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A UNIQUENESS THEOREM FOR FRANKLIN SERIES

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Abstract. In this paper we obtain, that if the partial sums $\sigma_{q_k}(x)$ of a Franklin series converge in measure to a function f, the ratio $\frac{q_{k+1}}{q_k}$ is bounded and the majorant of partial sums satisfies to a necessary condition, then the coefficients of the series are restored by the function f.

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1. Introduction

It is well known that there are trigonometric series converging almost everywhere to zero and having at least one non-zero coefficient. This also applies to the series in other classical orthogonal systems, for instance, to the series in Haar, Walsh and Franklin systems.

The uniqueness problem and reconstruction of coefficients of series by various orthogonal systems has been considered in a number of papers. Uniqueness theorems for almost everywhere convergent or summable trigonometric series were obtained in the papers [1] and [4], under some additional conditions imposed on the series. Results on uniqueness and restoration of coefficients for series by Haar, Franklin and generalized Haar systems have been obtained, for instance, in the papers [3],[6],[7] and [9]-[12] .

In this paper we will consider series by Franklin system.

The orthonormal Franklin system consists of piecewise linear and continuous functions. This system was constructed by Franklin [2] as the first example of a complete orthonormal system, which is a basis in $\mathbb{C}[0,1]$.

Let $n = 2^\mu + \nu$, $\mu \geq 0$, where $1 \leq \nu \leq 2^\mu$. Denote

$$s_{n,i} = \begin{cases} \frac{i}{2^{\mu+1}}, & \text{for } 0 \leq i \leq 2\nu, \\ \frac{i-\nu}{2^\mu}, & \text{for } 2\nu < i \leq n. \end{cases} \tag{1.1}$$

By S_n we denote the space of functions that are continuous and piecewise linear on $[0,1]$ with nodes $\{s_{n,i}\}_{i=0}^n$, that is $f \in S_n$ if $f \in C[0,1]$, and it is linear on each closed interval $[s_{n,i-1}, s_{n,i}]$, $i = 1, 2, \cdots, n$. It is clear, that $\dim S_n = n+1$, and the set $\{s_{n,i}\}_{i=0}^{n}$ is obtained by adding the point $s_{n,2\nu-1}$ to the set $\{s_{n-1,i}\}_{i=0}^{n-1}$. Hence, there exists a unique function $f_n \in S_n$, which is orthogonal to S_{n-1} and $\|f_n\|_2 = 1$. Setting $f_0(x) = 1,\ f_1(x) = \sqrt{3}(2x-1)$ for $x \in [0,1]$, we obtain an orthonormal system $\{f_n(x)\}_{n=0}^{\infty}$, which was defined equivalently by Franklin [2].

Here we quote a result by G. Gevorkyan [3] on restoration of coefficients of series by Franklin system.

Specifically, in [3] it was proved that if the Franklin series $\sum_{n=0}^{\infty} a_n f_n(x)$ converges a.e. to a function $f(x)$ and

$$\lim_{\lambda\to\infty}\left(\lambda \cdot |\{x \in [0,1] : \sup_{k\in\mathbb{N}} |S_k(x)| > \lambda\}|\right) = 0,$$

where

$$S_k(x) = \sum_{j=0}^{k} a_j f_j(x)$$

then the coefficients a_n of the Franklin series can be reconstructed by the following formula,

$$a_n = \lim_{\lambda\to\infty}\int_0^1 \left[f(x)\right]_\lambda f_n(x)dx,$$

where

$$\left[f(x)\right]_\lambda = \begin{cases} f(x), & \text{if } \ |f(x)| \le \lambda, \\ 0, & \text{if } \ |f(x)| > \lambda. \end{cases}$$

Similar result on uniqueness is also obtained for the Haar system (see [5]).

Afterwards Gevorkyan's result was extended by V. Kostin [10] to the series by generalized Haar system.

Consider the d-dimensional Franklin series

$$\sum_{\mathbf{n}\in\mathbb{N}_0^d} a_{\mathbf{n}} f_{\mathbf{n}}(\mathbf{x}),$$

where $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{N}_0^d$ is a vector with non-negative integer coordinates, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbf{x} = (x_1, \cdots, x_d) \in [0,1]^d$ and

$$f_{\mathbf{n}}(\mathbf{x}) = f_{n_1}(x_1) \cdot \ldots \cdot f_{n_d}(x_d).$$

The following theorem for multiple Franklin series was proved in [7].

Theorem A. *If the partial sums*

$$\sigma_{2^k}(\boldsymbol{x}) = \sum_{\boldsymbol{n}: n_i \le 2^k, i=1,\cdots,d} a_{\boldsymbol{n}} f_{\boldsymbol{n}}(\boldsymbol{x})$$

converge in measure to a function f and

$$\lim_{m\to\infty}\left(\lambda_m \cdot |\{\boldsymbol{x} \in [0,1]^d : \sup_k |\sigma_{2^k}(\boldsymbol{x})| > \lambda_m\}|\right) = 0$$

for some sequence $\lambda_m \to +\infty$, *then for any* $\boldsymbol{n} \in \mathbb{N}_0^d$

$$a_{\mathbf{n}} = \lim_{m\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_m} f_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}.$$

In this theorem instead of the partial sums $\sigma_{2^k}(\mathbf{x})$ one can take square partial sums $\sigma_{q_k}(\mathbf{x})$, where $\{q_k\}$ is any increasing sequence of natural numbers, for which the ratio q_{k+1}/q_k is bounded. The following theorem is proved in [11].

