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SOME ESTIMATES FOR THE SOLUTIONS OF THE FIRST ORDER NON-ALGEBRAIC CLASSES OF EQUATIONS

G. BARSEGIAN, F. MENG, W. YUAN

Guangzhou University, China[1]
Institue of mathematics of NAN Armenii
E-mails: *barsegiangrigor@yahoo.com; mfnfdbx@163.com; wjyuan1957@126.com*

Abstract. For some large classes of differential equations of the first order we give bounds for Ahlfors-Shimizu characteristics of meromorphic solutions in the complex plane of these equations. The considered equations largely generalize algebraic ones for which the obtained results imply the known Goldberg theorem. Characteristics of meromorphic solutions in a given domain weren't studied at all. We consider solutions in a given domain of some (large) equations and give bounds for Ahlfors-Shimizu characteristic for these solutions.

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1. Introduction

In this paper, we consider complex differential equations of the first order in two cases: for meromorphic solutions in the complex plane and in a given domain.

For algebraic equations, there is the classical Goldberg theorem related to meromorphic solutions in the complex plane. We study much larger equations; respectively our result implies as a particular case the mentioned Goldberg theorem.

Characteristics of meromorphic solutions in a given domain weren't studied. Recently G. Barsegian started similar studies; these studies were presented during his lectures in Guangzhou university in 2017. His aprroaches based on some new results related to arbitrary meromorphic functions in a given domain, see [4].

In this paper, we consider solutions in a given domain of some equations and give bounds for Ahlfors-Shimizu characteristic for the solutions.

2. Meromorphic solutions in the complex plane

We consider the following equation

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$$(*) \qquad \phi_0(z,w)\,(w')^m + \phi_1(z,w)\,(w')^{m-1} + \cdots + \phi_m(z,w) = 0,$$

where $\phi_i(z,w) = \sum_{\mu(i)=1}^{n(i)} \eta_{i,\mu(i)}(z)\chi_{i,\mu(i)}(w)$ for $i = 0,1,2,\dots,m$ and $\mu(i) = 1,2,\dots,n(i)$. Obviously we should exclude the case $\phi_0(z,w) \equiv 0$, since then the degree m reduces.

We put the following restrictions: all coefficient $\chi_{i,\mu(i)}(w)$ are meromorphic in $\mathbb{C}$, all coefficients $\eta_{i,\mu(i)}(z)$ with $i \neq 0$ are entire functions and all coefficients $\eta_{0,\mu(0)}(z)$ are polynomials. The equations $(*)$ with similar restrictions we will refer as $(F_p^{e,m}(\mathbb{C}))$.

Note that algebraic differential equations of the first order (see related studies in [8]) are particular cases of equations $(F_p^{e,m}(\mathbb{C}))$ when all mentioned above entire and meromorphic functions are polynomials.

For meromorphic function w in $\mathbb{C}$ we make use of classical Ahlfors-Shimizu characteristic

$$A(r) = A(r,w) = \frac{1}{\pi}\int\int_{D(r)} \frac{|w'|^2}{(1+|w|^2)^2}\,d\sigma,$$

where $D(r) = \{z : |z| < r\}$; for entire functions $\eta_{i,\mu(i)}$ we denote $M_i^{\mu(i)}(r) = \max_{z\in\partial D(r)} |\eta_{i,\mu(i)}(z)|$.

Theorem 2.1. *For any equation $(F_p^{e,m}(\mathbb{C}))$ with meromorphic solution $w(z)$ in the complex plane we have*

$$A(r) \le K_1 r^2 \max_{1\le i\le m}\left[\max_{\mu(i)} M_i^{\mu(i)}(r)\right]^{2/i}, \ \textit{for } r\to\infty,\ r\notin E, \tag{2.1}$$

where $K_1 < \infty$ is a constant independent of w and E is a set of finite logarithmic measure.

Corollary 2.1 ([7]). *Meromorphic solutions (in the complex plane) of algebraic differential equations are of finite order.*

Indeed, in this case all $M_i^{\mu(i)}(r)$ have polynomial growth so that Corollary 2.1 follows from (2.1). Thus Theorem 2.1 generalizes widely this old result in [7].

3. Meromorphic solutions in a given domain

Let D be a simply connected domain with smooth boundary ∂D of finite length.

