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SOME RESULTS ON THE PAINLEVÉ III DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract. In this paper, we investigate the following two Painlevé III equations: $\overline{w}\underline{w}(w^2-1)=w^2+\mu$ and $\overline{w}\underline{w}(w^2-1)=w^2-\lambda w$, where $\overline{w}:=w(z+1)$, $\underline{w}:=w(z-1)$ and μ ($\mu\neq-1$) and $\lambda\notin\{\pm1\}$ are constants. We discuss the equations of existence of rational solutions, of Borel exceptional values and the exponents of convergence of zeros, poles and fixed points of transcendental meromorphic solutions of these equations.

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1. Introduction

After the completion of the differential Nevanlinna theory, the value distribution of solutions of difference equations has received a considerable attention of a number of researchers. Halburd and Korhonen [1] abstracted the difference Painlevé II equation by using the value distribution theory. Chen and Shon [2] dealt with the properties of solutions of complex difference Riccati equations. It is an important discovery that difference Riccati equation plays an important role in the study of difference Painlevé equations.

We assume that the readers are familiar with the fundamental results and the standard notion of Nevanlinna's value distribution theory of meromorphic functions (see [3] – [5]).

Let w be a meromorphic function in the complex plane and let z be an arbitrary element in the complex plane. By $\rho(w)$, $\lambda(w)$ and $\lambda(1/w)$ we denote the order, the exponents of convergence of zeros and poles of w, respectively. The exponent of convergence of fixed points is defined by

$$\tau(w)=\limsup_{r\to\infty}\frac{\log N\left(r,\frac{1}{w-z}\right)}{\log r}.$$

The field of small functions of w is defined by

$$S(w)=\{\alpha\,\text{meromorphic}:T(r,\alpha)=S(r,w)\},$$

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where $S(r,w)$ is any quantity satisfying $S(r,w) = o\,(T(r,w))$ for all r outside a set of finite logarithmic measure. A meromorphic solution w is called *admissible* if all the coefficients of a difference equation are in the field $\mathcal{S}(w)$. For instance, all the non-rational meromorphic solutions of a difference equation which has only rational coefficients, are admissible.

Recently, Halburd and Korhonen [9], developing the Nevanlinna value distribution theory on difference expressions (see [6] – [8]), considered the following difference equation:

$$\overline{w} + \underline{w} = R(z,w), \tag{1.1}$$

where R is rational in w and is meromorphic in z with slow growth of coefficients. They proved that if the equation (1.1) has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation, or the equation (1.1) can be transformed to eight simple difference equations. These simple difference equations include the Painlevé I, II difference equations and some linear difference equations. We recall the family including Painlevé III difference equations.

Theorem A ([10]). *Assume that the equation:*

$$\overline{w}\underline{w} = R(z,w), \tag{1.2}$$

has an admissible meromorphic solution w of hyper-order less than one, where $R(z,w)$ is rational and irreducible in w and meromorphic in z. Then either w satisfies the following difference Riccati equation:

$$\overline{w} = \frac{\alpha w + \beta}{w + \gamma},$$

where α, β, $\gamma \in \mathcal{S}(w)$ are algebraic functions, or the equation (1.2) can be transformed to one of the following equations:

$$\overline{w}\underline{w} = \frac{\eta w^2 - \lambda w + \mu}{(w-1)(w-\nu)}, \tag{1.3a}$$

$$\overline{w}\underline{w} = \frac{\eta w^2 - \lambda w}{(w-1)}, \tag{1.3b}$$

$$\overline{w}\underline{w} = \frac{\eta(w-\lambda)}{(w-1)}, \tag{1.3c}$$

$$\overline{w}\underline{w} = hw^m. \tag{1.3d}$$

In (1.3a), the coefficients satisfy $\kappa^2\overline{\mu}\underline{\mu} = \mu^2$, $\overline{\lambda}\mu = \kappa\underline{\lambda}\overline{\mu}$, $\kappa\overline{\overline{\lambda}}\underline{\lambda} = \underline{\kappa}\lambda\overline{\lambda}$, and one of the following conditions:

$$(1)\ \eta \equiv 1,\ \overline{\nu}\underline{\nu} = 1,\ \kappa = \nu; \qquad (2)\ \overline{\eta} = \underline{\eta} = \nu,\ \kappa \equiv 1.$$

In (1.3b), $\eta\overline{\eta}=1$ *and* $\overline{\overline{\lambda}}\underline{\lambda}=\lambda\overline{\lambda}$.