Theorem B. *Let* $\{q_k\}$ *be an increasing sequence of natural numbers such that the ratio* q_{k+1}/q_k *is bounded. If the sums* $\sigma_{q_k}(\boldsymbol{x})$ *converge in measure to a function* f *and there exists a sequence* $\lambda_m \to +\infty$ *so that*

$$\lim_{m\to\infty} \left(\lambda_m \cdot |\{\boldsymbol{x} \in [0,1]^d : \sup_k |\sigma_{q_k}(\boldsymbol{x})| > \lambda_m\}| \right) = 0,$$

then for any $\boldsymbol{n} \in \mathbb{N}_0^d$

$$a_{\mathbf{n}} = \lim_{m\to\infty} \int_{[0,1]^d} \left[f(\mathbf{x})\right]_{\lambda_m} f_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}.$$

2. Lemmas and the main result

Let functions $h_m(x) : [0,1] \to \mathbb{R}$, satisfy the following conditions:

$$0 \le h_1(x) \le h_2(x) \le \cdots \le h_m(x) \le \cdots, \lim_{m\to\infty} h_m(x) = \infty, \tag{2.1}$$

there exists dyadic points

$$0 = t_{m,0} < t_{m,1} < t_{m,2} < \cdots < t_{m,n_m} = 1,$$

so that the intervals

$$I_k^m = [t_{m,k-1}, t_{m,k}), \qquad k = 1, \cdots, n_m,$$

are dyadic as well, i.e. I_k^m is of the form

$$\mathcal{D} = \left\{ \left[\frac{i}{2^j}, \frac{i+1}{2^j}\right), 0 \le i \le 2^j - 1, j \ge 0 \right\}$$

and the function $h_m(x)$ is constant on those intervals,

$$h_m(x) = \lambda_k^m, \qquad x \in I_k^m, \quad k = 1, \cdots, n_m.$$

Moreover

$$\inf_{m,k} \int_{I_k^m} h_m(x) dx = \inf_{m,k} |I_k^m| \lambda_k^m > 0, \tag{2.2}$$

$$\sup_{m,k} \left(\frac{\lambda_k^m}{\lambda_{k-1}^m} + \frac{\lambda_{k-1}^m}{\lambda_k^m} \right) < +\infty \tag{2.3}$$

and

$$\sup_{m,k} \left(\frac{|I_k^m|}{|I_{k-1}^m|} + \frac{|I_{k-1}^m|}{|I_k^m|} \right) < +\infty. \tag{2.4}$$

In other words, for any function h_m the interval $[0,1]$ can be partitioned into dyadic intervals, so that the values of the function on neighbouring intervals are equivalent to each other and so are the lengths of neighbouring intervals. The following theorem is proved in [9].

Theorem C. *Let $h_m(x)$ be sequence of functions satisfying conditions* $(2.1)-(2.3)$. *If the partial sums $\sigma_{2^\nu} = \sum_{n=0}^{2^\nu} a_n f_n$ converge in measure to a function f and*

$$\lim_{m\to\infty} \int_{\{x\in[0,1];\quad \sup_\nu |\sigma_\nu(x)|>h_m(x)\}} h_m(x)dx = 0,$$

then for any $n \in \mathbb{N}_0$

$$a_n = \lim_{m\to\infty} \int_0^1 \left[f(x)\right]_{h_m(x)} f_n(x)dx,$$

where

$$\left[f(x)\right]_{\lambda(x)} = \begin{cases} f(x), & \text{if} \quad |f(x)| \le \lambda(x), \\ 0, & \text{if} \quad |f(x)| > \lambda(x). \end{cases}$$

Now we are in position to state the main result of this paper.

Theorem 2.1. Let $h_m(x)$ be sequence of functions satisfying conditions $(2.1)-(2.3)$, and $\{q_k\}$ be an increasing sequence of natural numbers such that the ratio q_{k+1}/q_k is bounded. If the partial sums $\sigma_{q_k}(x)$ converge in measure to a function f and

$$\lim_{m\to\infty} \int_{\{x\in[0,1];\quad \sup_k |\sigma_{q_k}(x)|>h_m(x)\}} h_m(x)dx = 0, \tag{2.5}$$

then for any $n \in \mathbb{N}_0$

$$a_n = \lim_{m\to\infty} \int_0^1 \left[f(x)\right]_{h_m(x)} f_n(x)dx. \tag{2.6}$$

To prove Theorem 2.1 we will need the following two lemmas.

Lemma 2.1. *Let $0 = t_0 < t_1 < \cdots < t_n = 1$ and $h(x) = \lambda_k$, if $x \in I_k := [t_{k-1}, t_k)$ and $I_k \in \mathcal{D}$, when $k = 1, \cdots, n$. Moreover $\gamma > 0$*

$$\frac{1}{\gamma} \le \frac{\lambda_k}{\lambda_{k+1}} \le \gamma, \text{ when } k = 1, \cdots, n-1, \tag{2.7}$$

then there exists points $0 = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_s = 1$ such that $h(x) = \tilde{\lambda}_l$, $x \in \tilde{I}_l = [\tilde{t}_{l-1}, \tilde{t}_l) \in \mathcal{D}$, $l = 1, \cdots s$. Besides that

$$\frac{1}{2\gamma} \le \frac{|\tilde{I}_l|}{|\tilde{I}_{l+1}|} \le 2\gamma, \tag{2.8}$$

$$\frac{1}{\gamma} \le \frac{\tilde{\lambda}_l}{\tilde{\lambda}_{l+1}} \le \gamma, \text{ when } l = 1, \cdots, s-1, \tag{2.9}$$

$$\min_l \int_{\tilde{I}_l} h_m(x)dx = \min_k \int_{I_k} h_m(x)dx > 0. \tag{2.10}$$

The proof of the Lemma 2.1 can be found in [9], but we present it here for the sake of completeness.

