynomial $g(Rz) - \beta g(rz)$ has all its zeros in $|z| < 1$ for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$. Again by using Lemma 3.1, we have

$$\begin{aligned}\big|g(Rz) - \beta g(rz)\big| &\geq \big|g(Rz)\big| - |\beta|\big|g(rz)\big| \\ &> \left\{\Big(\frac{R+1}{r+1}\Big)^n - |\beta|\right\}\big|g(rz)\big|,\end{aligned}$$
$$\text{for} \quad |z| = 1 \quad \text{and} \quad R > r \geq 1.$$

That is

$$\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}\big|g(rz)\big| < \big|g(Rz) - \beta g(rz)\big|,$$
$$\text{for} \quad |z| = 1 \quad \text{and} \quad R > r \geq 1.$$

If γ is any complex number with $|\gamma| \leq 1$, then it follows by Rouche's theorem that all the zeros of $T(z) := g(Rz) - \beta g(rz) + \gamma\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}g(rz)$ lie in $|z| < 1$. Using Lemma 3.2, we get for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $|z| = 1$,

$$2\big|zD_\alpha T(z)\big| \geq n(|\alpha| - 1)\big|T(z)\big|.$$

Hence for any complex number λ with $|\lambda| < 1$, we have for $|z| = 1$,

$$2\big|zD_\alpha T(z)\big| > n|\lambda|(|\alpha| - 1)\big|T(z)\big|.$$

Therefore, it follows by Lemma 3.3, that all the zeros of

$$\begin{aligned} W(z) &:= 2zD_\alpha T(z) + n\lambda(|\alpha| - 1)T(z) \\ &= 2zD_\alpha g(Rz) + 2z\psi D_\alpha g(rz) + n\lambda\big(|\alpha| - 1\big)\big(g(Rz) + \psi g(rz)\big) \end{aligned} \tag{4.1}$$

lie in $|z| < 1$.

Replacing $g(z)$ by $f(z) - \delta F(z)$ and using definition of polar derivative gives

$$\begin{aligned} W(z) = 2z&\Big[n\Big\{f(Rz) - \delta F(Rz)\Big\} + (\alpha - Rz)\Big\{f(Rz) - \delta F(Rz)\Big\}'\Big] \\ &+ 2z\psi\Big[n\Big\{f(rz) - \delta F(rz)\Big\} + (\alpha - rz)\Big\{f(rz) - \delta F(rz)\Big\}'\Big] \\ &+ n\lambda(|\alpha| - 1)\Big\{f(Rz) - \delta F(Rz)\Big\} + n\lambda\psi(|\alpha| - 1)\Big\{f(rz) - \delta F(rz)\Big\}, \end{aligned}$$

which on simplification gives

$$\begin{aligned} W(z) &= 2z\Big[(n-m)f(Rz) + mf(Rz) + (\alpha - Rz)\big(f(Rz)\big)' - \delta\Big\{nF(rz) + (\alpha - rz)\big(F(Rz)\big)'\Big\}\Big] \\ &\quad + 2z\psi\Big[(n-m)f(rz) + mf(rz) + (\alpha - rz)\big(f(rz)\big)' - \delta\Big\{nF(rz) + (\alpha - rz)\big(F(rz)\big)'\Big\}\Big] \\ &\qquad + n\lambda(|\alpha| - 1)\Big\{f(Rz) - \delta F(Rz)\Big\} + n\lambda\psi(|\alpha| - 1)\Big\{f(rz) - \delta F(rz)\Big\} \\ &= 2z\Big\{(n-m)f(Rz) + D_\alpha f(Rz) - \delta D_\alpha F(Rz)\Big\} \\ &\quad + 2z\psi\Big\{(n-m)f(rz) + D_\alpha f(rz) - \delta D_\alpha F(rz)\Big\} \\ &\quad + n\lambda(|\alpha| - 1)\Big\{f(Rz) - \delta F(Rz)\Big\} + n\lambda\psi(|\alpha| - 1)\Big\{f(rz) - \delta F(rz)\Big\} \\ &= 2z\Big\{(n-m)f(Rz) + \psi(n-m)f(rz) + D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big\} \\ &\quad + n\lambda(|\alpha| - 1)f(Rz) + n\lambda\psi(|\alpha| - 1)f(rz) - \delta\Big\{2zD_\alpha F(Rz) + 2z\psi D_\alpha F(rz) \\ &\qquad + n\lambda\psi(|\alpha| - 1)F(Rz) + n\lambda\psi(|\alpha| - 1)f(rz)\Big\}. \end{aligned} \tag{4.2}$$

Since by (4.1), $W(z)$ has all its zeros in $|z| < 1$, therefore, by (4.2), we get for $|z| \geq 1$,

$$\left|z\Big[(n-m)\Big\{f(Rz)+\psi f(rz)\Big\}+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big]+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\psi f(rz)\Big\}\right|$$