In (1.3c), *the coefficients satisfy one of the following conditions:*

(1) $\eta\equiv 1$, *and either* $\lambda=\overline{\lambda}\underline{\lambda}$ *or* $\overline{\lambda}^{[3]}\underline{\lambda}_{[3]}=\overline{\overline{\lambda}}\underline{\underline{\lambda}}$;

(2) $\overline{\lambda}\underline{\lambda}=\overline{\overline{\lambda}}\underline{\underline{\lambda}}$, $\overline{\eta}\overline{\lambda}=\overline{\overline{\lambda}}\underline{\eta}$, $\eta\underline{\eta}=\overline{\overline{\eta}}\underline{\eta}_{[3]}$;

(3) $\overline{\overline{\eta}}\underline{\underline{\eta}}=\eta\underline{\eta}$, $\lambda=\underline{\eta}$;

(4) $\overline{\lambda}^{[3]}\underline{\lambda}_{[3]}=\overline{\overline{\lambda}}\underline{\underline{\lambda}}\lambda, \eta\lambda=\overline{\overline{\eta}}\underline{\underline{\eta}}$.

In (1.3d), $h\in\mathcal{S}(w)$ *and* $m\in\mathbb{Z}$, $|m|\le 2$.

The difference Painlevé III equations (1.3a)–(1.3d) have been studied recently by Zhang and Yang [11], and Zhang and Yi [12, 13], where a number of interesting results were obtained. In particular, Zhang and Yi [12] studied the following equation:

$$\overline{w}\underline{w}(w-1)^2=w^2-\lambda w+\mu, \tag{1.4}$$

where λ and μ are constants, and obtained the following two results.

Theorem B ([12]). *Let* $w(z)=\frac{P(z)}{Q(z)}$, *where* $P(z)$ *and* $Q(z)$ *are relatively prime polynomials of degrees* p *and* q, *respectively. If* $w(z)$ *is a solution of equation* (1.4), *then one of the following assertions holds:*

(i) $p=q$, $a^2(a-1)^2=a^2-\lambda a+\mu$, *where* $a=w(\infty)$;

(ii) $p<q$, $\lambda=\mu=0$, *and* $P(z)$ *is a constant.*

Example 1.1. The rational function $w(z)=\frac{1}{(z+1)^2}$ is a solution of the difference equation $\overline{w}\underline{w}(w-1)^2=w^2$. This shows that the conclusion (ii) of Theorem B may occur.

Theorem C ([12]). *If* w *is a transcendental meromorphic solution of equation* (1.4) *of finite order* $\rho(w)$, *then the following assertions hold:*

(i) $\tau(w)=\rho(w)$;

(ii) *If* $\lambda\mu\neq 0$, *then* $\lambda(w)=\rho(w)$.

Example 1.2. The function $w(z)=\sec^2\frac{\pi z}{2}$ is a solution of the difference equation $\overline{w}\underline{w}(w-1)^2=w^2$, and 0 is a Picard exceptional value of w. This shows that the condition $\lambda\mu\neq 0$ is necessary in assertion (ii) of Theorem C.

In this paper, motivated by the above theorems and equation (1.3a), we study two difference Painlevé III equations that follow. Observe first that if in equation (1.3a) of Theorem A, $\kappa=\nu=-1$ when both μ and λ are constants, then we have at least one of μ and λ to be 0 from $\overline{\lambda}\mu=\kappa\underline{\lambda}\overline{\mu}$. So, in Section 3, we discuss the question of existence of rational solutions of the following difference Painlevé III equation:

$$\overline{w}\underline{w}(w^2-1)=w^2+\mu, \tag{1.5}$$

where μ ($\mu \neq -1$) is a constant, and investigate the value distribution. In Section 4, we discuss the same questions, that is, the existence of rational solutions and the value distribution, of the following difference Painlevé III equation:

$$\overline{w}\underline{w}(w^2-1)=w^2-\lambda w, \tag{1.6}$$

where $\lambda(\lambda \neq \pm 1)$ is a constant.

The reminder of the paper is organized as follows. In Section 2, we state a number of auxiliary lemmas, which will be used to prove our main results. In Section 3, we study the equation (1.5). Section 4 is devoted to equation (1.6).

2. Auxiliary lemmas

In this section we state a number of auxiliary lemmas, which will be used to prove our main results. We first state the following lemma, which is a difference analogue of the logarithmic derivative lemma, and reads as follows.

Lemma 2.1. *Let f be a meromorphic function of finite order, and let c be a non-zero complex constant. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right)+m\left(r, \frac{f(z)}{f(z+c)}\right)=S(r,f).$$

In view of Lemma 2.1, we can obtain the following difference analogues of the Clunie and Mohon'ko lemmas (see [7, 8]).

