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SOLVABILITY OF FRACTIONAL MULTI-POINT BOUNDARY VALUE PROBLEMS WITH NONLINEAR GROWTH AT RESONANCE

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Abstract. This work is concerned with the solvability of multi-point boundary value problems for fractional differential equations with nonlinear growth at the resonance. Existence results are obtained with the use of the coincidence degree theory. As an application, we discuss an example to illustrate the obtained results.

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1. Introduction

This paper is devoted to the solvability of the following fractional multi-point boundary value problems (BVPs) at the resonance

$$
\begin{cases} \left(\phi(t)^C D^\alpha_{0+}u(t)\right)' = f\left(t, u(t), u'(t), u''(t), {}^C D^\alpha_{0+}u(t)\right), \quad t \in I = [0,1], \\ u(0) = 0,\ {}^C D^\alpha_{0+}u(0) = 0,\ u''(0) = \sum_{i=1}^{m} a_i u''(\xi_i),\ u'(1) = \sum_{j=1}^{l} b_j u'(\eta_j), \end{cases} \tag{1.1}
$$

where ${}^C D^\alpha_{0+}$ is the Caputo fractional derivative, $2 < \alpha \le 3$, $0 < \xi_1 < \cdots < \xi_m < 1$, $0 < \eta_1 < \cdots < \eta_l < 1$, $a_i, b_j \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, l$, $\phi(t) \in C^1([0,1])$, and $\mu = \min_{t\in I}\phi(t) > 0$. The nonlinearity is such that the following conditions are satisfied:

(H_0) $f : [0,1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is a Carathéodory function, that is,

(i) for each $x \in \mathbb{R}^4$, the function $t \longrightarrow f(t,x)$ is Lebesgue measurable;

(ii) for almost every $t \in [0,1]$, the function $t \longrightarrow f(t,x)$ is continuous on $\mathbb{R}^4$;

(iii) for each $r > 0$, there exists $\varphi_r(t) \in L^1([0,1], \mathbb{R})$ such that for a.e. $t \in [0,1]$ and every $|x| \le r$, we have $|f(t,x)| \le \varphi_r(t)$.

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The resonant conditions of (1.1) are as follows:

(H_1) $\sum_{i=1}^{m} a_i = 1, \quad \sum_{j=1}^{l} b_j = 1, \quad \sum_{j=1}^{l} b_j \eta_j = 1.$

This means that the linear operator $Lu = \left(\phi^C D_{0+}^{\alpha} u\right)'$ corresponding to the problem (1.1) has a nontrivial solution or, in a functional framework, L is not invertible, that is, dim kerL ≥ 1.

In order to be sure that the linear operator Q (to be specified later on) is well defined, we assume, in addition, that

(H_2) There exist $p, q \in \mathbb{Z}^+, q \geq p+1$ such that $\Delta(p,q) = d_{11}d_{22} - d_{12}d_{21}$, where

$$d_{11} = \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{s^p(\xi_i - s)^{\alpha-3}}{p\phi(s)} ds, \quad d_{21} = \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{s^q(\xi_i - s)^{\alpha-3}}{q\phi(s)} ds,$$

$$d_{12} = \int_0^1 \frac{s^p(1-s)^{\alpha-2}}{p\phi(s)} ds - \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{s^p(\eta_j - s)^{\alpha-2}}{p\phi(s)} ds,$$

$$d_{22} = \int_0^1 \frac{s^q(1-s)^{\alpha-2}}{q\phi(s)} ds - \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{s^q(\eta_j - s)^{\alpha-2}}{q\phi(s)} ds.$$

Note that $\Delta(p,q) \neq 0$ (see [19, 23]).

Fractional calculus is an extension of the ordinary differentiation and integration to arbitrary non-integer order. In particular, time fractional differential equations are used when attempting to describe the transport processes with long memory. Recently, the study of time fractional ordinary and partial differential equations has been received great attention by many researchers, both in theory and in applications. We refer the reader to the monographs [1, 2, 20, 26, 30, 34], the papers [35] – [39], and the references therein. The question of existence of solutions for fractional boundary-value problems at the resonance case has been extensively studied by many authors (see [5] – [8, 10, 12, 13, 14, 17, 18, 21, 22, 32], and the references therein. It is worth to mention that there are a number of papers dealing with the solutions of multi-point boundary value problems of fractional differential equations at the resonance (see [7, 8, 10, 17]).

In [8], Bai and Zhang considered a three-point boundary value problem of fractional differential equations with nonlinear growth given by

$$D_{0+}^{\alpha} u(t) = f\big(t, u(t), D_{0+}^{\alpha-1} u(t)\big), \quad t \in [0,1],$$

$$u(0) = 0, \quad u(1) = \sigma u(\eta),$$

where D_{0+}^{α} is the standard Riemann-Liouville derivative, $1 < \alpha \leq 2$, $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and $\sigma \in (0, \infty)$, $\eta \in (0, 1)$ are given constants such that $\sigma\eta^{\alpha-1} = 1$. The authors applied the coincidence degree theorem to prove existence of solutions. In [10], Chen and Tang have studied the following class of multi-point boundary value problems for fractional differential equations at the resonance by employing the coincidence degree theorem:

$$\Big(a(t)^C D^{\alpha}_{0^+}u(t)\Big)' = f\big(t, u(t), u'(t), {}^C D^{\alpha}_{0^+}u(t)\big), \quad t \in J,$$
$$u(0) = 0, \quad {}^C D^{\alpha}_{0^+}u(0) = 0, \quad u(1) = \sum_{j=1}^{m-1} \sigma_j u(\xi_j),$$

where $1 < \alpha \leq 2$, $f : [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions, $a(t) \in C^1([0,1])$, $\min_{t\in J} a(t) > 0$, $J = [0,1]$, $\sigma_j \in \mathbb{R}^*_+$, $\xi_j \in (0,1)$, $j = 1, \dots, m-1$, $m \in \mathbb{N}$, $m > 1$, and $\sum_{j=1}^{m-1} \sigma_j \xi_j = 1$. The results are obtained under the assumption that

$$\Lambda_0 = \sum_{j=1}^{m-1} \sigma_j \left(\xi_j \int_0^1 s(1-s)^{\alpha-1} \frac{1}{\phi(s)} ds - \int_0^{\xi_j} s(\xi_j - s)^{\alpha-1} \frac{1}{\phi(s)} ds \right) \neq 0.$$

