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SOME INEQUALITIES FOR RATIONAL FUNCTIONS WITH FIXED POLES

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Abstract. By using lemmas of Dubinin and Osserman some results for rational functions with fixed poles and restricted zeros are proved. The obtained results strengthen some known results for rational functions and, in turn, produce refinements of some polynomial inequalities as well.

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1. Introduction

Let $\mathbb{P}_n$ denote the class of all complex polynomials of degree at most n. If $P \in \mathbb{P}_n$, then concerning the estimate of $|P'(z)|$ on $|z| = 1$, we have

$$|P'(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The inequality (1.1) is a famous result due to Bernstein [3]. It is worth mentioning that in (1.1) equality holds if and only if $P(z)$ has all its zeros at the origin. So, it is natural to seek improvements under appropriate assumption on the zeros of $P(z)$. If we restrict ourselves to the class of polynomials $P(z)$ having no zeros in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{1.2}$$

whereas, if $P(z)$ has no zeros in $|z| > 1$, then by

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

The inequality (1.2) was conjectured by Erdös and later it was verified by Lax [7], whereas the inequality (1.3) is due to Turán [10].

Jain [6] had used a parameter β and proved an interesting generalization of (1.3). More precisely, Jain proved that if $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \le 1$, then for every β with $|\beta| \le 1$, we have

$$\max_{|z|=1} \left|zP'(z) + \frac{n\beta}{2} P(z)\right| \ge \frac{n}{2}\{1 + Re(\beta)\} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Li, Mohapatra and Rodriguez [12] gave a new perspective to inequalities (1.1) – (1.3), and extended them to rational functions with fixed poles. Essentially, in these inequalities they replaced the polynomial $P(z)$ by a rational function $r(z)$ with poles $a_1, a_2, ..., a_n$ all lying in $|z| > 1$, and z^n was replaced by a Blaschke product $B(z)$. Before proceeding towards their results, we first introduce the set of rational functions involved.

For $a_j \in \mathbb{C}$ with $j = 1, 2, ..., n$, we define

$$W(z) = \prod_{j=1}^{n}(z - a_j);\ B(z) = \prod_{j=1}^{n}\left(\frac{1 - \overline{a}_j z}{z - a_j}\right) = \frac{W^*(z)}{W(z)},$$

where

$$W^*(z) = z^n \overline{W\left(\frac{1}{\overline{z}}\right)}$$

and

$$\mathbb{R}_n = \mathbb{R}_n(a_1, a_2, ..., a_n) = \left\{\frac{P(z)}{W(z)} :\ P \in \mathbb{P}_n\right\}.$$

Then $\mathbb{R}_n$ is defined to be the set of rational functions with poles $a_1, a_2, ..., a_n$ at most and with finite limit at ∞. Note that $B(z) \in \mathbb{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. Also, for $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$, the conjugate transpose r^* of r is defined by $r^*(z) = B(z)\overline{r(\frac{1}{\overline{z}})}$.

In the past few years several papers pertaining to Bernstein-type inequalities for rational functions have appeared in the study of rational approximations (see [2], [4], [11] – [13]). For $r \in \mathbb{R}_n$, Li, Mohapatra and Rodriguez [12] proved the following, similar to (1.1), inequality for rational functions:

$$|r'(z)| \leq |B'(z)| \max_{|z|=1} |r(z)|. \tag{1.5}$$

As extensions of (1.2) and (1.3) to rational functions, Li, Mohapatra and Rodriguez also showed that if $r \in \mathbb{R}_n$, and $r(z) \neq 0$ in $|z| < 1$, then for $|z| = 1$,

$$|r'(z)| \leq \frac{|B'(z)|}{2} \max_{|z|=1} |r(z)|, \tag{1.6}$$

whereas, if $r \in \mathbb{R}_n$ has exactly n zeros in $|z| \leq 1$, then for $|z| = 1$,

$$|r'(z)| \geq \frac{|B'(z)|}{2} |r(z)|. \tag{1.7}$$

Very recently, Wali and Shah [13] proved an interesting refinement of (1.7). Namely, they proved that if $r \in \mathbb{R}_n$, and r has exactly n zeros in $|z| \leq 1$, where $r(z) = \frac{P(z)}{W(z)}$, with $P(z) = \sum_{j=0}^{n} c_j z^j$, then for $|z| = 1$,