Proof. Denote

$$c = \min_k \int_{I_k} h_m(x)dx = \min_k \lambda_k |I_k|,$$

and let $1 \leq k_0 \leq n$ such that $\lambda_{k_0}|I_{k_0}| = c$. From definition c follows that for any i, $-k_0 + 1 \leq i \leq n - k_0$ there exists $n_i \geq 0$ such that

$$2^{n_i} c \leq \lambda_{k_0+i}|I_{k_0+i}| < 2^{n_i+1} c. \tag{2.11}$$

Suppose that $n_0 = 0$ and denote

$$t_{i,j} = t_{k_0+i-1} + \frac{|I_{k_0+i}|}{2^{n_i}} j, \text{ when } j = 0, \cdots, 2^{n_i},$$

$$I_{i,j} = [t_{i,j-1}, t_{i,j}), \text{ and } \lambda_{i,j} = \lambda_i, \text{ when } j = 1, \cdots, 2^{n_i}.$$

Therefore

$$\int_{I_{0,1}} h_m(x)dx = c \leq \int_{I_{i,j}} h_m(x)dx = \lambda_{i,j}|I_{i,j}| < 2c.$$

From the definition $c, I_{i,j}$, (2.11) and (2.7) follows that

$$|I_{i,j}| = |I_{i,1}| < \frac{2c}{\lambda_{k_0+i}} \leq \frac{2c\gamma}{\lambda_{k_0+i-1}} \leq 2\gamma |I_{i-1,2^{n_{i-1}}}|,$$

similarly we obtain

$$|I_{i,j}| = |I_{i,1}| \geq \frac{c}{\lambda_{k_0+i}} \geq \frac{c}{\gamma\lambda_{k_0+i-1}} \geq \frac{1}{2\gamma}|I_{i-1,2^{n_{i-1}}}|.$$

From the last two inequalities follows that the ratio of the lengths of intervals $I_{i,j}$ with common endpoint is not greater than 2γ. By renumbering the intervals $\{I_{i,j}; -k_0 + 1 \leq i \leq n - k_0, 1 \leq j \leq 2^{n_i}\}$ in increasing order with respect to the left endpoint, we obtain the intervals $\tilde{I}_l, l = 1, \cdots, \sum_{i=-k_0+1}^{n-k_0} 2^{n_i}$, which satisfy the condition (2.8). From the definition $\tilde{I}_l$ it follows that the function $h_m(x)$ is constant,

$$h_m(x) = \tilde{\lambda}_l, \ \ x \in \tilde{I}_l$$

and from (2.7) we get (2.9), so $\frac{\tilde{\lambda}_l}{\tilde{\lambda}_{l+1}} = 1$ or there exists k, such that

$$\frac{\tilde{\lambda}_l}{\tilde{\lambda}_{l+1}} = \frac{\lambda_k}{\lambda_{k+1}}.$$

□

Lemma 2.2. *Let $h_m(x)$ be sequence of functions satisfying conditions* (2.1) − (2.3), *then there exists dyadic points $0 = \tilde{t}_{m,0} < \tilde{t}_{m,1} < \cdots < \tilde{t}_{m,\tilde{n}_m} = 1$ so that the* *intervals* $\tilde{I}_k^m = [\tilde{t}_{m,k-1}, \tilde{t}_{m,k}) \in \mathcal{D}$, $k = 1, \cdots, \tilde{n}_m$ *are dyadic as well and the function* $h_m(x)$ *is constant on those intervals,*

$$h_m(x) = \tilde{\lambda}_k^m, \text{ if } x \in \tilde{I}_k^m, \ k = 1, \cdots, \tilde{n}_m$$

and the conditions $(2.2) - (2.4)$ *are satisfied.*

3. The proof of the main theorem

Let $\{s_{n,i}\}_{i=0}^n$ be the points given in (1.1), $s_{n,-1} = 0$ and $s_{n,n+1} = 1$. Let us define the function $N_i^n(x)$ as follows. It is linear on intervals $[s_{n,j-1}, s_{n,j}], j = 1, 2, \cdots, n$, and

$$N_i^n(s_{n,j}) = \begin{cases} 1, & \text{if } i = j, \\ \\ 0, & \text{if } i \neq j, \end{cases} \qquad j = 0, 1, \cdots, n.$$

Let $\{q_k\}$ be an increasing sequence of natural numbers and M be a number satisfying the inequality

$$\frac{q_{k+1}}{q_k} \leq M, \text{ for all } k \in \mathbb{N}.$$

For any $j \in \{0, 1, \cdots, q_\nu\}$ denote

$$\Delta_j^\nu := [s_{q_\nu, j-1}, s_{q_\nu, j+1}],$$

$$M_j^{q_\nu}(x) := \frac{N_j^{q_\nu}(x)}{\|N_j^{q_\nu}(x)\|_1} = \frac{2}{|\Delta_j^\nu|} N_j^{q_\nu}(x).$$

Obviously

$$\frac{1}{2q_\nu} \leq |\Delta_j^\nu| \leq \frac{4}{q_\nu}, \tag{3.1}$$

$$\operatorname{supp} M_j^{q_\nu} = \Delta_j^\nu \qquad \text{and} \qquad \int_0^1 M_j^{q_\nu}(x)dx = 1.$$