Consider again equation $(*)$ by assuming that $w(z)$ is its meromorphic solutions in the closure $\bar{D} = D \cup \partial D$. In this case we assume that all $\eta_{i,\mu(i)}(z)$ are regular functions in $z \in \bar{D}$ and all $\chi_{i,\mu(i)}(w)$ are meromorphic functions in $w \in w(\bar{D})$. In addition we assume that $|\phi_0(z,w)| \ge c(D) = const > 0$ for $z \in \bar{D}$ and w with

$|w| < 10$. The equation with similar restrictions we will refer as $(F^{r,m}(D))$. In particular case when all $\chi_{i,\mu(i)}(w)$ are regular functions we will refer the equation as $(F^{r,r}(D))$.

For similar functions $w(z)$ as above, we consider *Ahlfors islands over the disk* $\Delta(\rho, a) = \{w : |w - a| < \rho\}$ which can be defined as those domains $\tilde{g}_k$ for $k = 1, 2, \ldots, n(D, \Delta(\rho, a), w)$, on the Riemann surface $\{w(z) : z \in \bar{D}\}$ which projected one to one and onto $\Delta(\rho, a)$ (see [1] or [9, Chapter 13]).

Defining $M_i^\mu(D) = \max_{z \in \partial D} |\eta_{i,\mu}(z)|$ and denoting by $S(D)$ the area of D, we formulate the following theorem.

Theorem 3.1. *Let $w(z)$ be a meromorphic in $\bar{D}$ solution of the equation $(F^{r,r}(D))$. Then for any set of disks $\Delta(\rho_\nu, a_\nu)$, $\nu = 1, 2, \ldots, q$, with non-intersecting closures we have*

$$\sum_{\nu=1}^{q} n(D, \Delta(\rho_\nu, a_\nu), w) \leq K_2 S(D), \tag{3.1}$$

where $K_2 < \infty$ is a constant independent of w; the constant depends only on the equation and D.

In the next result we make use of Ahlfors-Shimizu characteristic $A(D, w)$ (for arbitrary domain D) and another characteristics in Ahlfors theory of covering surfaces (see [1] and [9, Chapter 13])

$$L(D, w) = \int_{\partial D} \frac{|w'|}{(1 + |w|^2)} ds.$$

Theorem 3.2. *Let $w(z)$ be a meromorphic in $\bar{D}$ solution of the equation $(F^{r,m}(D))$ with $w(\bar{D})$ implying a disk $D(\varrho)$, where $\varrho = const > 0$. Then*

$$A(D, w) \leq K_3 S(D) + h_3 L(D, w), \tag{3.2}$$

where both constants K_3 and h_3 are independent on w; the constant depend only on the equation, D and ϱ.

4. Proofs

4.1. **Proof of Theorem 2.1.** We need some obvious comments.

In the case when the first coefficient $\phi_0(z, w)$ is nonconstant polynomial in z we can decompose $\phi_0(z, w)$ as $\Lambda_0(w) z^T + \Lambda_1(w) z^{T-1} + \cdots + \Lambda_T(w)$, $T \geq 1$, where $\Lambda_0(w)$ is a meromorphic function in w.

In the case when the first coefficient $\phi_0(z, w)$ does not depend on z we denote it by $\phi_0(w)$; obviously it is meromorphic in w.

All coefficients $\chi_{i,\mu(i)}(w)$ are meromorphic in the complex plane. This implies that for a fixed disk, say $D(10)$, any coefficients $\chi_{i,\mu(i)}(w)$, taken for any $i = 0, 1, 2, \ldots, m$, $\mu(i) = 1, 2, \ldots, n(i)$, have only a finite number of zeros in the disk $D(10)$. The same is true for the poles. The same is true for the zeros and poles of functions $\Lambda_0(w)$ and $\phi_0(w)$.

Now we take five non-passing through all these zeros and poles curves $\gamma_1, \ldots, \gamma_5$ in $D(10)$ with the distance between two different similar curves > 2. Then obviously all mentioned above functions do not vanish at any point $w = a \in \gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_5$ and there is a constant M such that the modules $|\cdots|$ of all mentioned above functions $\leq M$. Taking arbitrary five values $a_1, \ldots, a_5$ each belonging respectively to $\gamma_1, \ldots, \gamma_5$ we get the following statement.