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{\sqrt{|c_n|} - \sqrt{|c_0|}}{\sqrt{|c_n|}}\right\}|r(z)|. \tag{1.8}$$

In this paper, we establish some results for rational functions $r(z) = \frac{P(z)}{W(z)}$ with restricted zeros, where $P(z) = \sum_{j=0}^{n} c_j z^j$, by involving some coefficients of $P(z)$.

Our results strengthen some known inequalities for rational functions and, in turn, produce refinements of some polynomial inequalities as well.

2. Main results

In what follows we shall always assume that all the poles $a_1, a_2, ..., a_n$ of $r(z)$ lie in $|z| > 1$. In the case where all poles are in $|z| < 1$, we can obtain analogous results with suitable modifications.

Theorem 2.1. *Suppose that $r \in \mathbb{R}_n$, and all the n zeros of r lie in $|z| \leq 1$. If $r(z) = \frac{P(z)}{W(z)}$, where $P(z) = \sum_{j=0}^{n} c_j z^j$, then for every β with $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\left|zr'(z) + \frac{n\beta}{2}r(z)\right| \geq \frac{1}{2}\left\{|B'(z)| + nRe(\beta) + \frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right\}|r(z)|. \tag{2.1}$$

The result is best possible in the case $\beta = 0$, and in (2.1) equality holds for $r(z) = B(z) + \lambda$ with $|\lambda| = 1$.

We first discuss some consequences of Theorem 2.1. If we take $\alpha_j = \alpha$, $|\alpha| \geq 1$, for $j = 1, 2, ..., n$, then $W(z) = (z - \alpha)^n$ and $r(z) = \frac{P(z)}{(z-\alpha)^n}$, and hence we have

$$\begin{aligned} r'(z) &= \frac{(z-\alpha)^n P'(z) - n(z-\alpha)^{n-1}P(z)}{(z-\alpha)^{2n}} \\ &= -\left\{\frac{nP(z) - (z-\alpha)P'(z)}{(z-\alpha)^{n+1}}\right\} = \frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}, \end{aligned}$$

where $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ is the polar derivative of $P(z)$ with respect to point α. It generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Also, $W^*(z) = (1 - \overline{\alpha}z)^n$, which gives $B(z) = \left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^n$, implying that

$$B'(z) = \frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2 - 1)}{(z-\alpha)^{n+1}}.$$

With this choice, from (2.1) for $|z| = 1$, we get

$$\begin{aligned} &\left|zD_\alpha P(z) + \frac{n\beta}{2}(\alpha - z)P(z)\right| \\ &\geq \frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{|z-\alpha|} + nRe(\beta)|z-\alpha| + \frac{|c_n| - |c_0|}{|c_n| + |c_0|}|z-\alpha|\right\}|P(z)| \\ &\geq \frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{|\alpha|+1} + nRe(\beta)(|\alpha|-1) + \frac{|c_n| - |c_0|}{|c_n| + |c_0|}(|\alpha|-1)\right\}|P(z)| \\ &= \frac{|\alpha|-1}{2}\left\{n(1 + Re(\beta)) + \frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right\}|P(z)|. \end{aligned}$$

Thus, from Theorem 2.1 we immediately get the following result.

Corollary 2.1. *If $P(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for every $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \geq 1$ and $|\beta| \leq 1$, we have*

$$\max_{|z|=1}\left|zD_\alpha P(z)+\frac{n\beta}{2}(\alpha - z)P(z)\right| \geq \frac{|\alpha|-1}{2}\left\{n(1+Re(\beta))+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}\max_{|z|=1}|P(z)|. \tag{2.2}$$

Remark 2.1. *Since $|c_n| \geq |c_0|$ and hence for $\beta = 0$, the above corollary provides an improvement of a result due to Shah* [9].