Recall that

$$\sigma_{q_\nu}(x) = \sum_{n=0}^{q_\nu} a_n f_n(x).$$

Let's denote

$$\sigma^*(x) = \sup_\nu |\sigma_{q_\nu}(x)|,$$

and prove that for any j_0, ν_0 the following statement is true:

$$\int_0^1 \sigma_{q_{\nu_0}}(x) M_{j_0}^{q_{\nu_0}}(x)dx = \lim_{m\to\infty} \int_0^1 \left[f(x)\right]_{h_m(x)} M_{j_0}^{q_{\nu_0}}(x)dx.$$

For any $m \in \mathbb{N}$ denote

$$E_m := \{x \in \operatorname{supp}(M_{j_0}^{q_{\nu_0}}) = \Delta_{j_0}^{\nu_0} : \ \sigma^*(x) \geq h_m(x)\}.$$

From $(2.3), (2.4)$ it follows that there exists $\gamma > 0$ such that

$$\frac{\lambda_k^m}{\gamma} \leq \lambda_{k+1}^m \leq \gamma \lambda_k^m \quad \text{and} \quad \frac{|I_k^m|}{\gamma} \leq |I_{k+1}^m| \leq \gamma |I_k^m|. \tag{3.2}$$

Denote

$$\varepsilon_0 = \inf_{m,k} \int_{I_k^m} h_m(x)dx = \inf_{m,k} \lambda_k^m |I_k^m| > 0.$$

Let ε be an arbitrary positive number. Under the conditions of the theorem a number m_0 can be chosen to satisfy

$$2^7 M \int_{E_m} h_m(x)dx < \varepsilon,\ m \geq m_0.$$

Take

$$\varepsilon \leq \frac{2^3 \varepsilon_0}{\gamma}.$$

Let M_1 be a number such that

$$h_m(x) \geq M_1, \text{ for all } x \in [0,1], \text{ when } m \geq m_1,$$

then

$$M_1 |E_m| \leq \int_{E_m} h_m(x)dx < \frac{\varepsilon}{2^7 M}, \text{when } m \geq \max(m_0, m_1) =: m_2.$$

Therefore

$$|E_m| \leq \frac{\varepsilon}{2^7 M M_1},$$

let's take

$$M_1 = \frac{q_{\nu_0}\varepsilon}{2^2}.$$

Hence from (3.1) we obtain

$$|E_m| < \frac{2^2\varepsilon}{2^7 M q_{\nu_0}\varepsilon} = \frac{1}{2^5 M q_{\nu_0}} \leq \frac{|\Delta_{j_0}^{\nu_0}|}{2^4 M}. \tag{3.3}$$

Let's fix a number $m \geq m_2$ and prove that

$$|E_m \cap I_k^m| < \frac{|I_k^m|}{8M}, \quad k = 1, \cdots, n_m. \tag{3.4}$$

Suppose that there exists k_0, such that

$$|E_m \cap I_{k_0}^m| \geq \frac{|I_{k_0}^m|}{8M},$$

therefore

$$\varepsilon_0 \leq \lambda_{k_0}^m |I_{k_0}^m| \leq 8M \lambda_{k_0}^m |E_m \cap I_{k_0}^m| \leq 8M \int_{E_m} h_m(x)dx \leq \frac{\varepsilon}{2^4} \leq \frac{\varepsilon_0}{2\gamma} < \varepsilon_0,$$

which is a contradiction.

Note that for any $J \in \mathcal{D}$, which can represented in the form $\bigcup_{k=l}^{j} I_k^m$, from (3.4) we get

$$|J \cap E_m| = \sum_{k=l}^{j} |I_k^m \cap E_m| \leq \frac{1}{8M} \sum_{k=l}^{j} |I_k^m| = \frac{1}{8M}|J|,$$

therefore

$$|J \cap E_m| \leq \frac{|J|}{8M}, \quad \text{for any} \quad J = \bigcup_{k=l}^{j} I_k^m \in \mathcal{D}. \tag{3.5}$$

It is clear that if $J \in \mathcal{D}$ and $J \supset I_{k_0}^m$, then $J = \bigcup_{k=l}^{j} I_k^m$, when $l \le k_0 \le j$. Suppose $\nu \ge \nu_0$. We set

$$\Omega_\nu := \{A : A = [s_{q_\nu, j-1}, s_{q_\nu, j}] \quad \text{and} \quad A \subset \Delta_{j_0}^{\nu_0}\}.$$

Obviously

$$\frac{1}{2q_\nu} \le |A| \le \frac{2}{q_\nu}, \quad \text{for all} \quad A \in \Omega_\nu.$$

If $\nu = \nu_0$, then we set

$$\Omega_{\nu_0}^1 = \left\{A \in \Omega_{\nu_0} : |A \cap E_m| > \frac{1}{8M}|A|\right\}, \quad Q_{\nu_0} = \bigcup_{A \in \Omega_{\nu_0}^1} A,$$

and

$$\Omega_{\nu_0}^2 = \{A \in \Omega_{\nu_0} : A \not\subset Q_{\nu_0}\}, \quad P_{\nu_0} = \bigcup_{A \in \Omega_{\nu_0}^2} A.$$

From (3.3) we have, that

$$Q_{\nu_0} = \emptyset, \quad \text{and} \quad P_{\nu_0} = \operatorname{supp}(M_{j_0}^{q_{\nu_0}}).$$