(4.3)

$$\leq\left|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}\right|.$$

To see that the inequality (4.3) holds, note that if the inequality (4.3) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$\left|z_0\Big[(n-m)\,A+D_\alpha f(Rz_0)+\psi D_\alpha f(rz_0)\Big]+\frac{n\lambda}{2}(|\alpha|-1)\,A\right|$$

$$(4.4)\qquad >\left|z_0\Big\{D_\alpha F(Rz_0)+\psi D_\alpha F(rz_0)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}\right|,$$

where $A = f(Rz_0) + \psi f(rz_0)$. Now, because by hypothesis all the zeros of $F(z)$ lie in $|z| \leq 1$, the polynomial $F(Rz)$ has all its zeros in $|z| \leq \frac{1}{R} < 1$, and therefore, if we use Rouche's theorem and Lemmas 3.1 and 3.3 and argument similar to the above, we will get that all the zeros of

$$z\Big(D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}$$

lie in $|z| < 1$ for every $|\alpha| \geq 1, |\lambda| < 1$ and $R > r \geq 1$, that is

$$z\Big(D_\alpha F(Rz_0)+\psi D_\alpha F(rz_0)\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}\neq 0$$

for every z_0 with $|z_0| \geq 1$.
Therefore, if we take

$$\delta=\frac{z_0\Big[(n-m)\,A+D_\alpha f(Rz_0)+\psi D_\alpha f(rz_0)\Big]+\frac{n\lambda}{2}(|\alpha|-1)\,A}{z_0\Big(D_\alpha F(Rz_0)+\psi D_\alpha F(rz_0)\Big)+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz_0)+\psi F(rz_0)\Big\}},$$

then δ is a well-defined real or complex number, and in view of (4.4) we also have $|\delta| > 1$. Hence, with the choice of δ, we get from (4.2) that $W(z_0) = 0$ for some z_0, satisfying $|z_0| \geq 1$, which is clearly a contradiction to the fact that all the zeros of $W(z)$ lie in $|z| < 1$. Thus for every $R > r \geq 1, |\alpha| \geq 1, |\lambda| < 1$ and $|z| \geq 1$, inequality (4.3) holds and this completes the proof of
Theorem 1.1.

Proof of Corollary 2.1. Since the polynomial $f(z)$ does not vanish in $|z| < 1$, therefore, all the zeros of the polynomial $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})} \in \mathbb{P}_n$, lie in $|z| \leq 1$ and $|f(z)| = |Q(z)|$ for $|z| = 1$. Applying Theorem 1.1 with $F(z)$ replaced by $Q(z)$, the result follows.

Proof of Theorem 2.2. Since all the zeros of $F(z)$ lie in $|z| \leq 1$, for $R > r \geq 1, |\beta| \leq 1, |\gamma| \leq 1$, it follows as in the proof of Theorem 2.1, that all the zeros of

$$h(z) = F(Rz) - \beta F(rz) + \gamma\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}F(rz) = F(Rz) + \psi F(rz)$$

lie in $|z| \leq 1$. Hence by Lemma 3.2, we get for $|\alpha| \geq 1$,

$$2\big|zD_\alpha h(z)\big| \geq n(|\alpha|-1)\big|h(z)\big|, \quad \text{for} \quad |z| \geq 1.$$

This gives for every λ with $|\lambda| < 1$ and for $|z| \geq 1$

$$\left|z\Big\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\Big\}\right| - \frac{n|\lambda|}{2}(|\alpha|-1)\big|F(Rz) + \psi F(rz)\big| \geq 0. \tag{4.5}$$

Therefore, it is possible to choose the argument of λ in the right hand side of (4.3) such that for $|z| \geq 1$,

$$\begin{aligned}&\left|z\Big\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz) + \psi F(rz)\Big\}\right| \\ &= \left|z\Big\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\Big\}\right| - \frac{n|\lambda|}{2}(|\alpha|-1)\Big|F(Rz) + \psi F(rz)\Big|. \end{aligned} \tag{4.6}$$

Hence from (4.3), we get by using (4.6) for $|z| \geq 1$,

$$\begin{aligned}&\left|z\Big[(n-m)\Big\{f(Rz) + \psi f(rz)\Big\} + D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big]\right| \\ &\qquad - \frac{n|\lambda|}{2}(|\alpha|-1)\big|f(Rz) + \psi f(rz)\big| \\ &\leq \left|z\Big\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\Big\}\right| - \frac{n|\lambda|}{2}(|\alpha|-1)\Big|F(Rz) + \psi F(rz)\Big|. \end{aligned} \tag{4.7}$$

Letting $|\lambda| \to 1$ in (4.7), we immediately get (2.5) and this completes proof of Theorem 2.2 completely.

Proof of Corollary 2.5. By hypothesis, the polynomial $f(z)$ has all its zeros in $|z| \geq 1$, therefore, all the zeros of the polynomial $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})} \in \mathbb{P}_n$, lie in $|z| \leq 1$ and $|f(z)| = |Q(z)|$ for $|z| = 1$. Applying Theorem 2.2 with $F(z)$ replaced by $Q(z)$, the result follows.

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