Remark 2.2. *Dividing both sides of* (2.2) *by $|\alpha|$ and letting $|\alpha| \to \infty$, we obtain the following result, which as a special case, gives a strengthening of the classical Turán inequality* [10].

Corollary 2.2. *If $P(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq 1$. Then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, we have*

$$\max_{|z|=1}\left|zP'(z)+\frac{n\beta}{2}P(z)\right| \geq \frac{1}{2}\left\{n(1+Re(\beta))+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}\max_{|z|=1}|P(z)|. \tag{2.3}$$

Remark 2.3. *The above inequality for $\beta = 0$ was also independently proved by Dubinin* [5]. *Also, it is easy to see that the inequality* (2.3) *improves the inequality* (1.4) *as well.*

Taking $\beta = 0$ in Theorem 2.1, we get the following result.

Corollary 2.3. *Suppose $r \in \mathbb{R}_n$, and all the n zeros of r lie in $|z| \leq 1$. If $r(z) = \frac{P(z)}{W(z)}$, where $P(z) = \sum_{j=0}^{n} c_j z^j$, then for $|z| = 1$ we have*

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)|+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}|r(z)|.$$

The result is sharp and equality holds for $r(z) = B(z) + \lambda$ with $|\lambda| = 1$.

Remark 2.4. *Again, since $|c_n| \geq |c_0|$, it is easy to verify that*

$$\frac{|c_n|-|c_0|}{|c_n|+|c_0|} \geq \frac{\sqrt{|c_n|}-\sqrt{|c_0|}}{\sqrt{|c_n|}},$$

showing that Corollary 2.3 strengthens the inequality (1.8).

Instead of proving Theorem 2.1, we will prove the following more general result.

Theorem 2.2. *Suppose $r(z) = \frac{P(z)}{W(z)}$, where $P(z) = z^s\left(\sum_{j=0}^{n-s} c_{j+s} z^j\right)$, and all the zeros of r lie in $|z| \leq 1$ with a zero of multiplicity s at the origin. Then for every β with $|\beta| \leq 1$ and $|z| = 1$ we have*

$$\left|zr'(z)+\frac{n\beta}{2}r(z)\right| \geq \frac{1}{2}\left\{|B'(z)|+nRe(\beta)+s+\frac{|c_n|-|c_s|}{|c_n|+|c_s|}\right\}|r(z)|. \tag{2.4}$$

The result is best possible in the case $\beta = s = 0$, and equality in (2.4) holds for $r(z) = B(z) + \lambda$ with $|\lambda| = 1$.

Remark 2.5. *For $s = 0$, the inequality (2.4) reduces to (2.1).*

The next result generalizes the inequality (1.7).

Theorem 2.3. *Let $r \in \mathbb{R}_n$, and assume that r has all its zeros in $|z| \le 1$. Then for every β with $|\beta| \le 1$ and $|z| = 1$ we have*

$$\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| \ge \frac{1}{2}(1 - |\beta|)|r(z)|. \tag{2.5}$$

Equality in (2.5) holds when $\beta = 0$ for $r(z) = aB(z) + b$ with $|a| = |b|$.

The above inequality (2.5) will be a consequence of a more fundamental inequality presented by the following theorem.

Theorem 2.4. *Let $r \in \mathbb{R}_n$, and assume that r has all its zeros in $|z| \le 1$. Then for every β with $|\beta| \le 1$ and $|z| = 1$, we have*

$$\left|\frac{r'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(z)}{B(z)}\right| \ge \frac{1}{2}\left\{(1 - |\beta|)|r(z)| + \left(\left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right)\min_{|z|=1}|r(z)|\right\}. \tag{2.6}$$

Equality in (2.6) holds when $\beta = 0$ for $r(z) = aB(z) + b$ with $|a| = |b|$.