In [7], Bai and Zhang considered the solvability of the following fractional multi-point boundary value problems at the resonance with dim kerL $= 2$ by applying the coincidence degree theorem:

$$D^{\alpha}_{0^+}u(t) = f\big(t, u(t), D^{\alpha-2}_{0^+}u(t), D^{\alpha-1}_{0^+}u(t)\big), \quad t \in (0,1),$$
$$I^{\alpha-1}_{0^+}u(0) = 0, \quad D^{\alpha-1}_{0^+}u(0) = D^{3-\alpha}_{0^+}(\eta), \quad u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i),$$

where $2 < \alpha < 3$, $0 < \eta \leq 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, $m \geq 2$, $\sum_{i=1}^m \alpha_i \eta_i^{\alpha-1} = \sum_{i=1}^m \alpha_i \eta_i^{\alpha-2} = 1.$ $D^{\alpha}_{0^+}$ and $I^{\alpha}_{0^+}$ are the standard Riemann-Liouville fractional derivative and the fractional integral, respectively, and $f : [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions. The results are obtained under the assumption that

$$R = \frac{1}{\alpha}\eta^{\alpha} \frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} \Big[1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-2}\Big] - \frac{1}{\alpha-1}\eta^{\alpha-1} \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)} \Big[1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-1}\Big] \neq 0.$$

Jiang [17], by using the coincidence degree theorem, has obtained an existence result for the boundary value problems of fractional differential equations at the resonance with dim kerL $= 2$:

$$D^{\alpha}_{0^+}u(t) = f\big(t, u(t), D^{\alpha-1}_{0^+}u(t)\big), \quad \forall t \in J = [0,1],$$
$$u(0) = 0, \quad D^{\alpha-1}_{0^+}u(0) = \sum_{i=1}^{m} a_i D^{\alpha-1}_{0^+}(\xi_i), \quad D^{\alpha-2}_{0^+}u(0) = \sum_{j=1}^{n} b_j D^{\alpha-2}_{0^+}(\eta_j),$$

where $2 < \alpha < 3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, $\sum_{i=1}^m a_i = 1$, $\sum_{j=1}^n b_j = 1$, $\sum_{j=1}^n b_j \eta_j = 1$, and $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions. The results are obtained under the assumption that

$$\frac{1}{3}\Big(1 - \sum_{j=1}^{n} b_j \eta_j^3\Big) \sum_{i=1}^{m} a_i \xi_i - \frac{1}{2}\Big(1 - \sum_{j=1}^{n} b_j \eta_j^2\Big) \sum_{i=1}^{m} a_i \xi_i^2 \neq 0.$$

In this paper, we study problem (1.1), which allow f to have a nonlinear growth.

The rest of the paper is organized as follows. In Section 2, we introduce some notation, definitions and preliminary results, which will be used in the proofs of our main results (see [1, 2, 20, 26, 27, 28, 30, 34]). In Section 3, we state and prove our main results by applying the coincidence degree theorem. In Section 4 we provide an example.

2. Preliminaries

Definition 2.1. *Let $\alpha > 0$. For a function $u : (0, \infty) \longrightarrow \mathbb{R}$, the Riemann-Liouville fractional integral of order α of u is defined by*

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Remark 2.1. *The notation $I_{0+}^{\alpha} u(t) \mid_{t=0}$ means that the limit is taken at almost all points of the right-sided neighborhood $(0, \varepsilon)(\varepsilon > 0)$ of 0 as follows:*

$$I_{0+}^{\alpha} u(t) \mid_{t=0} = \lim_{t \to 0+} I_{0+}^{\alpha} u(t).$$

Generally, $I_{0+}^{\alpha} u(t) \mid_{t=0}$ is not necessarily equal to zero. For instance, let $\alpha \in (0,1)$ and $u(t) = t^{-\alpha}$. Then we have

$$I_{0+}^{\alpha} t^{-\alpha}|_{t=0} = \lim_{t \to 0+} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\alpha} ds = \Gamma(1-\alpha).$$

Definition 2.2. *Let $\alpha > 0$ and $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α. The Caputo fractional derivative of order α of a function $u : (0, \infty) \longrightarrow \mathbb{R}$ is given by*

$${}^{C}D_{0+}^{\alpha} u(t) = I_{0+}^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.2. *Let $\alpha, \eta > 0$ and $n = [\alpha] + 1$. Then the following relations hold:*

$${}^{C}D_{0+}^{\alpha} t^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)} t^{\eta-\alpha}, \quad (\eta > n-1),$$

and ${}^{C}D_{0+}^{\alpha} t^{k} = 0$, $(k = 0, \dots, n-1)$.

Lemma 2.3. *Let $\alpha, \beta \geq 0$, and $u \in L^1\big([0,1]\big)$. Then $I_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\alpha+\beta} u(t)$ and ${}^{C}D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$, for all $t \in [0,1]$*

Lemma 2.4. *Let $\alpha > 0$ and $n = [\alpha] + 1$, then*

$$I_{0+}^{\alpha}\, {}^{C}D_{0+}^{\alpha} u(t) = u(t) + \sum_{k=0}^{n-1} c_k t^k, \quad c_k \in \mathbb{R}.$$

Lemma 2.5. *Let $\alpha > 0$ and $n = [\alpha] + 1$. If ${}^{C}D^{\alpha}_{0^+}u(t) \in C[0,1]$, then $u(t) \in C^{n-1}([0,1])$.*

Proof. Let $v(t) \in C[0,1]$ be such that ${}^{C}D^{\alpha}_{0^+}u(t) = v(t)$. Then by Lemma 2.3, we have

$$u(t) = I^{\alpha}_{0^+}v(t) + \sum_{k=0}^{n-1} c_k t^k, \quad c_k \in \mathbb{R}.$$

It is easy to check that $u(t) \in C^{n-1}([0,1])$. □

Lemma 2.6. *Let $\alpha > 0$ and $u \in L^1\big([0,1],\mathbb{R}\big)$. Then for all $t \in [0,1]$ we have*

$$I^{\alpha+1}_{0^+}u(t) \leq \|I^{\alpha}_{0^+}u\|_{L^1}.$$

Proof. Let $u \in L^1\big([0,1],\mathbb{R}\big)$, then by Lemma 2.3 we have

$$I^{\alpha+1}_{0^+}u(t) = I^{1}_{0^+}I^{\alpha}_{0^+}u(t) = \int_0^t I^{\alpha}_{0^+}u(s)ds \leq \int_0^1 |I^{\alpha}_{0^+}u(s)|ds = \|I^{\alpha}_{0^+}u\|_{L^1}.$$

□

Lemma 2.7. *The fractional integral $I^{\alpha}_{0^+}$, $\alpha > 0$ is bounded in $L^1\big([0,1],\mathbb{R}\big)$, and*

$$\|I^{\alpha}_{0^+}u\|_{L^1} \leq \frac{\|u\|_{L^1}}{\Gamma(\alpha+1)}.$$

Proof. Let $u \in L^1\big([0,1],\mathbb{R}\big)$, then can write

$$\begin{aligned}\|I^{\alpha}_{0^+}u\|_{L^1} &= \int_0^1 |I^{\alpha}_{0^+}u(t)|dt \leq \frac{1}{\Gamma(\alpha)}\int_0^1\int_0^t (t-s)^{\alpha-1}|u(s)|dsdt \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^1 |u(s)|ds\int_s^1 (t-s)^{\alpha-1}dt \leq \frac{1}{\Gamma(\alpha+1)}\int_0^1 |u(s)|ds = \frac{\|u\|_{L^1}}{\Gamma(\alpha+1)}.\end{aligned}$$

□

Now we recall the coincidence degree continuation theorem and some related notions (for more details see [25]).