Lemma 2.2 ([8])**.** *Let f be a transcendental meromorphic solution of a finite order ρ for a difference equation of the form:*

$$U(z,f)P(z,f)=Q(z,f),$$

where $U(z,f)$, $P(z,f)$ and $Q(z,f)$ are difference polynomials such that the total degree $\deg_f U(z,f)=n$ in $f(z)$ and its shifts, and $\deg_f Q(z,f)\leq n$. If $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts, then, for each $\varepsilon>0$, we have

$$m\left(r, P(z,f)\right)=O\left(r^{\rho-1+\varepsilon}\right)+S(r,f),$$

possibly outside an exceptional set of a finite logarithmic measure.

Lemma 2.3 ([7, 8])**.** *Let w be a transcendental meromorphic solution of a finite order of the difference equation:*

$$P(z,w)=0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a)\not\equiv 0$ for a meromorphic function $a\in\mathcal{S}(w)$, then

$$m\left(r,\frac{1}{w-a}\right)=S(r,w).$$

Lemma 2.4 (See, e.g., [11, Theorem 3.1]). *Let w be a non-constant meromorphic solution of a finite order of equations (1.3a) – (1.3d) with constant coefficients, and let $m\neq 2$ in equation (1.3d). Then the following equalities hold:*

$$m(r,w)=S(r,w),\quad \lambda\left(\frac{1}{w}\right)=\rho(w).$$

We conclude this section by the following lemma.

Lemma 2.5 (See, e.g., [5, pp. 79–80]). *Let f_j $(j=1,\dots,n)$ $(n\geq 2)$ be meromorphic functions, and let g_j $(j=1,\dots,n)$ be entire functions. Assume that the following conditions are fulfilled:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)}\equiv 0$;
(ii) $g_h(z)-g_k(z)$ *is not a constant for* $1\leq h<k\leq n$;
(iii) $T(r,f_j)=S(r,e^{g_h(z)-g_k(z)})$ *for* $1\leq j\leq n$ *and* $1\leq h<k\leq n$.

Then $f_j(z)\equiv 0$, $j=1,\dots,n$.

3. Equation (1.5)

Theorem 3.1. *There is no any non-constant rational solution of equation (1.5).*

Proof. Assume the opposite that $w(z)=\frac{P(z)}{Q(z)}$ is a non-constant rational solution of equation(1.5), where $P(z)$ and $Q(z)$ are relatively prime polynomials of degrees p and q, respectively. Also, we assume that the leading coefficient of $P(z)$ is a $(a\neq 0)$ and the leading coefficient of $Q(z)$ is 1. Substituting $w(z)=\frac{P(z)}{Q(z)}$ into (1.5), we get

$$\frac{P(z+1)}{Q(z+1)}\frac{P(z-1)}{Q(z-1)}\left(\left(\frac{P(z)}{Q(z)}\right)^2-1\right)=\left(\frac{P(z)}{Q(z)}\right)^2+\mu. \tag{3.1}$$

We set $s=p-q$, and discuss the following three possible cases.

Case 1. Let $s>0$. Then $\frac{P(z)}{Q(z)}=az^s(1+o(1))$ as z tends to infinite and from (3.1), we get

$$a^2(z+1)^s(z-1)^s(1+o(1))\left(a^2z^{2s}(1+o(1))-1\right)=a^2z^{2s}(1+o(1))+\mu,$$

which is a contradiction as z tends to infinite.

Case 2. Let $s<0$. Now we have $\frac{P(z)}{Q(z)}=o(1)$ and $\frac{P(z+1)}{Q(z+1)}=o(1)$ as z tends to infinite. By (3.1), we obtain $\mu=0$. From (1.5), when $\mu=0$, we have

$$\overline{w}\underline{w}=\frac{w^2}{w^2-1}.$$

Let $w(z) = \frac{1}{f(z)}$. Substituting $w = \frac{1}{f}$ into the above equation, we obtain

$$\overline{f}\underline{f} = 1 - f^2.$$

Observing that the coefficients on the left- and right-hand sides of the above equation are $\frac{1}{a^2}$ and $-\frac{1}{a^2}$, respectively, we get $\frac{2}{a^2} = 0$, which is impossible.