Remark 2.6. *Theorem 2.4 is a refinement of Theorem 2.3, this can easily be seen by observing that $|1 + \frac{\beta}{2}| \ge |\frac{\beta}{2}|$ for $|\beta| \le 1$.*

Theorem 2.5. *Suppose $r \in \mathbb{R}_n$, and all the n zeros of r lie in $|z| \ge 1$. If $r(z) = \frac{P(z)}{W(z)}$, where $P(z) = \sum_{j=0}^{n} c_j z^j$, then for $|z| = 1$, we have*

$$|r'(z)| \le \frac{1}{2}\left\{|B'(z)| - \left(\frac{|c_0| - |c_n|}{|c_0| + |c_n|}\right)\frac{|r(z)|^2}{\| r(z) \|^2}\right\} \| r(z) \|, \tag{2.7}$$

where $\| r(z) \| = \max_{|z|=1} |r(z)|$. The result is best possible and equality in (2.7) holds for $r(z) = B(z) + \lambda$, $|\lambda| = 1$.

Remark 2.7. *Since all zeros of $r(z) = \frac{P(z)}{W(z)}$, and hence of $P(z) = \sum_{j=0}^{n} c_j z^j$, lie in $|z| \ge 1$, we have $|c_0| \ge |c_n|$, showing that Theorem 2.5 is an improvement of (1.6).*

3. Lemmas

In this section we state a number of lemmas, which will be used in the proofs of main results stated in Section 2.

Lemma 3.1. *(see [5]) If* $P(z)=\sum_{j=0}^{n}c_jz^j$ *is a polynomial of degree* n *having all its zeros in* $|z|\le 1$*, then at each point* z *of the circle* $|z|=1$ *at which* $P(z)\neq 0$*, we have*

$$Re\left(\frac{zP'(z)}{P(z)}\right)\ge \frac{n-1}{2}+\frac{|c_n|}{|c_n|+|c_0|}.$$

Lemma 3.2. *(see [2]) If* $|z|=1$*, then*

$$Re\left(\frac{zW'(z)}{W(z)}\right)=\frac{n-|B'(z)|}{2}.$$

Lemma 3.3. *(see [12]) If* $r\in\mathbb{R}_n$*, then for* $|z|=1$*, we have*

$$|r'(z)|+|(r^*(z))'|\le |B'(z)|\max_{|z|=1}|r(z)|.$$

Lemma 3.4. *Suppose* $r\in\mathbb{R}_n$ *is such that* $r(z)=\frac{P(z)}{W(z)}$*, where* $P(z)=\sum_{j=0}^{n}c_jz^j$*, and all the zeros of* r *lie in* $|z|>1$*. Then for* $|z|=1$*, we have*

$$Re\left(\frac{zr'(z)}{r(z)}\right)\le \frac{1}{2}\left\{|B'(z)|-\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right\}.$$

Proof. We have $r(z)=\frac{P(z)}{W(z)}$, where

$$P(z)=\sum_{j=0}^{n}c_jz^j=c_n\prod_{j=1}^{n}(z-z_j),$$

with $c_n\neq 0$ and $|z_j|>1$, $j=1,2,...,n$.
By direct calculation, we get

$$Re\left(\frac{zr'(z)}{r(z)}\right)=Re\left(\frac{zP'(z)}{P(z)}\right)-Re\left(\frac{zW'(z)}{W(z)}\right). \tag{3.1}$$

Let $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$, therefore, $P(z)=z^n\overline{Q(\frac{1}{\bar z})}$. Since $P(z)$ has all its zeros in $|z|>1$, it follows that $Q(z)$ has all its zeros in $|z|<1$, and hence

$$G(z)=\frac{Q(z)}{z^{n-1}\overline{Q(\frac{1}{\bar z})}}=\frac{zQ(z)}{P(z)}=\frac{\overline{c_n}}{c_n}z\prod_{j=1}^{n}\left(\frac{1-\bar z_jz}{z-z_j}\right) \tag{3.2}$$

is analytic in $|z|\le 1$ with $G(0)=0$ and $|G(z)|=1$ for $|z|=1$. Hence by a result of Osserman for the boundary Schwartz lemma [8], we have