Definition 2.3. *Let X and Y be real Banach spaces. A linear operator $L : dom\, L \subset X \longrightarrow Y$ is said to be a Fredholm operator of index zero if*

(1) *$Im\, L$ is a closed subset of Y;*
(2) *$dim\, ker\, L = codim\, Im\, L < \infty$.*

It follows from Definition 2.3 that there exist continuous projectors $P : X \longrightarrow X$ and $Q : Y \longrightarrow Y$ such that

$$KerL = ImP, \quad ImL = KerQ, \quad X = Ker\, L \oplus KerP, \quad Y = Im\, L \oplus Im\, Q.$$

Also, it follows that

$$L_p = L\,|_{dom\, L \bigcap Ker\, P} : dom\, L \bigcap Ker\, P \longrightarrow Im\, L$$

is invertible and its inverse is denoted by K_p.

Definition 2.4. *Let L be a Fredholm operator of index zero, and let Ω be an open bounded subset of X such that $dom\, L \bigcap \Omega \neq \emptyset$. Then the map $N : \overline{\Omega} \longrightarrow X$ will be called $L-$ compact on $\overline{\Omega}$ if*

(1) *$QN(\overline{\Omega})$ is bounded,*
(2) *$K_{P,Q}\, N = K_p(I - Q)N : \overline{\Omega} \longrightarrow X$ is compact.*

Theorem 2.8. *Let $L : dom\, L \subset X \longrightarrow Y$ be a Fredholm operator of index zero, and let $N : X \longrightarrow Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in \Big[\Big(dom\, L \backslash Ker\, L\Big) \bigcap \partial\Omega\Big] \times (0, 1)$.*
(2) *$Nx \notin Im\, L$ for every $x \in KerL \bigcap \partial\Omega$.*
(3) *$deg\, (QN\, |_{Ker\, L}, \Omega \bigcap Ker\, L, 0) \neq 0$, where $Q : Y \longrightarrow Y$ is a projection such that $Im\, L = Ker\, Q$.*

Then, the abstract equation $Lx = Nx$ has at least one solution in $dom\, L \bigcap \overline{\Omega}$.

For our purposes, the adequate functional space is:

$$X := \Big\{u :\ {}^{C}D^{\alpha}_{0+}u \in C\big([0, 1], \mathbb{R}\big),\ u \,\text{satisfies the boundary conditions of}\ (1.1)\ \Big\},$$

equipped with the norm:

$$\|u\|_X = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty + \|{}^{C}D^{\alpha}_{0+}u\|_\infty,$$

where

$$\|u\|_\infty = \max_{t\in[0,1]} |u(t)|.$$

By means of the functional analysis theory, we can prove that $(X, \|\cdot\|_X)$ is a Banach space. Let $Y = L^1[0, 1]$ be the space of real measurable functions $t \longrightarrow y(t)$ defined on $[0, 1]$ such that $t \longrightarrow |y(t)|$ is Lebesgue integrable. Then Y is a Banach space with the norm $\|y\|_{L^1} = \int_0^1 |y(t)|dt$. Define L to be the linear operator from $dom\, L \bigcap X$ to Y :

$$Lu = \big(\phi\, {}^{C}D^{\alpha}_{0+}u\big)', \quad u \in dom\, L.$$

where $dom\, L = \Big\{u \in X \mid {}^{C}D^{\alpha}_{0+}u(t) \,\text{is absolutely continuous on}\ [0,\ 1]\Big\}$, and define the operator $N : X \longrightarrow Y$ as follows:

$$Nu(t) = f\big(t, u(t), u'(t), u''(t), {}^{C}D^{\alpha}_{0+}u(t)\big), \quad t \in [0, 1].$$

Then the boundary value problem (1.1) can be written in the following form:

$$Lu = Nu, \quad u \in dom\, L.$$

To study the compactness of the operator N, we will need the following lemma.

Lemma 2.9. *A subset $U \subset X$ is a relatively compact set in X if and only if U is uniformly bounded and equicontinuous. Here the uniformly boundedness means that there exists $M > 0$ such that for every $u \in U$*

$$\|u\|_X = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty + \|{}^C D_{0^+}^\alpha u\|_\infty \leq M,$$

and the equicontinuity means that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$|u^{(i)}(t_1) - u^{(i)}(t_2)| < \varepsilon, \quad \forall u \in U,\ \forall t_1, t_2 \in I,\ |t_1 - t_2| < \delta,\ \forall i \in \{0, 1, 2\}.$$

and

$$|{}^C D_{0^+}^\alpha u(t_1) - {}^C D_{0^+}^\alpha u(t_2)| < \varepsilon, \quad \forall u \in U,\ \forall t_1, t_2 \in I,\ |t_1 - t_2| < \delta.$$

3. The main results

In this section we state and prove our main results.

Lemma 3.1. *Let $y \in Y$, $\phi(t) \in C^1[0,1]$, $\mu = \min_{t \in I} \phi(t) > 0$ and (H_1) hold, and let T_1, $T_2 : Y \longrightarrow Y$ be two linear operators defined by*

$$T_1(y) = \sum_{i=1}^m a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-3}}{\phi(s)} \int_0^s y(r)drds,$$

$$T_2(y) = \int_0^1 \frac{(1-s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)drds - \sum_{j=1}^l b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)drds.$$

Then $u \in X$ is a solution of the following linear fractional differential problem:

$$\begin{cases} \left(\phi(t) {}^C D_{0^+}^\alpha u(t)\right)' = y(t), \quad t \in I = [0,1], \\ u(0) = 0,\ {}^C D_{0^+}^\alpha u(0) = 0,\ u''(0) = \sum_{i=1}^m a_i u''(\xi_i),\ u'(1) = \sum_{j=1}^l b_j u'(\eta_j), \end{cases} \tag{3.1}$$

if and only if

$$u(t) = c_1 t + c_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds, \quad c_1, c_2 \in \mathbb{R}, \tag{3.2}$$

and

$$T_1(y) = T_2(y) = 0. \tag{3.3}$$

Proof. Let u be a solution of the problem (3.1). Then we have

$$\phi(t) {}^C D_{0^+}^\alpha u(t) = c + \int_0^t y(s)ds, \quad c \in \mathbb{R}.$$

Since ${}^C D_{0^+}^\alpha u(0) = 0$, we find

$${}^C D_{0^+}^\alpha u(t) = \frac{1}{\phi(t)} \int_0^t y(s)ds.$$

By Lemma 2.4, we get

$$u(t) = c_0 + c_1 t + c_2 t^2 + \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds, \quad c_0, c_1, c_2 \in \mathbb{R}.$$

Since $u(0) = 0$, we have

$$u(t) = c_1 t + c_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds, \quad c_1, c_2, \in \mathbb{R}.$$

