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ON THE DERIVATIVES OF THE HEUN FUNCTIONS

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Abstract. The Heun functions satisfy linear ordinary differential equations of second order with certain singularities in the complex plane. The first order derivatives of the Heun functions satisfy linear second order differential equations with one more singularity. In this paper we compare these equations with linear differential equations isomonodromy deformations of which are described by the Painlevé equations $P_{II} - P_{VI}$.

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1. Introduction

The general Heun equation is the most general second order linear Fuchsian ordinary differential equation with four regular singular points in the complex plane [2, 3, 4, 5]. Although it is a genaralization of the well-studied Gauss hypergeometric equation with three regular singularities, it is much more difficult to investigate properties of the Heun functions. The additional singularity causes many complications in comparison with the hypergeometric case (for instance, the solutions in general have no integral representations involving simpler mathematical functions). There also exist confluent Heun equations (see [3, 4]) which have irregular singularities. There are many studies on the properties of solutions of the Heun equations from different perspectives (see, for instance, [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the references therein). The Heun functions (and their confluent cases) appear extensively in many problems of mathematics, mathematical physics, physics and engineering (e.g., [18, 19, 20]). An extensive bibliography can be found at [1].

The general Heun equation is given by the following equation:

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-t}\right)\frac{du}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}u = 0, \tag{1.1}$$

where the parameters satisfy the Fuchsian relation

$$1 + \alpha + \beta = \gamma + \delta + \varepsilon. \tag{1.2}$$

This equation has four regular singular points at $z = 0$, 1, t and ∞. Its solutions, the Heun functions, are usually denoted by $u = H(t,\, q;\, \alpha,\, \beta,\, \gamma,\, \delta;\, z)$ assuming that ε is obtained from (1.2). The parameter q is referred to as the accessory parameter.

It is well-known that the derivative of the hypergeometric function ${}_2F_1$ is again a hypergeometric function with different values of the parameters. However, for the Heun function it is generally not the case. The first order derivative of the general Heun function satisfies a second order Fuchsian differential equation with five regular singular points [7, 8, 12]. It can be verified by direct computations that the function $v(z) = du/dz$, where $u = u(z)$ is a solution of (1.1), satisfies the following equation:

(1.3)
$$\frac{d^2v}{dz^2} + \left(\frac{\gamma+1}{z} + \frac{\delta+1}{z-1} + \frac{\varepsilon+1}{z-t} - \frac{\alpha\beta}{\alpha\beta z - q}\right)\frac{dv}{dz} + \frac{f(z)}{z(z-1)(z-t)(\alpha\beta z-q)}v = 0,$$

where $f(z) = z(\alpha\beta z - 2q)(\alpha\beta + \gamma + \delta + \varepsilon) + (q^2 + q(\gamma + t(\gamma+\delta) + \varepsilon) - \alpha\beta\gamma t)$. We see that an additional singularity at $z = q/(\alpha\beta)$ involving the accessory parameter is added.

It is known that in some cases equation (1.3) reduces to a Heun equation (1.1) with altered parameters [8]. Indeed, we can observe that in four cases when $q = 0$, $q = \alpha\beta$, $q = \alpha\beta t$ and $\alpha\beta = 0$ the additional singularity in (1.3) disappears and we obtain the Heun equation (1.1) with different parameters [8]. The equation for the derivatives of the Heun functions allows one to construct several new expansions of solutions of the Heun equations in terms of various special functions (e.g., hypergeometric functions) [7]. Similar results hold for confluent cases [12].

This paper is organized as follows. In Section 2 we give a list of all confluent Heun equations together with linear second order equations for the derivatives of the Heun functions. In Section 3 we briefly describe the theory of isomonodromy deformations of linear equations and show how the famous Painlevé equations appear in this context. Next, in Section 4 we present our main results. In particular, we will compare linear equations for the Heun derivatives with linear differential equations, isomonodromy deformations of which are described by the Painlevé equations.

2. Confluent Heun equations and equations for derivatives of confluent Heun functions

The general Heun equation is given by (1.1) together with (1.2) and the linear equation for the derivative of the Heun functions is (1.3).