$$|G'(z)|\ge \frac{2}{1+|G'(0)|},\quad for\quad |z|=1. \tag{3.3}$$

It easily follows from (3.2) that for $|z|=1$,

$$\frac{zG'(z)}{G(z)}=(n+1)-2Re\left(\frac{zP'(z)}{P(z)}\right). \tag{3.4}$$

Further, using (3.2), it can easy be verified that

$$\frac{zG'(z)}{G(z)}=1+\sum_{j=1}^{n}\frac{|z_j|^2-1}{|z-z_j|^2}.$$

Since $|z_j| > 1$ for $1 \leq j \leq n$, it follows from above that $\frac{zG'(z)}{G(z)}$ is real and positive. Also, taking into account that $|G(z)| = 1$ for $|z| = 1$, we have

$$\frac{zG'(z)}{G(z)} = \left|\frac{zG'(z)}{G(z)}\right| = |G'(z)| \quad \text{and} \quad |G'(0)| = \prod_{j=1}^{n}\left|\frac{1}{z_j}\right| = \left|\frac{c_n}{c_0}\right|.$$

Using these observations, from (3.3) and (3.4), we get for $P(z) \neq 0$ and $|z| = 1$,

$$(n+1) - 2Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{2}{1+\left|\frac{c_n}{c_0}\right|},$$

implying that

$$Re\left(\frac{zP'(z)}{P(z)}\right) \leq \frac{n+1}{2} - \frac{|c_0|}{|c_0|+|c_n|}. \tag{3.5}$$

Finally, using (3.5), Lemma 3.2 and (3.1), we get

$$Re\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{1}{2}\left\{|B'(z)| - \frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right\},$$

which completes the proof of of the lemma. □

Lemma 3.5. *Let $r, s \in \mathbb{R}_n$, and let all the n zeros of s lie in $|z| \leq 1$ and for $|z| = 1$,*

$$|r(z)| \leq |s(z)|.$$

Then for every $|\beta| \leq 1$ and $|z| = 1$, we have

$$|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)| \leq |B(z)s'(z) + \frac{\beta}{2}B'(z)s(z)|. \tag{3.6}$$

Equality in (3.6) holds for $r(z) = \mu s(z)$, $|\mu| = 1$.

Proof. The proof follows on the same lines as those given in the proof of Theorem 3.2 of Li [11]. Hence, we omit the details.

Lemma 3.6. *Let $r \in \mathbb{R}_n$, and let all the n zeros of r lie in $|z| \leq 1$. Then for every $|\beta| \leq 1$ and $|z| = 1$, we have*

$$|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)| \leq |B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)|. \tag{3.7}$$

Proof. Since $r^*(z) = B(z)\overline{r(1/\bar{z})}$, we have

$$|r^*(z)| = |r(z)| \text{ for } |z| = 1.$$

Also, since $r(z)$ has all its zeros in $|z| \leq 1$, we can apply Lemma 3.5 with $r(z)$ and $s(z)$ being replaced by $r^*(z)$ and $r(z)$, respectively, to obtain the result.

4. Proofs of theorems

Proof of Theorem 2.2. Since $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$, where $P(z)$ has all its zeros in $|z| \leq 1$ with a zero of multiplicity s at the origin, we can write

$$P(z) = z^s h(z), \tag{4.1}$$

where $h(z) = \sum_{j=0}^{n-s} c_{j+s} z^j$ is a polynomial of degree $n-s$ having all its zeros in $|z| \leq 1$.

From (4.1), we have

$$Re\left(\frac{zP'(z)}{P(z)}\right) = s + Re\left(\frac{zh'(z)}{h(z)}\right).$$

By a direct calculation, we obtain for every β with $|\beta| \leq 1$,

$$\frac{zr'(z)}{r(z)} + \frac{n\beta}{2} = \frac{zP'(z)}{P(z)} - \frac{zW'(z)}{W(z)} + \frac{n\beta}{2}.$$