By $u''(0) = \sum_{i=1}^m a_i u''(\xi_i)$ and $\sum_{i=1}^l a_i = 1$, we obtain

$$\sum_{i=1}^{l} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-3}}{\phi(s)} \int_0^s y(r)drds = 0.$$

From the conditions $u'(1) = \sum_{j=1}^l b_j u'(\eta_j)$ and $\sum_{j=1}^l b_j = \sum_{j=1}^l b_j \eta_j = 1$, we get

$$\int_0^1 \frac{(1-s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)drds - \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)drds = 0.$$

Thus, we have $T_1(y) = T_2(y) = 0$. On the other hand, if c_1, c_2 are arbitrary real constants and

$$u(t) = c_1 t + c_2 t^2 + \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds,$$

then clearly $u(0) = 0$, and by Lemma 2.2 and 2.3, we obtain

$$\begin{cases} & {}^C D_{0^+}^{\alpha} u(0) = 0 \\ \forall\ t \in [0,1], & \Big(\phi(t) {}^C D_{0^+}^{\alpha} u(t)\Big)' = y(t). \end{cases}$$

Taking into account that (3.3) holds, we get the following equations:

$$u''(0) - \sum_{i=1}^{m} a_i u''(\xi_i) = \frac{T_1(y)}{\Gamma(\alpha-2)} = 0, \quad u'(1) - \sum_{j=1}^{l} b_j u'(\eta_j) = \frac{T_2(y)}{\Gamma(\alpha-1)} = 0.$$

Thus, u is a solution of the problem (3.1). This completes the proof. □

Lemma 3.2. *Assume that the conditions $(H_0) - (H_2)$ hold.*

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds. \tag{3.4}$$

Furthermore, we have

$$\|K_p y\|_X \le \rho_1 \|y\|_{L^1}, \tag{3.5}$$

where

$$\rho_1 = \frac{1}{\mu}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + 1 \right). \tag{3.6}$$

Proof. It is clear that $Ker\, L = \big\{u \mid u(t) = c_1 t + c_2 t^2,\ c_1, c_2 \in \mathbb{R}\big\}$. Furthermore, Lemma 3.1 implies that

$$Im\, L = \big\{y \in Y \mid T_1(y) = T_2(y) = 0\big\}. \tag{3.7}$$

Consider a continuous linear mapping $Q : Y \longrightarrow Y$ defined by

$$Qy = Q_1(y)t^{p-1} + Q_2(y)t^{q-1}, \tag{3.8}$$

where p, q are given in (H_2), and

$$Q_1(y) = \frac{1}{\Delta(p,q)}\big(d_{22}T_1(y) - d_{21}T_2(y)\big),$$

$$Q_2(y) = \frac{1}{\Delta(p,q)}\big(- d_{12}T_1(y) + d_{11}T_2(y)\big).$$

We prove that $Ker\, Q = Im\, L$. Obviously, $ImL \subset Ker\, Q$. Also, if $y \in Ker\, Q$, then

$$\begin{cases} d_{22}T_1(y) - d_{21}T_2(y) = 0. \\ -d_{12}T_1(y) + d_{11}T_2(y) = 0. \end{cases} \tag{3.9}$$

The determinant of coefficients for (3.9) is $\Delta(p,q) \neq 0$. Therefore $T_1(y) = T_2(y) = 0$, implying that $y \in Im\, L$. Thus, $Ker\, Q \subset Im\, L$. Now, we show that $Q^2 y = Qy$, $y \in Y$. For $y \in Y$, we have

$$\begin{aligned} Q_1(Q_1(y)t^{p-1}) &= \frac{1}{\Delta(p,q)}\left[d_{22}T_1\left(Q_1(y)t^{p-1}\right) - d_{21}T_2\left(Q_1(y)t^{p-1}\right)\right] \\ &= \frac{1}{\Delta(p,q)}\big(d_{22}d_{11} - d_{21}d_{12}\big)Q_1 y = Q_1 y, \end{aligned}$$

and

$$\begin{aligned} Q_1\big(Q_2(y)t^{q-1}\big) &= \frac{1}{\Delta(p,q)}\left[d_{22}T_1\big(Q_2(y)t^{q-1}\big) - d_{21}T_2\big(Q_2(y)t^{q-1}\big)\right] \\ &= \frac{1}{\Delta(p,q)}\big(d_{22}d_{21} - d_{21}d_{22}\big)Q_2 y = 0. \end{aligned}$$

Similarly, we obtain

$$Q_2\big(Q_1(y)t^{p-1}\big) = 0, \quad Q_2\big(Q_2(y)t^{q-1}\big) = Q_2 y.$$

Therefore, we get

$$\begin{aligned} Q^2 y &= Q_1\big(Q_1(y)t^{p-1}\big)t^{p-1} + Q_1\big(Q_2(y)t^{q-1}\big)t^{p-1} \\ &\quad + Q_2\big(Q_1(y)t^{p-1}\big)t^{q-1} + Q_2\big(Q_2(y)t^{q-1}\big)t^{q-1} \\ &= Q_1(y)t^{p-1} + Q_2(y)t^{q-1} = Qy, \end{aligned}$$

showing that the operator Q is a projector.

Take $y \in Y$ of the form $y = \big(y - Qy\big) + Qy$ to obtain $\big(y - Qy\big) \in KerQ = ImL$ and $Qy \in ImQ$. Thus, $Y = Im\, Q + Im\, L$. Also, for any $y \in Im\, Q \cap Im\, L$, from $y \in Im\, Q$ there exist constants c_1, $c_2 \in \mathbb{R}$ such that $y(t) = c_1 t^{p-1} + c_2 t^{q-1}$, and from $y \in Im\, L$ we obtain

$$\begin{cases} d_{11}c_1 + d_{21}c_2 = 0, \\ d_{12}c_1 + d_{22}c_2 = 0. \end{cases} \tag{3.10}$$

The determinant of coefficients for (3.10) is $\Delta(p,q) \neq 0$. Therefore (3.10) has a unique solution $c_1 = c_2 = 0$, which implies that $Im\, Q \cap Im\, L = 0$. Then, we have

$$Y = Im\, Q \oplus Ker\, Q = Im\, Q \oplus Im\, L. \tag{3.11}$$

Thus, $dim\, Ker\, L = 2 = dim\, Im\, Q = codim\, Ker\, Q = codim\, Im\, L$, showing that L is a Fredholm operator of index zero.