Case 3. Let $s = 0$. Then $w(z) = \frac{P(z)}{Q(z)} = a + o(1)$ as z tends to infinity and from (3.1), we get

$$a^2(a^2 - 1) = a^2 + \mu, \tag{3.2}$$

where $a \notin \{0, \pm 1\}$. We rewrite (3.1) as follows:

$$\frac{P(z+1)}{Q(z+1)}\frac{P(z-1)}{Q(z-1)} = \frac{P^2(z) + \mu Q^2(z)}{P^2(z) - Q^2(z)}.$$

We assume that there is a point z_0 such that $P^2(z_0) + \mu Q^2(z_0) = 0$ and $P^2(z_0) - Q^2(z_0) = 0$. Since $\mu \neq -1$, we obtain $P(z_0) = 0$ and $Q(z_0) = 0$, which is a contradiction. Thus, the degrees of $P^2(z) + \mu Q^2(z)$ and $P^2(z) - Q^2(z)$ both are $2p$, and we have

$$(a^2 + \mu)\overline{P}\underline{P} = a^2(P^2 + \mu Q^2), \tag{3.3}$$

$$(a^2 - 1)\overline{Q}\underline{Q} = P^2 - Q^2. \tag{3.4}$$

Next, we assume $P = ar$, $p = n$. Then from (3.3) we have that

$$\mu Q^2 = \overline{r}\underline{r}\left(a^2 + \mu\right) - r^2 a^2, \tag{3.5}$$

where

$$r = z^n + A_{n-1}z^{n-1} + A_{n-2}z^{n-2} + A_{n-3}z^{n-3} + \cdots + A_1 z + A_0, \tag{3.6}$$

$$Q = z^n + B_{n-1}z^{n-1} + B_{n-2}z^{n-2} + B_{n-3}z^{n-3} + \cdots + B_1 z + B_0. \tag{3.7}$$

We rewrite (3.4) as follows:

$$(a^2 - 1)\overline{Q}\underline{Q} + Q^2 = P^2. \tag{3.8}$$

Substituting (3.6) and (3.7) into (3.5) and comparing the coefficients of terms z^{2n}, z^{2n-1}, z^{2n-2}, we obtain the following two equations:

$$\mu(B_{n-1} - A_{n-1}) = 0,$$
$$\mu(B_{n-1}^2 + 2B_{n-2}) = \mu(A_{n-1}^2 + 2A_{n-2} - n) - a^2 n.$$

If $\mu = 0$, then from the last equation we get $a^2 n = 0$, which is a contradiction. If $\mu \neq 0$, then the last two equations become

$$B_{n-1} = A_{n-1}, \quad B_{n-2} = A_{n-2} - \frac{n\left(a^2 + \mu\right)}{2\mu}.$$

By the same way, we substitute (3.6) and (3.7) into (3.8), and compare the coefficients of terms z^{2n}, z^{2n-1}, z^{2n-2}, to obtain

$$B_{n-1} = A_{n-1}, \quad B_{n-2} = A_{n-2} - \frac{n\left(1-a^2\right)}{2a^2}.$$

So, we get $a^4 = \mu(1-2a^2)$. On the other hand, from (3.2) we have $\mu = a^2\left(a^2-2\right)$. It is obvious that $a^2 = 1$, which is a contradiction. □

Theorem 3.2. *If w is a transcendental meromorphic solution of equation* (1.5) *of a finite order $\rho(w) > 0$, then the following assertions hold:*

(i) $\lambda\left(\frac{1}{w}\right) = \tau(w) = \rho(w)$;

(ii) *when $\mu \neq 0$, we have $\lambda(w) = \rho(w)$;*

(iii) *w has at most one non-zero Borel exceptional value.*

Proof. Denote $\phi(z) = w(z) - z$, and observe that $\phi(z)$ is a transcendental meromorphic function and $T(r,\phi) = T(r,w) + S(r,w)$. Substituting $w(z) = \phi(z) + z$ into (1.5), we obtain

$$\left(\overline{\phi} + z + 1\right)\left(\underline{\phi} + z - 1\right)\left((\phi+z)^2 - 1\right) = (\phi+z)^2 + \mu.$$

Denote

$$P(z,\phi) = \left(\overline{\phi} + z + 1\right)\left(\underline{\phi} + z - 1\right)\left((\phi+z)^2 - 1\right) - (\phi+z)^2 - \mu,$$

and observe that $P(z,0) = \left(z^2-1\right)^2 - z^2 - \mu \not\equiv 0$. From Lemma 2.3, we get

$$m\left(r, \frac{1}{w-z}\right) = m\left(r, 1/\phi\right) = S(r,\phi),$$

implying that $N\left(r, \frac{1}{w-z}\right) = T\left(r,w\right) + S(r,w)$, and hence $\tau(w) = \rho(w)$.

In view of Lemma 2.4 we have $m(r,w) = S(r,w)$. Then, the equality $\lambda\left(\frac{1}{w}\right) = \rho(w)$ holds.

To prove the assertion (ii), for $\mu \neq 0$, we denote

$$P_1(z,w) = \overline{w}\underline{w}(w^2-1) - w^2 - \mu,$$

and observe that $P_1(z,0) = -\mu \not\equiv 0$. Then, from Lemma 2.3, we obtain $m(r,1/w) = S(r,w)$, implying that $\lambda(w) = \rho(w)$.