Proposition 4.1. *There are five values $a_\nu \in D(10)$, $\nu = 1, \ldots, 5$; with non-intersecting closures of $\Delta(1, a_\nu)$, $\nu = 1, \ldots, 5$, such that*

1. All those functions $\chi_{i,\mu(i)}(w)$ which include variable w does not vanish at any point $w = a \in (a_1, \ldots, a_5)$ consequently

$$\Phi_i = \max_{1 \leq i \leq m} \sum_{\mu(i)=1}^{n(i)} |\chi_{i,\mu(i)}(a_\nu)| \leq n(i)M < \infty,$$

where Φ_i depends only on $a_1, \ldots, a_5$ and the involved coefficients;

2. Function $\phi_0(w)$ do not vanish at any point $w = a \in (a_1, \ldots, a_5)$, respectively we have

$$\Phi_0 = \min_{1 \leq \nu \leq q} |\phi_0(a_\nu)| > 0,$$

where Φ_0 is a constant depending only on $a_1, \ldots, a_5$ and ϕ_0;

3. Function $\Lambda_0(w)$ does not vanish at any point $w = a \in (a_1, \ldots, a_5)$, respectively we have

$$\Phi_\Lambda = \min_{1 \leq \nu \leq 5} |\Lambda_0(a_\nu)| > 0,$$

where Φ_Λ is a constant depending only on $a_1, \ldots, a_5$ and $\Lambda_0(w)$.

We need the following result.

Theorem A ([2, Theorem 1]). *For any meromorphic function w* in $\mathbb{C}$, *any set $a_1, a_2, \ldots, a_q \in \mathbb{C}$, $q > 4$, of distinct values and any monotonically decreasing on $[0, \infty)$ function $\psi(r)$ with $\psi(r) \to 0$ as $r \to \infty$, there is a set $E \subset [0, \infty)$ of finite logarithmic measure and for every $r \notin E$* there is *a subset $\{z_k^*(a_\nu)\} \subset D(r)$, $1 \leq \nu \leq q$, $1 \leq k \leq n^*(r, a_\nu)$, of the a_ν-points of w* for which

$$|w'(z_k^*(a_\nu))| \geq \psi(r) \frac{A^{1/2}(r)}{r}, \ 1 \leq \nu \leq q, \ 1 \leq k \leq n^*(r, a_\nu), \tag{4.1}$$

and

$$\sum_{\nu=1}^{q} n^*(r,a_\nu) \ge (q-4)A(r) - o[A(r)], \quad r\to\infty, \quad r\notin E. \tag{4.2}$$

A slight modification (see [2]) replaces $\psi(r)$ by a positive constant: *given positive* ε, $0<\varepsilon<1/2$, *there is a constant* $K=K\{a_1,a_2,\dots,a_q,\varepsilon)>0$ *such that* (4.1) *becomes*

$$|w'(z_k^*(a_\nu))| \ge K\frac{A^{1/2}(r)}{r}, \ 1\le\nu\le q, \quad 1\le k\le n^*(r,a_\nu) \tag{4.3}$$

for a set of a_ν*-points which satisfy*

$$\sum_{\nu=1}^{q} n^*(r,a_\nu) \ge (q-4-\varepsilon)A(r) - o[A(r)], \quad r\to\infty, \quad r\notin E. \tag{4.4}$$

Now we apply Theorem A and Proposition 4.1 to solutions w in the complex plane of equations $(F_p^{e,m}(\mathbb{C}))$.

Due to definitions and Theorem A we have the following.

Property 4.1. *Let* $a_1,\dots,a_5$ *be the points mentioned in Proposition 4.1. Then in any* $D(r)$ *with* $r\notin E$, *there is a set* $Z(r)$ *consisting of* $\sum_{\nu=1}^{5} n^*(r,a_\nu)$ *points* $z_k^*(a_\nu)$, $1\le\nu\le5$, $1\le k\le n^*(r,a_\nu)$, *such that at each similar point we have inequality (4.1) for* $1\le\nu\le5$, $1\le k\le n^*(r,a_\nu)$, *and we have also*

$$\sum_{\nu=1}^{5} n^*(r,a_\nu) \ge (1-\varepsilon)A(r) - o[A(r)], \quad r\to\infty, \quad r\notin E.$$

Due to the last inequality for $r\to\infty$, $r\notin E$, we have $\sum_{\nu=1}^{5} n^*(r,a_\nu)\to\infty$ and since the points $z_k^*(a_\nu)$ cannot have any limit point in any finite disk we obtain the following.