Let a mapping $P : X \longrightarrow X$ be defined by

$$Pu(t) = u'(0)t + \frac{u''(0)}{2}t^2. \tag{3.12}$$

We note that P is a linear continuous projector and $Im\, P = Ker\, L$. It follows from $u = (u - Pu) + Pu$ that $X = Ker\, P + Ker\, L$. By simple calculation, we obtain that $KerL \cap KerP = \{0\}$, and hence

$$X = Ker\, L \oplus Ker\, P. \tag{3.13}$$

Define $K_p : Im\, L \longrightarrow dom\, L \cap Ker\, P$ as follows:

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds.$$

Now, we show that K_p is the inverse of $L\mid_{dom\, L \cap Ker\, P}$. In fact, for $u \in dom\, L \cap Ker\, P$, we have

$$\begin{aligned}(K_p L)u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s \left(\phi\, {}^C D_{0+}^{\alpha} u\right)'(r)drds = I_{0+}^{\alpha}\, {}^C D_{0+}^{\alpha} u(t) \\ &= u(t) + u(0) + u'(0)t + \frac{u''(0)}{2}t^2.\end{aligned}$$

In view of $u \in dom\, L \cap Ker\, P$, we have $u(0) = 0$ and $Pu = 0$. Thus

$$(K_p L)u(t) = u(t), \tag{3.14}$$

and for $y \in Im\; L$, we find

$$(LK_p)y(t) = L(K_p y)(t) = \left[\phi(t)\, {}^C D_{0+}^{\alpha} I_{0+}^{\alpha} \left(\frac{I_{0+}^1 y}{\phi}\right)(t)\right]' = y(t).$$

Thus, $K_p = \left(L\mid_{dom\, L \cap Ker\, P}\right)^{-1}$. Again, for each $y \in Im\; L$, in view of Lemmas 2.3, 2.6 and 2.7, we can write

$$\begin{aligned}\|K_p y\|_X &= \sum_{i=0}^{2} \max_{t \in I} \left|(K_p y)^{(i)}(t)\right| + \max_{t \in I} \left|{}^C D_{0+}^{\alpha}(K_p y)(t)\right| \\ &= \sum_{i=0}^{2} \max_{t \in I} \left| I_{0+}^{\alpha - i} \left(\frac{I_{0+}^1 y}{\phi}\right)(t)\right| + \max_{t \in I} \left|\left(\frac{I_{0+}^1 y}{\phi}\right)(t)\right| \\ &\le \sum_{i=0}^{2} \max_{t \in I} \left|\frac{I_{0+}^{\alpha+1-i} y(t)}{\mu}\right| + \max_{t \in I} \left|\frac{I_{0+}^1 y(t)}{\mu}\right|\end{aligned}$$

$$\leq \sum_{i=0}^{2} \frac{\|y\|_{L^1}}{\mu\Gamma(\alpha+1-i)} + \frac{\|y\|_{L^1}}{\mu} \leq \rho_1 \|y\|_{L^1},$$

and the result follows. □

Lemma 3.3. *Suppose that Ω is an open bounded subset of X such that $dom\, L \bigcap \overline{\Omega} \neq \emptyset$. Then N is L-compact on $\overline{\Omega}$.*

Proof. It is clear that $QN(\overline{\Omega})$ and $K_p(I-Q)N(\overline{\Omega})$ are bounded, due to the fact that f satisfies the Carathéodory conditions. Using the Lebesgue dominated convergence theorem, we can easily show that QN and $K_{P,Q}N = K_p(I-Q)N : \overline{\Omega} \longrightarrow X$ are continuous. By the hypothesis (iii) on the function f, there exists a constant $M > 0$, such that $|(I-Q)N(u(t))| \leq M$, for all $u \in \Omega$ and $t \in [0,1]$. For $i = 0,1,2$, $0 \leq t_1 \leq t_2 \leq 1$, and $u \in \Omega$, we can write

$$\begin{aligned}
&\Big|\big(K_{P,Q}Nu\big)^{(i)}(t_2) - \big(K_{P,Q}Nu\big)^{(i)}(t_1)\Big| \\
=&\frac{1}{\Gamma(\alpha-i)}\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-i-1}}{\phi(s)}\int_0^s (I-Q)Nu(r)drds\right. \\
&\left. - \int_0^{t_1}\frac{(t_1-s)^{\alpha-i-1}}{\phi(s)}\int_0^s (I-Q)Nu(r)drds\right| \\
\leq&\frac{M}{\mu\Gamma(\alpha-i)}\left\{\int_0^{t_1}(t_2-s)^{\alpha-i-1} - (t_1-s)^{\alpha-i-1}ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-i-1}ds\right\} \\
=&\frac{M}{\mu\Gamma(\alpha+1-i)}(t_2^{\alpha-i} - t_1^{\alpha-i}).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\big|{}^CD^\alpha_{0^+}K_{P,Q}Nu(t_2) - {}^CD^\alpha_{0^+}K_{P,Q}Nu(t_1)\big| \\
=&\left|\frac{1}{\phi(t_2)}\int_0^{t_2}(I-Q)Nu(s)ds - \frac{1}{\phi(t_1)}\int_0^{t_1}(I-Q)Nu(s)ds\right| \\
=&\left|\left(\frac{1}{\phi(t_2)} - \frac{1}{\phi(t_1)}\right)\int_0^{t_1}(I-Q)Nu(s)ds + \frac{1}{\phi(t_2)}\int_{t_1}^{t_2}(I-Q)Nu(s)ds\right| \\
\leq&\frac{M}{\mu^2}\big|\phi(t_2) - \phi(t_1)\big| + \frac{M}{\mu}\big(t_2 - t_1\big).
\end{aligned}$$

Since t^α, $t^{\alpha-1}$, $t^{\alpha-2}$ and $\phi(t)$ are uniformly continuous on $[0,1]$, we conclude that $K_p(I-Q)N : \overline{\Omega} \longrightarrow X$ is compact. □

Now we are in position to state the main result of this paper.

Theorem 3.4. *Assume that, in addition to $(H_0)-(H_2)$, the following conditions hold.*

(H_3) *There exists a Carathéodory function $\Phi : [0,1] \times (\mathbb{R}^+)^4 \longrightarrow \mathbb{R}^+$ that is nondecreasing with respect to the last four arguments and satisfies the inequality:*

$$\Big|f\big(t,x_0,x_1,x_2,x_3\big)\Big| \leq \Phi\big(t,|x_0|,|x_1|,|x_2|,|x_3|\big).$$

(H_4) $\lim_{r\to\infty}\sup \frac{1}{r}\int_0^1 \big|\Phi\big(s,r,r,r,r\big)\big|ds < \frac{1}{\rho_1+\rho_2}$ *where* ρ_1 *is defined by* (3.6), *and*

$$\rho_2 = \frac{1}{\mu}\left(\frac{2}{\Gamma(\alpha)} + \frac{5}{\Gamma(\alpha-1)}\right).$$

(H_5) *There exists a constant* $A>0$ *such that for* $u \in dom\, L\backslash Ker\, L$, *if* $|u^{'}(t)|>A$ *or* $|u^{''}(t)|>A$ *for all* $t\in[0,1]$, *then* $T_1(Nu)\neq 0$ *or* $T_2(Nu)\neq 0$.