Therefore for $0 \leq \theta < 2\pi$ by Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned} Re\left(\frac{zr'(z)}{r(z)} + \frac{n\beta}{2}\right)\Bigg|_{z=e^{i\theta}} &= Re\left(\frac{zP'(z)}{P(z)}\right)\Bigg|_{z=e^{i\theta}} - Re\left(\frac{zW'(z)}{W(z)}\right)\Bigg|_{z=e^{i\theta}} + \frac{n}{2}Re(\beta) \\ &= s + Re\left(\frac{zh'(z)}{h(z)}\right)\Bigg|_{z=e^{i\theta}} - Re\left(\frac{zW'(z)}{W(z)}\right)\Bigg|_{z=e^{i\theta}} + \frac{n}{2}Re(\beta) \\ &\geq \left(s + \frac{n-s-1}{2} + \frac{|c_n|}{|c_n|+|c_s|}\right) - \left(\frac{n-|B'(e^{i\theta})|}{2}\right) + \frac{n}{2}Re(\beta) \\ &= \frac{1}{2}\left\{|B'(e^{i\theta})| + s + \frac{|c_n|-|c_s|}{|c_n|+|c_s|} + nRe(\beta)\right\}, \end{aligned}$$

for the points $e^{i\theta}, 0 \leq \theta < 2\pi$, other then the zero of $r(z)$. Hence, we have

$$\left|e^{i\theta}r'(e^{i\theta}) + \frac{n}{2}\beta r(e^{i\theta})\right| \geq \frac{1}{2}\left\{|B'(e^{i\theta})| + s + \frac{|c_n|-|c_s|}{|c_n|+|c_s|} + nRe(\beta)\right\}|r(e^{i\theta})|, \tag{4.2}$$

for the points $e^{i\theta}, 0 \leq \theta < 2\pi$, other then the zeros of $r(z)$.

Since (4.2) is true for the points $e^{i\theta}, 0 \leq \theta < 2\pi$, which are the zeros of $r(z)$ as well, it follows that

$$\left|zr'(z) + \frac{n}{2}\beta r(z)\right| \geq \frac{1}{2}\left\{|B'(z)| + s + \frac{|c_n|-|c_s|}{|c_n|+|c_s|} + nRe(\beta)\right\}|r(z)|,$$

for $|z| = 1$ and for every β with $|\beta| \leq 1$. This completes the proof of the theorem.

Proof of Theorem 2.3. By a direct calculation (see, e.g., [12], p. 529), one can obtain

$$|(r^*(z))'| = |B'(z)r(z) - r'(z)B(z)| \text{ for } |z| = 1,$$

and hence, using the fact that $|B(z)| = 1$ for $|z| = 1$, we get

$$|(r^*(z))'| \geq |B'(z)||r(z)| - |r'(z)|.$$

This gives for $|z|=1$,

$$|r'(z)|+|(r^*(z))'|\geq |B'(z)||r(z)|. \tag{4.3}$$

Next, for any $|\beta|\leq 1$, we have

$$\begin{aligned}&\left|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)\right|+\left|B(z)(r^*(z))'+\frac{\beta}{2}B'(z)r^*(z)\right|\\&\geq |B(z)|\,|r'(z)|+|B(z)||(r^*(z))'|-\left|\frac{\beta}{2}\right||B'(z)||r(z)|-\left|\frac{\beta}{2}\right||B'(z)||r^*(z)|,\end{aligned}$$

and hence, by using (4.3) and the fact that $|r(z)|=|r^*(z)|$ for $|z|=1$, we obtain

$$\begin{aligned}&\left|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)\right|+\left|B(z)(r^*(z))'+\frac{\beta}{2}B'(z)r^*(z)\right|\\&\geq |r'(z)|+|(r^*(z))'|-|\beta||B'(z)||r(z)|\geq |B'(z)||r(z)|-|\beta||B'(z)||r(z)|. \end{aligned}\tag{4.4}$$

Now, by Lemma 3.6, we have for $|z|=1$,

$$\left|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)\right|\geq\left|B(z)(r^*(z))'+\frac{\beta}{2}B'(z)r^*(z)\right|. \tag{4.5}$$

The inequalities (4.4) and (4.5) together yield to

$$\left|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)\right|\geq\frac{|B'(z)|}{2}(1-|\beta|)|r(z)|, \tag{4.6}$$

for $|z|=1$ and $|\beta|\leq 1$.