Now we proceed to prove the assertion (iii) of the theorem. To this end, we assume that a and b are two non-zero finite Borel exceptional values of w, and set

$$f(z) = \frac{w(z)-a}{w(z)-b}. \tag{3.9}$$

Then, we have $\rho(f) = \rho(w)$, $\lambda(f) = \lambda(w-a) < \rho(f)$ and $\lambda(1/f) = \lambda(w-b) < \rho(f)$. Since f is of finite order, we suppose that

$$f(z) = g(z)e^{dz^n}, \tag{3.10}$$

where d ($d \neq 0$) is a constant, n ($n \geq 1$) is an integer, and $g(z)$ is a meromorphic function satisfying the condition:

$$\rho(g) < \rho(f) = n. \tag{3.11}$$

Then, we have

$$f(z+1) = g(z+1)g_1(z)e^{dz^n}, \quad f(z-1) = g(z-1)g_2(z)e^{dz^n}, \tag{3.12}$$

where $g_1(z) = e^{ndz^{n-1}+\cdots+d}$ and $g_2(z) = e^{-ndz^{n-1}+\cdots+(-1)^n d}$. From (3.9) we get $w = \dfrac{bf-a}{f-1}$. Next, in view of (1.5), (3.9) to (3.12), we can write

$$A(z)e^{4dz^n} + B(z)e^{3dz^n} + C(z)e^{2dz^n} + D(z)e^{dz^n} + E = 0, \tag{3.13}$$

where

$$\begin{aligned}
A(z) &= \left[b^4 - 2b^2 - \mu\right] g^2\overline{g}g_1\underline{g}g_2,\\
B(z) &= \left[-2b^2(ab-1) + 2ab + 2\mu\right] g\overline{g}g_1\underline{g}g_2\\
&\quad + \left[-ab(b^2-1) + b^2 + \mu\right] g^2(\overline{g}g_1 + \underline{g}g_2),\\
C(z) &= \left[b^2(a^2-1) - a^2 - \mu\right] \overline{g}\underline{g}g_1g_2 + \left[a^2b^2 - a^2 - b^2 - \mu\right] g^2\\
&\quad - \left[-2ab(ab-1) + 2ab + 2\mu\right] g(\overline{g}g_1 + \underline{g}g_2),\\
D(z) &= \left[-a^3b + ab + a^2 + \mu\right] (\overline{g}g_1 + \underline{g}g_2) + 2(-a^3b + a^2 + ab + \mu)g,\\
E &= a^4 - 2a^2 - \mu.
\end{aligned}$$

Applying Lemma 2.5 to (3.13) and taking into account (3.11), we see that all the coefficients vanish. Since a and b are non-zero constants, we deduce from $A(z) = 0$ and $E = 0$ that

$$a^4 - 2a^2 = \mu, \;\; b^4 - 2b^2 = \mu. \tag{3.14}$$

Then, we have $(a^2 - b^2)(a^2 + b^2 - 2) = 0$. Now we discuss the following two cases.

Case 1. Let $a^2 = b^2$. Due to $a \neq b$, we get $a = -b$. Denote $G = g$, $G_1 = \overline{g}g_1$ and $G_2 = \underline{g}g_2$. From $B(z) = 0$, $D(z) = 0$, we have

$$\begin{gathered}
2(b^4 + \mu)G_1G_2 = (-b^4 - \mu)G(G_1 + G_2),\\
2(a^4 + \mu)G = (-a^4 - \mu)(G_1 + G_2).
\end{gathered}$$

Noting that $\mu \neq -1$, we get $b^4 + \mu \neq 0$ and $a^4 + \mu \neq 0$ by (3.14). Thus, we have

$$2G_1G_2 = -G(G_1 + G_2), \;\; 2G = -(G_1 + G_2).$$

From the last two equations, we obtain

$$G^2 = G_1 G_2,\ 4G_1 G_2 = (G_1 + G_2)^2.$$

So, we have $-G = G_1 = G_2$ and $\overline{f} = \underline{f} = -f$. From (3.9), the equality $a = -b$ and the above equation, we get

$$\overline{w} = \underline{w} = \frac{a^2}{w}.$$

Hence, from (1.5) we get $a^4(w^2 - 1) = w^4 + \mu w^2$. Therefore, w is a constant, which is a contradiction.