Now suppose we have defined the sets $\Omega_{\nu'}^1$, $\Omega_{\nu'}^2$, $Q_{\nu'}$ for all $\nu' < \nu$. Let's denote

$$\Omega_\nu^1 = \left\{A \in \Omega_\nu : |A \cap E_m| > \frac{1}{8M}|A| \quad \text{and} \quad A \not\subset \bigcup_{\nu' < \nu} Q_{\nu'}\right\}, \tag{3.6}$$

$$Q_\nu = \bigcup_{A \in \Omega_\nu^1} A, \quad \Omega_\nu^2 = \left\{A \in \Omega_\nu : A \not\subset \bigcup_{\nu' \le \nu} Q_{\nu'}\right\}, \quad P_\nu = \bigcup_{A \in \Omega_\nu^2} A.$$

Thus we have defined the families $\Omega_\nu^1, \Omega_\nu^2$ and the sets Q_ν, P_ν, satisfying to the following conditions,

$$\Omega_\nu^1 \subset \Omega_\nu, \quad \Omega_\nu^2 \subset \Omega_\nu,$$

$$\operatorname{supp}(M_{j_0}^{q_{\nu_0}}) = P_\nu \cup \left(\bigcup_{\nu' \le \nu} Q_{\nu'}\right), \quad P_\nu \cap \left(\bigcup_{\nu' \le \nu} Q_{\nu'}\right) = \emptyset, \tag{3.7}$$

$$Q_{\nu'} \cap Q_{\nu''} = \emptyset, \quad \text{if} \quad \nu' \ne \nu''. \tag{3.8}$$

From (3.6) and (3.8) we obtain

$$\left|\bigcup_{\nu' \le \nu} Q_{\nu'}\right| < 8M|E_m|, \quad \text{for any} \quad \nu.$$

Now let us prove that for any $A \in \Omega_\nu^1$, $\nu \ge \nu_0$, there exists k such that $A \subset I_k^m$. Otherwise, there exists k_0 such that $A \supset I_{k_0}^m$. Since $A = \bigcup_{k=l}^{j} I_k^m$, $l \le k_0 \le j$, therefore from (3.5) we get $|A \cap E_m| \le |A|/8M$, but $A \in \Omega_\nu^1$. For any $\nu > \nu_0$ denote

$$J_\nu = \{j : \Delta_j^\nu \cap Q_\nu \ne \emptyset, \quad \Delta_j^\nu \subset P_{\nu-1}\}.$$

Now let us prove that

$$|\sigma_{q_\nu}(x)| \le 3h_m(x), \quad \text{if} \quad x \in \Delta_j^\nu, j \in J_\nu. \tag{3.9}$$

Suppose $A \in \Omega_\nu^1$, with $A \subset \Delta_j^\nu$, therefore $A \subset I_l^m$ for some l. Let's prove that

$$\Delta_j^\nu \subset I_k^m \cup I_{k+1}^m, \quad \text{when} \quad k = l-1 \quad \text{or} \quad k = l. \tag{3.10}$$

Without loss of generality suppose that $\Delta_j^\nu \supset I_{l+1}^m$. From (3.2) we get

$$2|A| = |\Delta_j^\nu| > |I_{l+1}^m| \ge \frac{|I_l^m|}{\gamma},$$

therefore

$$\varepsilon_0 \le |I_l^m|\lambda_l^m < 2\gamma|A|\lambda_l^m \le 16\gamma M|A \cap E_m|\lambda_l^m \le 16\gamma M \int_{E_m} h_m(x)dx$$

$$< \frac{16\gamma M\varepsilon}{2^7 M} \le \frac{16 \cdot 2^3\gamma\varepsilon_0}{2^7\gamma} = \varepsilon_0,$$

which is a contradiction.

Let Δ_1 and Δ_2 be respectively the left and right halves of the interval Δ_j^ν, $\Delta_1 \subset \Delta_j^\nu$, $\Delta_2 \subset \Delta_j^\nu$. From (3.10) we get, that there exists l_1, l_2 such that $\Delta_1 \subset I_{l_1}^m$, $\Delta_2 \subset I_{l_2}^m$, it is clear that $|l_1 - l_2| \le 1$. Therefore

$$h_m(x) = \lambda_{l_j}^m, \qquad x \in \Delta_j, j = 1, 2. \tag{3.11}$$

Since $\Delta_1, \Delta_2 \subset \Delta_j^\nu \subset P_{\nu-1}, (j \in J_\nu)$, then there exists $\tilde{\Delta}_1, \tilde{\Delta}_2 \in \Omega_{\nu-1}$, so that $\Delta_i \subset \tilde{\Delta}_i \subset P_{\nu-1},\ i = 1, 2$, we get that

$$|\Delta_i \cap E_m| \le |\tilde{\Delta}_i \cap E_m| \le \frac{1}{8M}|\tilde{\Delta}_i| \le \frac{1}{8M} \cdot \frac{2}{q_{\nu-1}} \le \frac{1}{4q_\nu} \le \frac{|\Delta_i|}{2}. \tag{3.12}$$

Suppose that $x \in \Delta_1$, (the case $x \in \Delta_2$ is considered similarly). Since $\sigma_{q_\nu}(x)$ is a linear function on $\Delta_1 = [\alpha, \beta]$, we have set

$$I := \{t \in \Delta_1 : |\sigma_{q_\nu}(t)| < \lambda_{l_1}^m\}$$

is an interval. From (3.11) and (3.12) we get

$$|I| = |\{t \in \Delta_1 : |\sigma_{q_\nu}(t)| < h_m(t)\}| \ge |\Delta_1 \cap E_m^c| \ge \frac{1}{2}|\Delta_1|. \tag{3.13}$$