Finally, taking into account that $|B'(z)|\neq 0$ and $|B(z)|=1$ for $|z|=1$, from (4.6), we get

$$\left|\frac{r'(z)}{B'(z)}+\frac{\beta}{2}\frac{r(z)}{B(z)}\right|\geq\frac{1}{2}(1-|\beta|)|r(z)|,$$

for $|z|=1$ and $|\beta|\leq 1$. □

Proof of Theorem 2.4. Observe first that if $r(z)$ has some zeros on $|z|=1$, then $\min\limits_{|z|=1}|r(z)|=0$, and in this case, the result follows from Theorem 2.3.

So, henceforth, we assume that all the zeros of $r(z)$ lie in $|z|<1$. Let $m:=\min\limits_{|z|=1}|r(z)|$. Clearly $m>0$, and we have $|\lambda m|<|r(z)|$ on $|z|=1$ for any λ with $|\lambda|<1$. By Rouche's theorem, the rational function $G(z)=r(z)+\lambda m$ has all its zeros in $|z|<1$. Let $H(z)=B(z)\overline{G(1/\bar{z})}=r^*(z)+\bar{\lambda}mB(z)$, then $|H(z)|=|G(z)|$ for $|z|=1$. Applying Lemma 3.6, for any β with $|\beta|\leq 1$ and $|z|=1$, we get

$$\begin{aligned}&\left|B(z)\Big((r^*(z))'+\bar{\lambda}B'(z)m\Big)+\frac{\beta}{2}B'(z)\Big(r^*(z)+\bar{\lambda}B(z)m\Big)\right|\\&\leq\left|B(z)r'(z)+\frac{\beta}{2}B'(z)\Big(r(z)+\lambda m\Big)\right|,\end{aligned}\tag{4.7}$$

implying that

$$\Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z) + \bar{\lambda}\Big(1+\frac{\beta}{2}\Big)B(z)B'(z)m\Big|$$
$$\le \Big|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| + \Big|\frac{\beta}{2}\Big|\,|\lambda|m|B'(z)| \tag{4.8}$$

for $|z|=1, |\beta|\le 1$ and $|\lambda|<1$.
Choosing the arguments of λ on the left hand side of (4.8) to satisfy

$$\Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z) + \bar{\lambda}\Big(1+\frac{\beta}{2}\Big)B(z)B'(z)m\Big|$$
$$= \Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\Big| + |\lambda|m\Big|1+\frac{\beta}{2}\Big|\,|B(z)B'(z)|, \tag{4.9}$$

in view of (4.8), (4.9) and the fact that $|B(z)|=1$ for $|z|=1$, we get

$$\Big|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\Big| \ge \Big|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\Big|$$
$$+ |\lambda||B'(z)|\bigg\{\Big|1+\frac{\beta}{2}\Big| - \Big|\frac{\beta}{2}\Big|\bigg\}m. \tag{4.10}$$

Finally, letting $|\lambda|\to 1$ in (4.10) and adding $|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)|$ to both sides, and using (4.4), we get the required assertion. Theorem 2.4 is proved.

Proof of Theorem 2.5. Since $r(z)=\frac{P(z)}{W(z)}$, where $P(z)=\sum_{j=0}^{n} c_j z^j$ and $r(z)$ has all its zeros in $|z|\ge 1$, and also $r^*(z)=B(z)\overline{r(1/\bar{z})}$, we have

$$z(r^*(z))' = zB'(z)\overline{r(\frac{1}{\bar{z}})} - \frac{B(z)}{z}\overline{r'(\frac{1}{\bar{z}})},$$

and therefore, for $|z|=1$ (so that $z=\frac{1}{\bar{z}}$), we get

$$|(r^*(z))'| = \Big|zB'(z)\overline{r(z)} - B(z)\overline{zr'(z)}\Big| = |B(z)|\Big|\frac{zB'(z)}{B(z)}\overline{r(z)} - \overline{zr'(z)}\Big|. \tag{4.11}$$