(H_6) *There exists a constant* $B>0$ *such that for any* $c_1,\, c_2 \in \mathbb{R}$, *if* $|c_1|>B, |c_2|>B$, *then either*

$$T_1N\big(c_1t+c_2t^2\big)+T_2N\big(c_1t+c_2t^2\big)<0,$$

or

$$T_1N\big(c_1t+c_2t^2\big)+T_2N\big(c_1t+c_2t^2\big)>0.$$

Then, the problem (1.1) *has at least one solution.*

Remark 3.5. A sufficient condition for (H_3) to be satisfied is the existence of functions $\theta_i(t)\in Y,\ i=0,\dots,5$ and a constant $\nu\in(0,1)$ such that for all $x_0,x_1,x_2,x_3\in\mathbb{R}$ and $t\in[0,1]$ the nonlinearity f verifies one of the following growth conditions:

$$\Big|f\big(t,x_0,x_1,x_2,x_3\big)\Big| \le \sum_{i=0}^{3}\theta_i(t)|x_i| + \theta_4(t)|x_0|^{\nu}+\theta_5(t),$$

$$\Big|f\big(t,x_0,x_1,x_2,x_3\big)\Big| \le \sum_{i=0}^{3}\theta_i(t)|x_i| + \theta_4(t)|x_1|^{\nu}+\theta_5(t),$$

$$\Big|f\big(t,x_0,x_1,x_2,x_3\big)\Big| \le \sum_{i=0}^{3}\theta_i(t)|x_i| + \theta_4(t)|x_2|^{\nu}+\theta_5(t),$$

$$\Big|f\big(t,x_0,x_1,x_2,x_3\big)\Big| \le \sum_{i=0}^{3}\theta_i(t)|x_i| + \theta_4(t)|x_3|^{\nu}+\theta_5(t).$$

In this case, (H_4) reduces to the following:

(H_4^*) $\sum_{i=0}^{3}\|\theta_i\|_{L^1} < \frac{1}{\rho_1+\rho_2}$.

Proof of Theorem 3.4. Consider the set

$$\Omega_1 = \Big\{u\in dom\, L\backslash Ker\, L \mid Lu=\lambda Nu,\ \lambda\in[0,1]\Big\},$$

and observe that for $u\in\Omega_1$, we have $Lu=\lambda Nu$. Thus, $\lambda\neq 0$, $Nu\in Im\, L = Ker\, Q\subset Y$, and hence, $Q(Nu)=0$, that is, $T_1(Nu)=T_2(Nu)=0$. It follows from condition (H_5) that there exist $t_1,t_2\in[0,1]$, such that $|u'(t_1)|\le A, |u''(t_2)|\le A$.

If $t_1=t_2=0$, then we have $|u'(0)|\le A, |u''(0)|\le A$. Otherwise, in view of $Lu=\lambda Nu$, we obtain

$$u(t) = u'(0)t + \frac{u''(0)}{2}t^2 + \frac{\lambda}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s Nu(r)drds.$$

If $t_2 \neq 0$, then

$$u''(t_2) = u''(0) + \frac{\lambda}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s Nu(r)drds,$$

and, together with $|u''(t_2)| \leq A$, we get

$$|u''(0)| \leq |u''(t_2)| + \frac{1}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s |Nu(r)|drds \leq A + \frac{\|Nu\|_{L^1}}{\mu\Gamma(\alpha-1)}.$$

Consequently, we have

$$|u''(0)| \leq A + \frac{1}{\mu\Gamma(\alpha-1)} \|Nu\|_{L^1}. \tag{3.15}$$

If $t_1 \neq 0$, then

$$u'(t_1) = u'(0) + u''(0)t_1 + \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s Nu(r)drds,$$

and, according to (3.15) and $|u'(t_1)| \leq A$, we get

$$\begin{aligned}|u'(0)| &\leq |u'(t_1)| + |u''(0)| + \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s |Nu(r)|drds \\ &\leq 2A + \frac{1}{\mu}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right) \|Nu\|_{L^1}.\end{aligned}$$

Therefore

$$|u'(0)| \leq 2A + \frac{1}{\mu}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right) \|Nu\|_{L^1}. \tag{3.16}$$

Next, for $u \in \Omega_1$, we get

$$\begin{aligned}\|Pu\|_X &= \sum_{i=0}^{2} \max_{t\in[0,1]} \left|(Pu)^{(i)}(t)\right| + \max_{t\in[0,1]} \left|{}^C D^{\alpha}_{0^+}(Pu)(t)\right| \\ &\leq 2|u'(0)| + 3|u''(0)|.\end{aligned}$$

From (3.15) and (3.16), we obtain

$$\|Pu\|_X \leq 7A + \rho_2 \|Nu\|_{L^1}. \tag{3.17}$$

Again, for all $u \in \Omega_1$, we have $(I-P)u \in dom\, L \cap Ker\, P$, and hence, by (3.14) and (3.5), we find

$$\|(I-P)u\|_X = \|K_p L(I-P)u\|_X \leq \rho_1 \|L(I-P)u\|_{L^1} = \rho_1 \|Lu\|_{L^1} \leq \rho_1 \|Nu\|_{L^1}. \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$\|u\|_X \leq \|Pu\|_X + \|(I-P)u\|_X \leq 7A + (\rho_1+\rho_2)\|Nu\|_{L^1}. \tag{3.19}$$

On the other hand, from (H_3), we have

$$\begin{aligned}\|Nu\|_{L^1} &= \int_0^1 \left|f\left(s, u(s), u'(s), u''(s), {}^C D^{\alpha}_{0^+}u(s)\right)\right| ds \\ &\leq \int_0^1 \left|\Phi\left(s, u(s), u'(s), u''(s), {}^C D^{\alpha}_{0^+}u(s)\right)\right| ds\end{aligned}$$

$$\leq \int_0^1 \Big|\Phi\big(s, \|u\|_X, \|u\|_X, \|u\|_X, \|u\|_X\big)\Big| ds. \tag{3.20}$$

Because the function Φ is Carathéodory, the function $\Psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, given by $\Psi(r) = \frac{1}{r}\int_0^1 \big|\Phi\big(s,r,r,r,r\big)\big| ds$, is well defined. Let $l = \lim_{r\to\infty} \sup \Psi(r)$. By (H_4) we have $0 < l < \frac{1}{\rho_1+\rho_2}$, and hence, for each $0 < \varepsilon < \frac{1}{\rho_1+\rho_2} - l$, there exists r_ε such that $r \geq r_\varepsilon \Longrightarrow \Psi(r) < l + \varepsilon$. If $\|u\|_X \geq r_\varepsilon$, then $\Psi(\|u\|_X) < \frac{1}{\rho_1+\rho_2}$, and hence, (3.20) implies that

$$\|Nu\|_{L^1} \leq (l+\varepsilon)\|u\|_X. \tag{3.21}$$

Therefore, in view of (3.19) and (3.21), we obtain

$$r_\varepsilon \leq \|u\|_X \leq \frac{7A}{1 - \big(\rho_1+\rho_2\big)(l+\varepsilon)}.$$