Since $\sigma_{q_\nu}(t)$ is linear, then

$$|\sigma'_{q_\nu}(t)| < \frac{2\lambda_{l_1}^m}{\frac{1}{2}(\beta - \alpha)} = \frac{4\lambda_{l_1}^m}{\beta - \alpha}. \tag{3.14}$$

From (3.14) we get

$$|\sigma_{q_\nu}(\alpha)| < \lambda_{l_1}^m + \frac{4\lambda_{l_1}^m}{\beta - \alpha} \cdot \frac{\beta - \alpha}{2} = 3\lambda_{l_1}^m,$$

similarly we obtain

$$|\sigma_{q_\nu}(\beta)| < 3\lambda_{l_1}^m.$$

Using the last inequalities and (3.10), we get

$$|\sigma_{q_\nu}(t)| < 3h_m(t), \quad t \in [\alpha, \beta] = \Delta_1.$$

Similarly we obtain (according to definition of P_ν), that if $\Delta_j^\nu \subset P_\nu$, then

$$|\sigma_{q_\nu}(x)| \le 3h_m(x), \quad \text{if} \quad x \in \Delta_j^\nu \subset P_\nu.$$

Now let's define by induction expansions ψ_n for $M_{j_0}^{q_{\nu_0}}$,

$$M_{j_0}^{q_{\nu_0}} = \psi_n = \sum_{\nu \le n} \sum_{j \in J_\nu} \alpha_{\nu,j}^n M_j^{q_\nu} + \sum_{j:\Delta_j^n \subset P_n} \alpha_j^n M_j^{q_n}, \tag{3.15}$$

where

$$\sum_{\nu \le n} \sum_{j \in J_\nu} \alpha_{\nu,j}^n + \sum_{j:\Delta_j^n \subset P_n} \alpha_j^n = 1, \quad \alpha_{\nu,j}^n \ge 0,\ \alpha_j^n \ge 0. \tag{3.16}$$

Since $P_{\nu_0} = \operatorname{supp}(M_{j_0}^{q_{\nu_0}})$, then $\psi_{\nu_0} = M_{j_0}^{q_{\nu_0}}$. Suppose we have defined expansions $\psi_{\nu_0}, \cdots, \psi_n$, satisfying (3.15) and (3.16). Clearly for any $\Delta_j^n \subset P_n$ we have

$$M_j^{q_n}(x) = \sum_{\nu:\Delta_\nu^{n+1} \subset \operatorname{supp} M_j^{q_n}} \beta_\nu M_\nu^{q_{n+1}}(x), \quad \beta_\nu \ge 0. \tag{3.17}$$

Note that if $\Delta_j^n \subset P_n$ and $\Delta_\nu^{n+1} \subset \operatorname{supp} M_j^{q_n} = \Delta_j^n$, then either $\Delta_\nu^{n+1} \cap Q_{n+1} \ne \emptyset$ and, therefore $\nu \in J_{n+1}$, or $\Delta_\nu^{n+1} \subset P_{n+1}$. Therefore, inserting the expressions (3.17) in (3.15) and grouping similar terms, we obtain

$$M_{j_0}^{q_{\nu_0}} = \psi_{n+1} = \sum_{\nu \le n+1} \sum_{j \in J_\nu} \alpha_{\nu,j}^{n+1} M_j^{q_\nu} + \sum_{j:\Delta_j^{n+1} \subset P_{n+1}} \alpha_j^{n+1} M_j^{q_{n+1}}.$$

Since the integrals of all functions $M_j^{q_\nu}$ are 1, we get that

$$\sum_{\nu \le n+1} \sum_{j \in J_\nu} \alpha_{\nu,j}^{n+1} + \sum_{j:\Delta_j^{n+1} \subset P_{n+1}} \alpha_j^{n+1} = 1,$$

therefore for any n

$$(\sigma_{q_n}, M_{j_0}^{q_{\nu_0}}) = \sum_{\nu \le n} \sum_{j \in J_\nu} \alpha_{\nu,j}^n (\sigma_{q_n}, M_j^{q_\nu}) + \sum_{j:\Delta_j^n \subset P_n} \alpha_j^n (\sigma_{q_n}, M_j^{q_n}).$$

Note that

$$(f_p, M_j^{q_\nu}) = \int_0^1 f_p(x) M_j^{q_\nu}(x) dx = 0, \quad \text{if} \quad \nu \ge \nu_0 \quad \text{and} \quad p > q_\nu.$$

Therefore

$$(\sigma_{q_n}, M_j^{q_\nu}) = \sum_{p=0}^{q_n} a_p (f_p, M_j^{q_\nu}) = \sum_{p=0}^{q_\nu} a_p (f_p, M_j^{q_\nu}) = (\sigma_{q_\nu}, M_j^{q_\nu}). \tag{3.18}$$

Hence we have

$$\int_0^1 \sigma_{q_{\nu_0}}(t) M_{j_0}^{q_{\nu_0}}(t) dt - \int_0^1 \left[f(t)\right]_{h_m(t)} M_{j_0}^{q_{\nu_0}}(t) dt = (\sigma_{q_n} - \left[f\right]_{h_m}, M_{j_0}^{q_{\nu_0}}) \tag{3.19}$$

$$= \sum_{\nu\le n}\sum_{j\in J_\nu}\alpha_{\nu,j}^n(\sigma_{q_\nu}-\left[f\right]_{h_m}, M_j^{q_\nu}) + \sum_{j:\Delta_j^n\subset P_n}\alpha_j^n(\sigma_{q_n}-\left[f\right]_{h_m}, M_j^{q_n}) =: I_n^1 + I_n^2.$$