The confluent Heun equation is written as

$$\frac{d^2u}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\varepsilon\right)\frac{du}{dz}+\frac{\alpha z-q}{z(z-1)}u=0 \tag{2.1}$$

and the linear equation for the function $v=du/dz$ is given by

$$\frac{d^2v}{dz^2}+\left(\frac{\gamma+1}{z}+\frac{\delta+1}{z-1}+\varepsilon-\frac{\alpha}{\alpha z-q}\right)\frac{dv}{dz}+\frac{g(z)}{z(z-1)(\alpha z-q)}v=0, \tag{2.2}$$

where $g(z)=(\alpha+\varepsilon)(\alpha z^2-2qz)+(q^2-(\gamma+\delta-\varepsilon)q+\alpha\gamma)$.

The double-confluent Heun equation is

$$\frac{d^2u}{dz^2}+\left(\frac{\gamma}{z^2}+\frac{\delta}{z}+\varepsilon\right)\frac{du}{dz}+\frac{\alpha z-q}{z^2}u=0 \tag{2.3}$$

and the linear equation for the function $v=du/dz$ is given by

$$\frac{d^2v}{dz^2}+\left(\frac{\gamma}{z^2}+\frac{\delta+2}{z}+\varepsilon-\frac{\alpha}{\alpha z-q}\right)\frac{dv}{dz}+\frac{h(z)}{z^2(\alpha z-q)}v=0, \tag{2.4}$$

where $h(z)=(\alpha+\varepsilon)(\alpha z^2-2qz)+(q^2-\delta q-\alpha\gamma)$.

The bi-confluent Heun equation is

$$\frac{d^2u}{dz^2}+\left(\frac{\gamma}{z}+\delta+\varepsilon z\right)\frac{du}{dz}+\frac{\alpha z-q}{z}u=0 \tag{2.5}$$

and the linear equation for the function $v=du/dz$ is given by

$$\frac{d^2v}{dz^2}+\left(\frac{\gamma+1}{z}+\delta+\varepsilon z-\frac{\alpha}{\alpha z-q}\right)\frac{dv}{dz}+\frac{k(z)}{z(\alpha z-q)}v=0, \tag{2.6}$$

where $k(z)=(\alpha+\varepsilon)z(\alpha z-2q)+(q^2-\delta q-\alpha\gamma)$.

The tri-confluent Heun equation is

$$\frac{d^2u}{dz^2}+\left(\gamma+\delta z+\varepsilon z^2\right)\frac{du}{dz}+(\alpha z-q)u=0 \tag{2.7}$$

and the linear equation for the function $v=du/dz$ is given by

$$\frac{d^2v}{dz^2}+\left(\gamma+\delta z+\varepsilon z^2-\frac{\alpha}{\alpha z-q}\right)\frac{dv}{dz}+\frac{p(z)}{(\alpha z-q)}v=0, \tag{2.8}$$

where $p(z)=(\alpha+\varepsilon)(\alpha z^2-2qz)+(q^2-\delta q-\alpha\gamma)$.

3. Isomonodromic deformations of linear equations and the Painlevé equations

In this section we briefly review the theory of isomonodromic deformations of linear second order differential equations following [21, 22, 23]. We shall use notation similar to [22].

The isomonodromic deformations of linear second order differential equations of the form

$$\frac{d^2v}{dz^2}+p_1(z)\frac{dv}{dz}+p_2(z)v=0, \tag{3.1}$$

with p_1, p_2 being rational functions of z and parameters of deformation $t_1, \ldots, t_n$, are governed by a completely integrable Hamiltonian system of partial differential equations with respect to the parameters. When there is one parameter of deformation, t, the Painlevé equations $P_I - P_{VI}$ appear as the compatibility condition of the extended linear system consisting of equation (3.1) and equaton

$$\frac{\partial v}{\partial t} = a(z,t)\frac{\partial v}{\partial z} + b(z,t)v. \tag{3.2}$$

The Painlevé equations $P_I - P_{VI}$ are nonlinear second order differential equations with the so-called Painlevé property. They have many interesting properties and appear in many areas of mathematics. See, for instance, [24, 25, 21] and numerous references therein. The completely integrable Hamiltonian system is then equivalent to a Painlevé equation for one of the variables. Below we shall present necessary formulas for equations $P_{II} - P_{VI}$.