Consequently, we have

$$\|u\|_X \leq \max\left\{ r_\varepsilon, \frac{7A}{1-(l+\varepsilon)\big(\rho_1+\rho_2\big)} \right\} = \frac{7A}{1-(l+\varepsilon)\big(\rho_1+\rho_2\big)}. \tag{3.22}$$

Since (3.22) is valid for all $0 < \varepsilon < \frac{1}{\rho_1+\rho_2} - l$, we get

$$\|u\|_X \leq \frac{7A}{1 - l\big(\rho_1+\rho_2\big)}.$$

So, Ω_1 is bounded. Denote

$$\Omega_2 = \Big\{ u \in Ker\, L \mid Nu \in Im\, L \Big\},$$

and observe that for $u \in \Omega_2$, we have $u \in Ker\, L = \big\{ u \mid u(t) = c_1 t + c_2 t^2,\ c_1, c_2 \in \mathbb{R} \big\}$, and $Q(Nu) = 0$, that is,

$$T_1 N\big(c_1 t + c_2 t^2\big) = T_2 N\big(c_1 t + c_2 t^2\big) = 0.$$

From condition (H_6), we get $|c_1| \leq B, |c_2| \leq B$. Hence, Ω_2 is bounded. Define

$$\Omega_3 := \Big\{ u \in Ker\, L \mid -\lambda J u + (1-\lambda) QNu = 0,\ \lambda \in [0,1] \Big\}$$

provided that the first part of condition (H_6) holds, or

$$\Omega_3 := \Big\{ u \in Ker\, L \mid -\lambda J u + (1-\lambda) QNu = 0,\ \lambda \in [0,1] \Big\}$$

provided that the second part of (H_6) holds, where $J : Ker\, L \longrightarrow Im\, Q$ is the linear isomorphism given by

$$J\big(c_1 t + c_2 t^2\big) = \omega_1 t^{p-1} + \omega_2 t^{q-1}, \quad c_1, c_2 \in \mathbb{R}, \tag{3.23}$$

with

$$\omega_1 = \frac{1}{\Delta(p,q)}\big(d_{22}|c_1| - d_{21}|c_2|\big), \quad \omega_2 = \frac{1}{\Delta(p,q)}\big(- d_{12}|c_1| + d_{11}|c_2|\big).$$

Without loss of generality, we assume that the first part of (H_6) holds.
In fact $u \in \Omega_3$, means that $u = c_1 t + c_2 t^2$ and $-\lambda Ju + (1-\lambda)QNu = 0$. Then we obtain

$$-\lambda J\big(c_1 t + c_2 t^2\big) + (1-\lambda)QN\big(c_1 t + c_2 t^2\big) = 0. \tag{3.24}$$

If $\lambda = 0$, then $|c_1| \le B, |c_2| \le B$. If $\lambda = 1$, then

$$\begin{cases} d_{22}|c_1| - d_{21}|c_2| = 0 \\ -d_{12}|c_1| + d_{11}|c_2| = 0. \end{cases} \tag{3.25}$$

The determinant of coefficients for (3.25) is $\Delta(p,q) \neq 0$. Thus, the system (3.25) has only zero solution, that is, $c_1 = c_2 = 0$.
Otherwise, if $\lambda \neq 0$ and $\lambda \neq 1$, in view of (3.23), the equation (3.24) becomes

$$\lambda\big(\omega_1 t^{p-1} + \omega_2 t^{q-1}\big) = (1-\lambda)\Big(Q_1 N\big(c_1 t + c_2 t^2\big)t^{p-1} + Q_2 N\big(c_1 t + c_2 t^2\big)t^{q-1}\Big).$$

Hence

$$\begin{cases} \lambda\omega_1 = (1-\lambda)Q_1\big(c_1 t + c_2 t^2\big), \\ \lambda\omega_2 = (1-\lambda)Q_2\big(c_1 t + c_2 t^2\big). \end{cases}$$

Thus, we have

$$\begin{cases} \lambda|c_1| = (1-\lambda)T_1 N\big(c_1 t + c_2 t^2\big), \\ \lambda|c_2| = (1-\lambda)T_2 N\big(c_1 t + \delta_2 t^2\big). \end{cases}$$

Then, we get

$$\lambda\big(|\delta_1| + |\delta_2|\big) = (1-\lambda)\Big(T_1 N\big(\delta_1 t + \delta_2 t^2\big) + T_2 N\big(\delta_1 t + \delta_2 t^2\big)\Big) < 0.$$

By the first part of condition (H_6), we have $|\delta_1| \le B, |\delta_2| \le B$. Hence, Ω_3 is bounded.

Now, we proceed to show that all the conditions of Theorem 2.8 are satisfied. Let Ω be a bounded open set of X containing $\bigcup_{i=1}^{3} \overline{\Omega}_i$. By Lemma 3.3, N is L-compact on $\overline{\Omega}$. Because Ω_1 and Ω_2 are bounded sets, we have

(1) $Lu \neq \lambda Nu$ for each $(u,\lambda) \in \Big[\Big(domL \backslash KerL\Big) \cap \partial\Omega\Big] \times (0,1)$;
(2) $Nu \notin ImL$ for each $u \in KerL \cap \partial\Omega$.

To show that the condition (3) of Theorem 2.8 is satisfied, we define

$$H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu,$$

and observe that, because Ω_3 is bounded, then we have

$$H(u,\lambda) \neq 0, \quad \forall u \in KerL \bigcap \partial\Omega.$$

Appealing to the homotopy property of the degree, we obtain

$$\begin{aligned} deg\Big(QN\,|_{kerL}, \Omega \bigcap KerL, 0\Big) &= deg\Big(H(\cdot,0), \Omega \bigcap KerL, 0\Big) \\ = deg\Big(H(\cdot,1), \Omega \bigcap KerL, 0\Big) &= deg\Big(\pm J, \Omega \bigcap KerL, 0\Big) \neq 0. \end{aligned}$$

Thus, the condition (3) of Theorem 2.8 is also satisfied.

Finally, we can apply Theorem 2.8, to conclude that the abstract equation $Lu = Nu$ has at least one solution in dom $\mathrm{L}\bigcap\overline{\Omega}$, and hence, the boundary value problem (1.1) has at least one solution in X. Theorem 3.4 is proved. □

4. An example

To illustrate our main result, we discuss an example.

Example 4.1. *Let us consider the following fractional boundary value problem*

$$\begin{aligned} &\Big(\phi(t)^C D_{0+}^{\frac{5}{2}} u(t)\Big)' = f\big(t, u(t), u'(t), u''(t), {}^C D_{0+}^{\frac{5}{2}} u(t)\big),\ t\in[0,1] \\ &u(0) = {}^C D_{0+}^{\alpha} u(0) = 0,\ u''(0) = -u''\left(\frac{1}{3}\right) + 2u''\left(\frac{1}{6}\right), \\ &u'(1) = -2u'\left(\frac{1}{4}\right) + 3u'\left(\frac{1}{2}\right). \end{aligned} \tag{4.1}$$

where $\phi(t) = e^{t-3}$ and

$$f\big(t, x_0, x_1, x_2, x_3\big) = x_2 + \cos x_3\big(1 - \sin x_1\big) + \sqrt{|x_2|}.$$

Now show that the conditions of Theorem 3.4 are fulfilled.