Case 2. Let $a^2 + b^2 = 2$. When $B(z) = 0$ and $D(z) = 0$, then using arguments similar to those applied in Case 1, we get

$$2G_1 G_2 = -G(G_1 + G_2),\ 2G = -(G_1 + G_2).$$

Noting that $\mu \neq -1$, the above equations also lead to a contradiction by the similar reasoning as in Case 1. This completes the proof of the theorem. □

4. Equation (1.6)

Theorem 4.1. *Let $w(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are relatively prime polynomials of degrees p and q, respectively. If $w(z)$ is a non-constant rational solution of equation* (1.6)*, then*

$$p = q,\ a(a^2 - 1) = a - \lambda, \text{ where } a = \pm\frac{\sqrt{6}}{3},\ \lambda = \frac{4a}{3}.$$

Proof. For $p \neq q$, the proof of the theorem is similar to that of Cases 1 and 2 in Theorem 3.1, so we only prove the theorem for $p = q$. We assume that the leading coefficient of $P(z)$ is a ($a \neq 0$), and the leading coefficient of $Q(z)$ is 1. Substituting $w(z) = \frac{P(z)}{Q(z)}$ into (1.6), we get

$$\frac{P(z+1)}{Q(z+1)}\frac{P(z-1)}{Q(z-1)}\left(\left(\frac{P(z)}{Q(z)}\right)^2 - 1\right) = \left(\frac{P(z)}{Q(z)}\right)^2 - \lambda\frac{P(z)}{Q(z)}. \tag{4.1}$$

When $p = q$, we have $\frac{P(z)}{Q(z)} = a + o(1)$ and $\frac{P(z+1)}{Q(z+1)} = a + o(1)$ as z tends to infinite. Then, from (4.1) we get the following equation

$$a(a^2 - 1) = a - \lambda, \tag{4.2}$$

where $a \notin \{0, \pm 1\}$.

We rewrite (4.1) as follows:

$$\frac{P(z+1)}{Q(z+1)}\frac{P(z-1)}{Q(z-1)} = \frac{P^2(z) - \lambda P(z)Q(z)}{P^2(z) - Q^2(z)}.$$

Arguments, similar to those applied in the proof of Theorem 3.1 (Case 3), can be used to conclude that the degrees of $P^2(z) - \lambda P(z)Q(z)$ and $P^2(z) - Q^2(z)$ both are $2p$ for $\lambda \neq \pm 1$. Hence, we have

$$(a^2 - \lambda a)\overline{P}\underline{P} = a^2(P^2 - \lambda PQ), \tag{4.3}$$

$$(a^2 - 1)\overline{Q}\underline{Q} = P^2 - Q^2. \tag{4.4}$$

Next, we assume $P = ar$, $p = n$, and use (4.3) to obtain

$$\lambda r Q = \overline{r}\underline{r}\,(\lambda - a) + ar^2, \tag{4.5}$$

where

$$r = z^n + A_{n-1}z^{n-1} + A_{n-2}z^{n-2} + A_{n-3}z^{n-3} + \cdots + A_1 z + A_0, \tag{4.6}$$

$$Q = z^n + B_{n-1}z^{n-1} + B_{n-2}z^{n-2} + B_{n-3}z^{n-3} + \cdots + B_1 z + B_0. \tag{4.7}$$

We rewrite (4.4) as follows:

$$(a^2 - 1)\overline{Q}\underline{Q} + Q^2 = P^2. \tag{4.8}$$

Substituting (4.6) and (4.7) into (4.5) and comparing the coefficients of terms z^{2n}, z^{2n-1}, z^{2n-2}, we obtain the following two equations:

$$\lambda(B_{n-1} - A_{n-1}) = 0,$$
$$\lambda(B_{n-2} + A_{n-1}B_{n-1} + A_{n-2}) = \lambda(A_{n-1}^2 + 2A_{n-2} - n) + an.$$

For $\lambda = 0$, from the last equation we get $an = 0$, which is a contradiction. For $\lambda \neq 0$, the last two equations become

$$B_{n-1} = A_{n-1}, \quad B_{n-2} = A_{n-2} + \frac{n\,(a - \lambda)}{\lambda}.$$

By the same way, we substitute (4.6) and (4.7) into (4.8), and compare the coefficients of terms z^{2n}, z^{2n-1}, z^{2n-2}, to obtain

$$B_{n-1} = A_{n-1}, \quad B_{n-2} = A_{n-2} + \frac{n\left(a^2 - 1\right)}{2a^2}.$$

So, we get $2a^3 = \lambda(3a^2 - 1)$. And from (4.2), we have $\lambda = 2a - a^3$. By the above equations, we have $\left(3a^2 - 2\right)\left(a^2 - 1\right) = 0$. Since $a^2 \neq 1$, we get $a = \pm\frac{\sqrt{6}}{3}$ and $\lambda = \pm\frac{4\sqrt{6}}{9}$. Therefore $\dfrac{\lambda}{a} = \dfrac{4}{3}$. □