To get the sixth Painlevé equation one chooses

$$p_1(z,t) = \frac{1-\kappa_0}{z} + \frac{1-\kappa_1}{z-1} + \frac{1-\theta}{z-t} - \frac{1}{z-\lambda}, \tag{3.3}$$

$$p_2(z,t) = \frac{\kappa}{z(z-1)} - \frac{t(t-1)H_{VI}}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}, \tag{3.4}$$

where

$$\begin{aligned} t(t-1)H_{VI} &= \lambda(\lambda-1)(\lambda-t)\mu^2 \\ &\quad -\{\kappa_0(\lambda-1)(\lambda-t) + \kappa_1\lambda(\lambda-t) + (\theta-1)\lambda(\lambda-1)\}\mu + \kappa(\lambda-t). \end{aligned}$$

Then the compatibility between (3.1) and (3.2) with certain $a(z,t)$ and $b(z,t)$ (see [21, 22, 23] for details) leads to the Hamiltonian system

$$\frac{d\lambda}{dt} = \frac{\partial H_{VI}}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H_{VI}}{\partial \lambda}$$

and by eliminating the function μ one can get the sixth Painlevé equation

$$\begin{aligned} \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\ &\quad + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha_6 + \beta_6\frac{t}{\lambda^2} + \gamma_6\frac{t-1}{(\lambda-1)^2} + \delta_6\frac{t(t-1)}{(\lambda-t)^2}\right), \end{aligned} \tag{3.5}$$

where

$$\alpha_6 = \frac{1}{2}\kappa_\infty^2, \quad \beta_6 = -\frac{1}{2}\kappa_0^2, \quad \gamma_6 = \frac{1}{2}\kappa_1^2, \quad \delta_6 = \frac{1}{2}(1-\theta^2)$$

and

$$\kappa = \frac{1}{4}(\kappa_0 + \kappa_1 + \theta - 1)^2 - \frac{1}{4}\kappa_\infty^2.$$

To get the fifth Painlevé equation one chooses

$$p_1(z,t) = \frac{1-\kappa_0}{z} + \frac{\eta t}{(z-1)^2} + \frac{1-\theta}{z-1} - \frac{1}{z-\lambda}, \tag{3.6}$$

$$p_2(z,t) = \frac{\kappa}{z(z-1)} - \frac{tH_V}{z(z-1)^2} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}, \tag{3.7}$$

where

$$tH_V = \lambda(\lambda-1)^2\mu^2 - \{\kappa_0(\lambda-1)^2 + \theta\lambda(\lambda-1) - \eta t\lambda\}\mu + \kappa(\lambda-1).$$

Then similarly to the previous case the corresponding Hamiltonian system with the Hamiltonian H_V leads to the fifth Painlevé equation

$$\begin{aligned}\frac{d^2\lambda}{dt^2} &= \left(\frac{1}{2\lambda} + \frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(\alpha_5\lambda + \frac{\beta_5}{\lambda}\right) \\ &\quad + \gamma_5\frac{\lambda}{t} + \delta_5\frac{\lambda(\lambda+1)}{\lambda-1},\end{aligned} \tag{3.8}$$

where

$$\alpha_5 = \frac{1}{2}\kappa_\infty^2, \quad \beta_5 = -\frac{1}{2}\kappa_0^2, \quad \gamma_5 = (1+\theta)\eta, \quad \delta_5 = \frac{1}{2}\eta^2$$

and

$$\kappa = \frac{1}{4}(\kappa_0+\theta)^2 - \frac{1}{4}\kappa_\infty^2.$$

To get the fourth Painlevé equation one chooses

$$p_1(z,t) = \frac{1-\kappa_0}{z} - \frac{z+2t}{2} - \frac{1}{z-\lambda}, \tag{3.9}$$

$$p_2(z,t) = \frac{1}{2}\theta_\infty - \frac{H_{IV}}{2z} + \frac{\lambda\mu}{z(z-\lambda)}, \tag{3.10}$$

where

$$H_{IV} = 2\lambda\mu^2 - (\lambda^2 + 2t\lambda + 2\kappa_0)\mu + \theta_\infty\lambda.$$

Then the corresponding Hamiltonian system with the Hamiltonian H_{IV} leads to the fourth Painlevé equation

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 + \frac{3}{2}\lambda^3 + 4t\lambda^2 + 2(t^2-\alpha_4)\lambda + \frac{\beta_4}{\lambda}, \tag{3.11}$$

where

$$\alpha_4 = -\kappa_0 + 2\theta_\infty + 1, \quad \beta_4 = -2\kappa_0^2.$$

The standard third Painlevé equation is given by

$$\frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{\alpha_3\lambda^2+\beta_3}{t} + \gamma_3\lambda^3 + \frac{\delta_3}{\lambda}. \tag{3.12}$$