Using (3.9) and (3.18), for I_n^1 we will have the estimate

$$|I_n^1| = \left|\sum_{\nu\le n}\sum_{j\in J_\nu}\alpha_{\nu,j}^n(\sigma_{q_\nu}-\left[f\right]_{h_m}, M_j^{q_\nu})\right| \le \sum_{\nu\le n}\sum_{j\in J_\nu}\alpha_{\nu,j}^n(|\sigma_{q_\nu}|+h_m, M_j^{q_\nu})$$

$$\le 4\sum_{\nu\le n}\sum_{j\in J_\nu}\alpha_{\nu,j}^n(h_m, M_j^{q_\nu}) = 4(h_m, \sum_{\nu\le n}\sum_{j\in J_\nu}\alpha_{\nu,j}^n M_j^{q_\nu}).$$

By

$$\sum_{\nu\le n}\sum_{j\in J_\nu}\alpha_{\nu,j}^n M_j^{q_\nu} \le M_{j_0}^{q_{\nu_0}},$$

we have

$$|I_n^1| \le 4\int\limits_{\bigcup_{\nu\le n}\bigcup_{j\in J_\nu}\Delta_j^\nu} h_m(t)M_{j_0}^{q_{\nu_0}}(t)dt.$$

Denote

$$J_\nu^1 := \{j\in J_\nu : \exists k \ \ s.t.\ \ \Delta_j^\nu\subset I_k^m\},\quad J_\nu^2 := J_\nu\setminus J_\nu^1,$$

$$A_n := \bigcup_{\nu\le n}\bigcup_{j\in J_\nu^1}\Delta_j^\nu,\ \ B_n := \bigcup_{\nu\le n}\bigcup_{j\in J_\nu^2}\Delta_j^\nu.$$

It is easy to notice that

$$|I_n^1| \le \left(\int_{A_n} h_m(t)M_{j_0}^{q_{\nu_0}}(t)dt + \int_{B_n} h_m(t)M_{j_0}^{q_{\nu_0}}(t)dt\right) \le C\left(\int_{A_n} h_m(t)dt + \int_{B_n} h_m(t)dt\right)$$

$$=: C(I_n^3 + I_n^4).$$

From (3.10) we get that for any $j\in J_\nu^2$ there exists k, such that

$$\Delta_j^\nu \subset I_k^m\cup I_{k+1}^m,$$

and the definitions Ω_ν^1 and Q_ν, we obtain that for any k there exists $(\nu(k), j(k))$ pair, such that

$$j(k)\in J_\nu^2 \quad\text{and}\quad \Delta_{j(k)}^{\nu(k)}\subset I_k^m\cup I_{k+1}^m.$$

Applying (3.2) we get

$$I_n^4 \le \sum_{k=1}^{n_m}(\lambda_k^m+\lambda_{k+1}^m)|\Delta_{j(k)}^{\nu(k)}| \le (\gamma+1)\sum_{k=1}^{n_m}\lambda_k^m|\Delta_{j(k)}^{\nu(k)}|,$$

and from (3.6) we get

$$|\Delta_{j(k)}^{\nu(k)}| \le 2|Q_{\nu(k)}\cap(I_k^m\cup I_{k+1}^m)| \le 2|\bigcup_{\nu\le n}Q_\nu\cap(I_k^m\cup I_{k+1}^m)|.$$

Therefore

$$I_n^4 \le 2(\gamma+1)\left(\sum_{k=1}^{n_m}\lambda_k^m\left|\bigcup_{\nu\le n}Q_\nu\cap I_k^m\right| + \gamma\sum_{k=1}^{n_m}\lambda_{k+1}^m\left|\bigcup_{\nu\le n}Q_\nu\cap I_{k+1}^m\right|\right)$$

$$= 2(\gamma+1)^2 \sum_{k=1}^{n_m} \lambda_k^m \left| \bigcup_{\nu \le n} Q_\nu \cap I_k^m \right| =: 2(\gamma+1)^2 I_n^5.$$

Using (3.6) we can estimate I_n^5 as follows,

$$I_n^5 \le 8M \sum_{k=1}^{n_m} \lambda_k^m \left| E_m \cap \left(\bigcup_{\nu \le n} Q_\nu \right) \cap I_k^m \right| \le 8M \sum_{k=1}^{n_m} \lambda_k^m \left| E_m \cap I_k^m \right| = 8M \int_{E_m} h_m(t)dt$$

$$< \frac{8M\varepsilon}{2^7 M} = \frac{\varepsilon}{2^4}.$$

If $j \in J_\nu^1$,then there exists k, such that $\Delta_j^\nu \subset I_k^m$, therefore

$$|\Delta_j^\nu| \le 2|\Delta_j^\nu \cap Q_\nu|,$$

from the last inequality we get,

$$|A_n \cap I_k^m| \le 2| \bigcup_{\nu \le n} Q_\nu \cap I_k^m|.$$

Therefore

$$I_n^3 = \int_{A_n} h_m(t)dt = \sum_{k=1}^{n_m} \lambda_k^m |A_n \cap I_k^m| \le 2 \sum_{k=1}^{n_m} \lambda_k^m | \bigcup_{\nu \le n} Q_\nu \cap I_k^m| = 2 I_n^5.$$