Corresponding to the notation of the problem (1.1), we have that $\alpha = \frac{5}{2},\ l = 2,\ m = 2,\ a_1 = -1,\ a_2 = 2,\ \xi_1 = \frac{1}{3},\ \xi_2 = \frac{1}{6},\ b_1 = -2,\ b_2 = 3,\ \eta_1 = \frac{1}{4},\ \eta_2 = \frac{1}{2},\ \mu = \min_{t\in I}\phi(t) = e^{-3} > 0$. Then we have $a_1 + a_2 = b_1 + b_2 = 1$, $b_1\eta_1 + b_2\eta_2 = 1$. Thus, the condition (H_1) is satisfied.

Also, we find

$$T_1(y) = -\int_0^{\frac{1}{3}}\left(\frac{1}{3} - s\right)^{-\frac{1}{2}} e^{3-s}\int_0^s y(r)drds + 2\int_0^{\frac{1}{6}}\left(\frac{1}{6} - s\right)^{-\frac{1}{2}} e^{3-s}\int_0^s y(r)drds,$$

$$\begin{aligned} T_2(y) = &\int_0^1 (1-s)^{\frac{1}{2}} e^{3-s}\int_0^s y(r)drds - 2\int_0^{\frac{1}{4}}\left(\frac{1}{4} - s\right)^{\frac{1}{2}} e^{3-s}\int_0^s y(r)drds \\ &+ 3\int_0^{\frac{1}{2}}\left(\frac{1}{2} - s\right)^{\frac{1}{2}} e^{3-s}\int_0^s y(r)drds. \end{aligned}$$

By simple calculations, we get

$$\Delta(1,2) = \begin{vmatrix} \frac{-761}{993} & \frac{-301}{982} \\ \frac{1545}{311} & \frac{463}{431} \end{vmatrix} = \frac{263}{376} \neq 0,$$

Therefore, the condition (H_2) holds.

On the other hand, we have

$$\Big|f\big(t, x_0, x_1, x_2, x_3\big)\Big| \leq |x_2| + \sqrt{|x_2|} + 2.$$

It is easy to see that the condition (H_3) holds, where

$$\theta_0(t)=\theta_1(t)=\theta_3(t)=0,\ \theta_2(t)=1,\ \theta_4(t)=\frac{1}{2},\ \theta_5(t)=2,\ \nu=\frac{1}{2}.$$

Next, we have

$$(\rho_1+\rho_2)\sum_{i=0}^{3}\|\theta_i\|_{L^1}=e^{-3}\left(\frac{1}{\Gamma(3.5)}+\frac{3}{\Gamma(2.5)}+\frac{6}{\Gamma(1.5)}+1\right)=\frac{833}{1620}<1.$$

Therefore, the condition (H_4^*) holds.

Let $A=9$ and assume that $|u''(t)|>9$ holds for all $t\in[0,1]$. Then, by the continuity of $u''(t)$, we have either $u''(t)>9$ for all $t\in[0,1]$, or $u''(t)<-9$ for all $t\in[0,1]$. If $u''(t)>9$, then for all $t\in[0,1]$ we obtain

$$\begin{aligned}
T_2(y)=&\int_0^1(1-s)^{\frac{1}{2}}e^{3-s}\int_0^s\Big(u''(r)+\cos{}^CD_{0+}^{\alpha}u(r)\big(1-\sin u'(r)\big)+\sqrt{|u''(r)|}\Big)drds\\
&-2\int_0^{\frac{1}{4}}\Big(\frac{1}{4}-s\Big)^{\frac{1}{2}}e^{3-s}\int_0^s\Big(u''(r)+\cos{}^CD_{0+}^{\alpha}u(r)\big(1-\sin u'(r)\big)+\sqrt{|u''(r)|}\Big)drds\\
&+3\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-s\Big)^{\frac{1}{2}}e^{3-s}\int_0^s\Big(u''(r)+\cos{}^CD_{0+}^{\alpha}u(r)\big(1-\sin u'(r)\big)+\sqrt{|u''(r)|}\Big)drds.\\
&\geq 5\int_0^1 s(1-s)^{\frac{1}{2}}e^{3-s}ds-14\int_0^{\frac{1}{4}}s\Big(\frac{1}{4}-s\Big)^{\frac{1}{2}}e^{3-s}ds+15\int_0^{\frac{1}{2}}s\Big(\frac{1}{2}-s\Big)^{\frac{1}{2}}e^{3-s}ds\\
&\geq\frac{7280}{257}.
\end{aligned}$$

If $u''(t)<-9$, then for all $t\in[0,1]$ we obtain

$$\begin{aligned}
T_2(y)=&\int_0^1(1-s)^{\frac{1}{2}}e^{3-s}\int_0^s\Big(u''(r)+\cos{}^CD_{0+}^{\alpha}u(r)\big(1-\sin u'(r)\big)+\sqrt{|u''(r)|}\Big)drds\\
&-2\int_0^{\frac{1}{4}}\Big(\frac{1}{4}-s\Big)^{\frac{1}{2}}e^{3-s}\int_0^s\Big(u''(r)+\cos{}^CD_{0+}^{\alpha}u(r)\big(1-\sin u'(r)\big)+\sqrt{|u''(r)|}\Big)drds\\
&+3\int_0^{\frac{1}{2}}\Big(\frac{1}{2}-s\Big)^{\frac{1}{2}}e^{3-s}\int_0^s\Big(u''(r)+\cos{}^CD_{0+}^{\alpha}u(r)\big(1-\sin u'(r)\big)+\sqrt{|u''(r)|}\Big)drds.\\
&\leq -4\int_0^1 s(1-s)^{\frac{1}{2}}e^{3-s}ds+14\int_0^{\frac{1}{4}}s\Big(\frac{1}{4}-s\Big)^{\frac{1}{2}}e^{3-s}ds-12\int_0^{\frac{1}{2}}s\Big(\frac{1}{2}-s\Big)^{\frac{1}{2}}e^{3-s}ds\\
&\leq-\frac{12329}{544}.
\end{aligned}$$

So, the condition (H_5) is satisfied.

Let $B=1$ and $c_1,c_2\in\mathbb{R}$ be such that $|c_1|>1$, $|c_2|>1$. Then we have

$$T_1N\big(c_1t+c_2t^2\big)+T_2N\big(c_1t+c_2t^2\big)=\big(2|c_2|+\sqrt{2|c_2|}\big)(d_{11}+d_{12})<0.$$

So, the condition (H_6) is satisfied.

Thus, all the assumptions of Theorem 3.4 are satisfied, and hence, the problem (4.1) has at least one solution.

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