However, for our purpose it is more convenient to consider an equation which can be obtained from (3.12) by changing $\lambda(t) \to \lambda(t^2)/t$ and by renaming the new variable $\tau = t^2$ as t again. This equation is given by

$$\frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{\alpha_3\lambda^2+\gamma_3\lambda^3}{4t^2} + \frac{\beta_3}{4t} + \frac{\delta_3}{4\lambda}. \tag{3.13}$$

Equation (3.13), which will be denoted by P'_{III}, appears in the result of isomonodromic deformations of the linear equation (3.1) with

$$p_1(z,t) = \frac{\eta_0 t}{z^2} + \frac{1-\theta_0}{z} - \eta_\infty - \frac{1}{z-\lambda}, \tag{3.14}$$

$$p_2(z,t) = \frac{\eta_\infty(\theta_0+\theta_\infty)}{2z} - \frac{tH'_{III}}{z^2} + \frac{\lambda\mu}{z(z-\lambda)}, \tag{3.15}$$

where

$$tH'_{III} = \lambda^2\mu^2 - \{\eta_\infty\lambda^2 + \theta_0\lambda - \eta_0 t\}\mu + \frac{1}{2}\eta_\infty(\theta_0+\theta_\infty)\lambda$$

and the parameters are related by

$$\alpha_3 = -4\eta_\infty\theta_\infty,\ \ \beta_3 = 4\eta_0(1+\theta_0),\ \ \gamma_3 = 4\eta_\infty^2,\ \ \delta_3 = -4\eta_0^2.$$

Finally, the second Painlevé equation

$$\frac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \alpha_2 \tag{3.16}$$

appears in the result of isomonodromic deformations of the linear equation (3.1) with

$$p_1(z,t) = -2z^2 - t - \frac{1}{z-\lambda}, \tag{3.17}$$

$$p_2(z,t) = -(2\alpha_2+1)z - 2H_{II} + \frac{\mu}{z-\lambda}, \tag{3.18}$$

where

$$H_{II} = \frac{1}{2}\mu^2 - \left(\lambda^2 + \frac{1}{t}\right)\mu - \left(\alpha_2 + \frac{1}{2}\right)\lambda. \tag{3.19}$$

4. Main results

In this section we compare equations for the derivatives of the Heun functions with the linear differential equations whose isomonodromy deformations are governed by the Painlevé equations $P_{II} - P_{VI}$.

Let us consider the equation for the derivative of the general Heun function (1.3). By choosing parameters

$$\alpha\beta = \kappa_0 + \kappa_1 + \theta + \kappa,\ \ \beta = \frac{1}{2}(\pm\kappa_\infty - 1 - \kappa_0 - \kappa_1 - \theta),$$
$$\gamma = -\kappa_0,\ \ \delta = -\kappa_1,\ \ \varepsilon = -\theta,\ \ q = \alpha\beta\lambda,$$

we can calculate that the resulting equation is the same as equation (3.1) with (3.3), (3.4) and the expression for H_{VI} provided that

$$\mu = \frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda-1} + \frac{\theta}{\lambda-t}.$$

If now λ and μ are viewed as functions of t, substituting this condition into the Hamiltonian system leading to the sixth Painlevé equation, we find that λ satisfies

the Riccati equation

$$\frac{d\lambda}{dt} = \frac{\kappa_0 t - (1 + \kappa_0 + (\kappa_0 + \kappa_1)t + \theta)\lambda + (1 + \kappa_0 + \kappa_1 + \theta)\lambda^2}{t(t-1)}$$

and $\kappa_0 + \kappa_1 + \theta + \kappa = 0$. This gives classical solutions of the sixth Painlevé equation provided that $\kappa_0 = \pm\kappa_\infty - \theta - \kappa_1 - 1$. However, with this additional condition on the parameters we have $\alpha\beta = 0$ and $q = 0$.