Taking into account that (see [12], formula (15))

$$\frac{zB'(z)}{B(z)} = |B'(z)| > 0,$$

from (4.11) for $|z|=1$ with $r(z)\ne 0$, we get

$$\Big|\frac{z(r^*(z))'}{r(z)}\Big|^2 = \Big||B'(z)| - \frac{zr'(z)}{r(z)}\Big|^2$$
$$= |B'(z)|^2 + \Big|\frac{zr'(z)}{r(z)}\Big|^2 - 2|B'(z)|Re\Big(\frac{zr'(z)}{r(z)}\Big),$$

which, in view of Lemma 3.4, for $|z|=1$ with $r(z)\ne 0$, gives

$$\Big|\frac{z(r^*(z))'}{r(z)}\Big|^2 \ge |B'(z)|^2 + \Big|\frac{zr'(z)}{r(z)}\Big|^2 - |B'(z)|\bigg\{|B'(z)| - \frac{|c_0|-|c_n|}{|c_0|+|c_n|}\bigg\}$$
$$= \Big|\frac{zr'(z)}{r(z)}\Big|^2 + \Big(\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\Big)|B'(z)|.$$

This implies for $|z|=1$ that

$$|r'(z)|^2+\left(\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right)|B'(z)||r(z)|^2\le|(r^*(z))'|^2.$$

Combining this with Lemma 3.3, for $|z|=1$ we get

$$\begin{aligned}|r'(z)|+&\left\{|r'(z)|^2+\left(\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right)|B'(z)||r(z)|^2\right\}^{\frac{1}{2}}\\&\le|r'(z)|+|(r^*(z))'|\le|B'(z)|\ \|\ r(z)\ \|,\end{aligned}$$

or equivalently,

$$\begin{aligned}&|r'(z)|^2+\left(\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right)|B'(z)||r(z)|^2\\&\le|B'(z)|^2\ \|\ r(z)\ \|^2\ -2|B'(z)||r'(z)|\ \|\ r(z)\ \|+|r'(z)|^2,\end{aligned}$$

which, in view of the fact that $|B'(z)|\neq 0$, after simplification, for $|z|=1$ gives

$$|r'(z)|\le\frac{1}{2}\left\{|B'(z)|-\left(\frac{|c_0|-|c_n|}{|c_0|+|c_n|}\right)\frac{|r(z)|^2}{\|\ r(z)\ \|^2}\right\}\ \|\ r(z)\ \|\ .$$

This completes the proof of the theorem.

Remark 4.1. *From inequality* (4.10)*, for* $|z|=1$ *and for every* $|\beta|\le 1$*, we have*

$$\begin{aligned}&\left|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)\right|-\left|B(z)(r^*(z))'+\frac{\beta}{2}B'(z)r^*(z)\right|\\&\ge|B'(z)|\left\{\left|1+\frac{\beta}{2}\right|-\left|\frac{\beta}{2}\right|\right\}\min_{|z|=1}|r(z)|.\end{aligned}\tag{4.12}$$

Since $|B'(z)|\neq 0$ for $|z|=1$, from (4.12) we get the following inequality

$$\min_{|z|=1}\left\{\left|\frac{r'(z)}{B'(z)}+\frac{\beta}{2}\frac{r(z)}{B(z)}\right|-\left|\frac{(r^*(z))'}{B'(z)}+\frac{\beta}{2}\frac{r^*(z)}{B(z)}\right|\right\}\ge\left(\left|1+\frac{\beta}{2}\right|-\left|\frac{\beta}{2}\right|\right)\min_{|z|=1}|r(z)|.\tag{4.13}$$

Taking $\beta=0$ in (4.13), we get

$$\min_{|z|=1}\left\{\left|\frac{r'(z)}{B'(z)}\right|-\left|\frac{(r^*(z))'}{B'(z)}\right|\right\}\ge\min_{|z|=1}|r(z)|,$$

yielding

$$\min_{|z|=1}\left|\frac{r'(z)}{B'(z)}\right|\ge\min_{|z|=1}|r(z)|.\tag{4.14}$$

Clearly, the inequality (4.14) gives a generalization of the corresponding result for polynomials (see [1], Theorem 1).

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