Theorem 4.2. *If w is a transcendental meromorphic solution of equation* (1.6) *of a finite order $\rho(w) > 0$, then the following assertions hold:*

(i) $\lambda\left(\dfrac{1}{w}\right) = \tau(w) = \rho(w)$*;*

(ii) *when $\lambda \neq 0$, we have $\lambda(w) = \rho(w)$;*

(iii) *w has at most one non-zero Borel exceptional value.*

Proof. Denote $\phi(z) = w(z) - z$, and observe that $\phi(z)$ is a transcendental meromorphic function and $T(r,\phi) = T(r,w) + S(r,w)$. Substituting $w(z) = \phi(z) + z$ into (1.6), we obtain $\left(\overline{\phi} + z + 1\right)\left(\underline{\phi} + z - 1\right)\left((\phi + z)^2 - 1\right) = (\phi + z)^2 - \lambda\,(\phi + z)$. Denote

$$P(z,\phi) = \left(\overline{\phi} + z + 1\right)\left(\underline{\phi} + z - 1\right)\left((\phi + z)^2 - 1\right) - (\phi + z)^2 + \lambda\,(\phi + z)\,,$$

and observe that $P(z,0) = \left(z^2 - 1\right)^2 - z^2 + \lambda z \not\equiv 0$. Then, from Lemma 2.3, we obtain

$$m\left(r, \frac{1}{w - z}\right) = m\left(r, 1/\phi\right) = S(r,\phi),$$

implying that $N\left(r, \frac{1}{w - z}\right) = T\left(r,w\right) + S(r,w)$, and hence $\tau(w) = \rho(w)$.

We deduce from Lemma 2.4 that $m(r,w) = S(r,w)$. Then, the equality $\lambda\left(\frac{1}{w}\right) = \rho(w)$ holds.

To prove the assertion (ii), for $\lambda \neq 0$, we rewrite (1.6) as follows:

$$\overline{w}\underline{w} = \frac{w^2 - \lambda w}{w^2 - 1}.$$

Let $w(z) = \frac{1}{f(z)}$. Substituting $w = \frac{1}{f}$ into the last equality, we get

$$\overline{f}\underline{f}f\lambda = \overline{f}\underline{f} - 1 + f^2.$$

From Lemma 2.2, we obtain $m(r, 1/w) = S(r,w)$. Therefore, $\lambda(w) = \rho(w)$.

Now we proceed to prove the assertion (iii) of the theorem. To this end, we assume that a and b are two non-zero finite Borel exceptional values of w, and set

$$f(z) = \frac{w(z) - a}{w(z) - b}. \tag{4.9}$$

Then, we have $\rho(f) = \rho(w)$, $\lambda(f) = \lambda(w - a) < \rho(f)$ and $\lambda(1/f) = \lambda(w - b) < \rho(f)$. Since f is of finite order, we suppose that

$$f(z) = g(z)e^{dz^n}, \tag{4.10}$$

where d $(d \neq 0)$ is a constant, n $(n \geq 1)$ is an integer, and $g(z)$ is a meromorphic function satisfying the condition:

$$\rho(g) < \rho(f) = n. \tag{4.11}$$

Then, we have

$$f(z+1) = g(z+1)g_1(z)e^{dz^n}, \quad f(z-1) = g(z-1)g_2(z)e^{dz^n}, \tag{4.12}$$

where $g_1(z) = e^{ndz^{n-1} + \cdots + d}$ and $g_2(z) = e^{-ndz^{n-1} + \cdots + (-1)^n d}$. From (4.9), we get $w = \dfrac{bf - a}{f - 1}$. In view of (1.6), (4.9) to (4.12), we can write

$$A(z)e^{4dz^n} + B(z)e^{3dz^n} + C(z)e^{2dz^n} + D(z)e^{dz^n} + E = 0, \tag{4.13}$$

where

$$\begin{aligned}
A(z) &= \left[b^4-2b^2+b\lambda\right] g^2\overline{g}g_1\underline{g}g_2,\\
B(z) &= \left[-2b^2(ab-1)+2ab-\lambda(a+b)\right] g\overline{g}g_1\underline{g}g_2\\
&\quad+\left[-ab(b^2-1)+b(b-\lambda)\right] g^2(\overline{g}g_1+\underline{g}g_2),\\
C(z) &= \left[b^2(a^2-1)-a^2+a\lambda\right] \overline{g}\underline{g}g_1g_2+\left[a^2b^2-a^2-b^2+b\lambda\right] g^2\\
&\quad+\left[2ab(ab-1)-2ab+\lambda(a+b)\right] g(\overline{g}g_1+\underline{g}g_2),\\
D(z) &= \left[-2a^3b+2ab+2a^2-\lambda(a+b)\right] g+(-a^3b+a^2+ab-a\lambda)(\overline{g}g_1+\underline{g}g_2),\\
E &= a^4-2a^2+a\lambda.
\end{aligned}$$