In the equation for the derivative of the confluent Heun function (2.2) we first make the change of variables $v(z) \to (1 - z/(z-1))^\sigma v(z/(z-1))$, renaming the new independent variable as z again, then put

$$\gamma = -\kappa_0, \quad \delta = \kappa_0 + \theta + 2\sigma, \quad \varepsilon = -t\eta,$$
$$\sigma = -\frac{1}{2}(\kappa_0 \pm \kappa_\infty + \theta), \quad q = \frac{\alpha\lambda}{\lambda - 1}, \quad \alpha = \frac{1}{2}t\eta(2 + \kappa_0 \pm \kappa_\infty + \theta).$$

The resulting equation is the same as equation (3.1) with (3.6), (3.7) and the expression for H_V provided that

$$\mu = \frac{\kappa_0}{\lambda} - \frac{t\eta}{(\lambda-1)^2} + \frac{\theta - \kappa_0 \pm \kappa_\infty}{2(\lambda - 1)}.$$

Substituting this condition into the Hamiltonian system leading to the fifth Painlevé equation, we see that λ satisfies the Riccati equation

$$t\frac{d\lambda}{dt} \pm \kappa_\infty\lambda^2 - (\pm\kappa_\infty - \kappa_0 - t\eta) - \kappa_0 = 0$$

and $\eta(2 + \kappa_0 \pm \kappa_\infty + \theta) = 0$. Again, with this additional condition on the parameters we have $\alpha = 0$ and $q = 0$.

In the equation for the derivative of the bi-confluent Heun function (2.6) we take

$$\gamma = -\kappa_0, \quad \delta = -t, \quad q = \alpha\lambda, \quad \alpha = \frac{\theta_\infty + 1}{2}, \quad \varepsilon = -\frac{1}{2}.$$

The resulting equation is the same as equation (3.1) with (3.9), (3.10) and the expression for H_{IV} provided that

$$\mu = t + \frac{\kappa_0}{\lambda} + \frac{\lambda}{2}.$$

Substituting this condition into the Hamiltonian system leading to the fourth Painlevé equation, we find that λ satisfies the Riccati equation

$$\frac{d\lambda}{dt} = \lambda^2 + 2t\lambda + 2\kappa_0$$

and $\theta_\infty + 1 = 0$. Again, with this additional condition on the parameters we have $\alpha = 0$ and $q = 0$.

In the equation for the derivative of the double-confluent Heun function (2.6) we take

$$\gamma = t\eta_0, \quad \delta = -1 - \theta_0, \quad q = \alpha\lambda, \quad \alpha = \frac{1}{2}\eta_\infty(\theta_0 + \theta_\infty + 2), \quad \varepsilon = -\eta_\infty.$$

The resulting equation is the same as equation (3.1) with (3.14), (3.15) and the expression for H'_{III} provided that

$$\mu = \eta_\infty - \frac{t\eta_0}{\lambda^2} + \frac{\theta_0 + 1}{\lambda}.$$

Substituting this condition into the Hamiltonian system leading to the modified third Painlevé equation P'_{III}, we find that λ satisfies the Riccati equation

$$t\frac{d\lambda}{dt} = \eta_\infty \lambda^2 + (\theta_0 + 2)\lambda - t\eta_0$$

and $\eta_\infty(\theta_0 + \theta_\infty + 2) = 0$. Again, with this additional condition on the parameters we have $\alpha = 0$ and $q = 0$.

In the equation for the derivative of the tri-confluent Heun function (2.8) we take

$$\gamma = -t, \;\; \delta = 0, \;\; q = \alpha\lambda, \;\; \alpha = 1 - 2\alpha_2, \;\; \varepsilon = -2.$$

The resulting equation is the same as equation (3.1) with (3.17), (3.18) and the expression for H_{II} provided that

$$\mu = 2\lambda^2 + t.$$

Substituting this condition into the Hamiltonian system leading to the second Painlevé equation, we see that λ satisfies the Riccati equation

$$2\frac{d\lambda}{dt} = 2\lambda^2 + t$$

and $2\alpha_2 = 1$. Again, with this additional condition on the parameters we have $\alpha = 0$ and $q = 0$.

Hence, we see that in all cases we can reduce equations for the derivatives of the Heun functions to certain linear equations, isomonodromy deformations of which lead to the Painlevé equations with an additional constraint on λ and μ. However, in order to get classical solutions of the Painlevé equations we need an additional constraint on the parameters. Therefore, those linear equations isomonodromy deformations of which are described by classical solutions of the Painlveé equations cannot be obtained from the equations for the derivatives of the Heun functions.

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