So

$$|I_n^1| \le C \left(\frac{(\gamma+1)^2 \varepsilon}{2^3} + \frac{2\varepsilon}{2^4} \right) = \frac{C\varepsilon(1+(\gamma+1)^2)}{2^3} = \varepsilon C_\gamma. \tag{3.20}$$

Now let us estimate I_n^2. Since

$$\sum_{j:\Delta_j^n \subset P_n} \alpha_j^n M_j^{q_n} \le M_{j_0}^{q_{\nu_0}}, \quad \text{then}$$

$$|I_n^2| \le (|\sigma_{q_n} - \big[f\big]_{h_m}|, \sum_{j:\Delta_j^n \subset P_n} \alpha_j^n M_j^{q_n}) \le \int\limits_{\bigcup_{j:\Delta_j^n \subset P_n} \Delta_j^n} |\sigma_{q_n}(t) - \big[f(t)\big]_{h_m(t)}| M_{j_0}^{q_{\nu_0}}(t)dt$$

$$\le C \int\limits_{\bigcup_{j:\Delta_j^n \subset P_n} \Delta_j^n} |\sigma_{q_n}(t) - \big[f(t)\big]_{h_m(t)}| dt.$$

Denote

$$C_n = \bigcup_{j:\Delta_j^n \subset P_n} \Delta_j^n \cap E_m, \quad D_n = \bigcup_{j:\Delta_j^n \subset P_n} \Delta_j^n \cap E_m^c \cap \{t, |\sigma_{q_n}(t) - f(t)| \le \varepsilon\},$$

$$F_n = \bigcup_{j:\Delta_j^n \subset P_n} \Delta_j^n \cap E_m^c \cap \{t, |\sigma_{q_n}(t) - f(t)| > \varepsilon\}.$$

It is clear see that

$$C_n \cup D_n \cup F_n = \bigcup_{j:\Delta_j^n \subset P_n} \Delta_j^n \ \text{ and } \ |f(t)| \le h_m(t) \ \text{ a. e., } \ \text{when } \ t \in D_n \cup F_n \subset E_m^c.$$

Therefore

$$|I_n^2| \le C\left(\int_{C_n} |\sigma_{q_n}(t) - \big[f(t)\big]_{h_m(t)}|dt + \int_{D_n} |\sigma_{q_n}(t) - f(t)|dt + \int_{F_n} |\sigma_{q_n}(t) - f(t)|dt\right)$$

$$=: C(I_n^6 + I_n^7 + I_n^8).$$

If $t \in C_n$, then

$$|\sigma_{q_n}(t) - \big[f(t)\big]_{h_m(t)}| \le |\sigma_{q_n}(t)| + |\big[f(t)\big]_{h_m(t)}| \le 4h_m(t),$$

and

$$I_n^6 \le 4\int_{C_n} h_m(t)dt \le 4\int_{E_m} h_m(t)dt \le \frac{2^2\varepsilon}{2^7 M} = \frac{\varepsilon}{2^5 M} < \varepsilon.$$

From definition of D_n it follows that if $t \in D_n$, then

$$|\sigma_{q_n}(t) - f(t)| \le \varepsilon, \text{ therefore } I_n^7 \le \int_{D_n} \varepsilon \le \varepsilon.$$

Since $\sigma_{q_n}(x)$ converge in measure to the function f, then there exists n such that

$$|\{t, |\sigma_{q_n}(t) - f(t)| > \varepsilon\}| < \frac{\varepsilon}{\max\{h_m(t),\ t \in [0,1]\}},$$

and

$$|\sigma_{q_n}(t) - f(t)| \le |\sigma_{q_n}(t)| + |f(t)| \le 4h_m(t), \text{ for a. e., } t \in F_n \subset E_m^c.$$

Therefore

$$I_n^8 \le 4\int_{F_n} h_m(t)dt \le 4\max\{h_m(t),\ t \in [0,1]\} \cdot |\{t, |\sigma_{q_n}(t) - f(t)| > \varepsilon\}| < 4\varepsilon.$$

So $|I_n^2| \le 6\varepsilon$, therefore by (3.19) and (3.20), we get

$$\left|(\sigma_{q_{\nu_0}}, M_{j_0}^{q_{\nu_0}}) - \int_0^1 \big[f(t)\big]_{h_m(t)} M_{j_0}^{q_{\nu_0}}(t)dt\right| \le C_\gamma \varepsilon.$$

Now let's prove that for any $n \in \mathbb{N}_0$ the coefficient a_n can be reconstructed by (2.6). Take arbitrary n and choose ν so that $q_\nu \ge n$, then $f_n \in S_{q_\nu}$. Taking into account that the system of functions $\{M_j^{q_\nu}\}_{j\in\{0,1,\cdots,q_\nu\}}$ is a basis in the space S_{q_ν}, one can find number $\beta_j, j \in \{0,1,\cdots,q_\nu\}$, such that

$$f_n(x) = \sum_{j\in\{0,1,\cdots,q_\nu\}} \beta_j M_j^{q_\nu}(x).$$

Therefore

$$a_n = (\sigma_{q_\nu}, f_n) = \sum_{j=0}^{q_\nu} \beta_j (\sigma_{q_\nu}, M_j^{q_\nu}) = \sum_{j=0}^{q_\nu} \beta_j \lim_{m\to\infty} \int_0^1 \big[f(x)\big]_{h_m(x)} M_j^{q_\nu}(x)dx$$

$$= \lim_{m\to\infty} \int_0^1 \big[f(x)\big]_{h_m(x)} f_n(x)dx.$$

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