Applying Lemma 2.5 to (4.13) and taking into account (4.11), we see that all the coefficients vanish. Since a and b are non-zero constants, we deduce from $A(z)=0$ and $E=0$ that

$$a^3-2a=-\lambda,\ b^3-2b=-\lambda. \tag{4.14}$$

Then, we have $(a-b)(a^2+ab+b^2-2)=0$. Since $a\neq b$, it follows that $a^2+b^2+ab=2$. By (4.14), a and b are distinct zeros of the equation $z^3-2z+\lambda=0$.

According to the algebraic basic theorem, the above equation has three solutions. Denoting by x the third solution, and using the relationship between roots and coefficients, we obtain $abx=-\lambda$, $ab+ax+bx=-2$, $a+b+x=0$, implying that

$$x=-\frac{\lambda}{ab},\ a+b=-x=\frac{\lambda}{ab},\ ab+(a+b)x=ab-\frac{\lambda^2}{a^2b^2}=-2.$$

So, we have

$$ab(a+b)=\lambda,\ 2ab+a^2b^2=(a+b)\lambda, a^2+b^2+ab=2.$$

Denote $G=g$, $G_1=\overline{g}g_1$ and $G_2=\underline{g}g_2$. From $B(z)=0$, $D(z)=0$ and the above equations, we have

$$\begin{aligned}
(2b^2-2ab^3-a^2b^2)G_1G_2 &= (2ab^3+a^2b^2-ab-b^2)G(G_1+G_2),\\
(2a^2-2a^3b-a^2b^2)G &= (2a^3b+a^2b^2-ab-a^2)(G_1+G_2).
\end{aligned}$$

Because

$$\frac{G_1G_2}{G(G_1+G_2)}=\frac{2ab^3+a^2b^2-ab-b^2}{2b^2-2ab^3-a^2b^2}=\frac{b^2-ab}{2b^2-2ab^3-a^2b^2}-1.$$

By $a^2+b^2+ab=2$, we gain $2b^2-2ab^3-a^2b^2=b^3(b-a)$, and hence, we have

$$\frac{G_1G_2}{G(G_1+G_2)}=\frac{1}{b^2}-1.$$

Thus, we get

$$G_1G_2=\left(\frac{1}{b^2}-1\right)G(G_1+G_2),\ G=\left(\frac{1}{a^2}-1\right)(G_1+G_2).$$

Noting that $\lambda \neq \pm 1$, by (4.14), we get $a^2 \neq 1$ and $b^2 \neq 1$. Moreover, since the last two equations are homogeneous, there exist two non-zero constants α and β, such that $G_1 = \alpha G$ and $G_2 = \beta G$. Then, we have

$$\alpha\beta = \frac{a^2 - a^2b^2}{b^2 - a^2b^2}. \tag{4.15}$$

On the other hand, combining (4.10) and (4.12), we get $\overline{f} = \alpha f$, $\underline{f} = \beta f$, which yields $\alpha\beta = 1$. Thus, by (4.15), we have $a^2 = b^2$. When $a = b$, then we get a contradiction. So, we have only to consider the case $a = -b$. From $B(z) = 0$, $D(z) = 0$ and $a = -b$, we have

$$\begin{aligned} 2b^4G_1G_2 &= (-b^4 + b\lambda)G(G_1 + G_2), \\ 2a^4G &= (-a^4 + a\lambda)(G_1 + G_2), \end{aligned} \tag{4.16}$$

implying that

$$(-b^4 - b\lambda)G_1G_2 = (-b^4 + b\lambda)G^2. \tag{4.17}$$

Since the last equation is homogeneous, there exist two non-zero constants α and β, such that $G_1 = \alpha G$ and $G_2 = \beta G$. Then, we have

$$\alpha\beta(b^3 + \lambda) = b^3 - \lambda. \tag{4.18}$$

On the other hand, combining (4.11) and (4.13), we get $\overline{f} = \alpha f$, $\underline{f} = \beta f$, which yields $\alpha\beta = 1$. Thus by (4.18), we have $\lambda = 0$, and, in view of (4.16) and (4.17), we infer that $2G = -(G_1 + G_2)$ and $G_1G_2 = G^2$. Then, $G_1 = G_2 = -G$. Thus, we have $\alpha = \beta = -1$ and $\overline{f} = \underline{f} = -f$, and by the similar reasoning as in Case 1 of the proof of Theorem 3.1, we get a contradiction. □

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