Property 4.2. *For any constant* $H>1$ *there is a constant* $r(H)$ *such that any disk* $D(r)$ *with* $r\ge r(H)$, $r\notin E$, *implies a point* $z_k^*(a_\nu)$ *occurring in Property 4.1 and satisfying also* $|z_k^*(a_\nu)|>H$.

Now we take any point $z_k^*(a_\nu)$ satisfying Property 4.2 and put it into equation $(*)$. We have

$$\phi_0(z_k^*(a_\nu), w(z_k^*(a_\nu)))\,(w'(z_k^*(a_\nu)))^m + \phi_1(z_k^*(a_\nu), w(z_k^*(a_\nu)))\,(w'(z_k^*(a_\nu)))^{m-1} +$$

$$\cdots + \phi_m(z_k^*(a_\nu), w(z_k^*(a_\nu))) = 0. \tag{4.5}$$

It is well known (see [10, Section III, Problem 21]) that all roots of an algebraic equation

$$z^m + b_1 z^{m-1} + \cdots + b_m = 0$$

are contained in the disk $|z| \leq \max_{1\leq i\leq m}(m|b_i|)^{1/i}$.

Applying this to (4.5) we get

$$|w'(z_k^*(a_\nu))| \leq \max_{1\leq i\leq m}\left[m\frac{\phi_i(z_k^*(a_\nu), w(z_k^*(a_\nu)))}{\phi_0(z_k^*(a_\nu), w(z_k^*(a_\nu)))}\right]^{1/i}. \tag{4.6}$$

Notice that item 1 in Proposition 4.1 is valid for $w = a = w(z_k^*(a_\nu))$ (since $w(z_k^*(a_\nu)) \in (a_1, \dots, a_5)$), so that we have

$$|\phi_i(z_k^*(a_\nu), w(z_k^*(a_\nu)))| \leq n(i)\Phi_i \max_{\mu(i)} M_i^{\mu(i)}(r), i = 1, 2, \dots, m. \tag{4.7}$$

Now we need to consider below bounds for ϕ_0 which in our case is polynomial in z and meromorphic in w.

We can have only the following cases for $\phi_0(z, w)$:

(case 1) there are non-constant polynomial coefficients $\eta_{0,\mu(0)}(z)$;

(case 2) all $\eta_{0,\mu(0)}(z)$ are constants however not all $\chi_{0,\mu(0)}(w)$ are constants;

(case 3) all $\eta_{0,\mu(0)}(z)$ are constants and, in addition, all $\chi_{0,\mu(0)}(w)$ are constants.

In the first case we can decompose $\phi_0(z, w)$ (as $\Lambda_0(w)z^T + \Lambda_1(w)z^{T-1} + \dots + \Lambda_T(w)$, $T \geq 1$) and note that at the pair (z, w), where $w \in (a_1, \dots, a_5)$ and $|z|$ is enough large, the term $\Lambda_0(a_\nu)\left(z_k^*(a_\nu)\right)^T$ become dominant in $\phi_0(z_k^*(a_\nu), a_\nu)$, so that we have $|\phi_0(z_k^*(a_\nu), a_\nu)| \geq (1/2)|\Lambda_0(a_\nu)||z_k^*(a_\nu)|^T$ for $|z_k^*(a_\nu)| > r_0$; here obviously r_0 depends on ϕ_0 and values $a_1, \dots, a_5$. Consequently taking $r(H)$ (in Property 4.2) equals to r_0 and taking into account item 3 in Proposition 4.1 (i.e. $\Phi_\Lambda = \min_{1\leq\nu\leq q}|\Lambda_0((a_\nu)| > 0$) and also Property 4.2 we obtain the following assertion: for any disk $D(r)$ with $r \geq r(H)$, $r \notin E$, there is a value $a_\nu \in (a_1, \dots, a_5)$ and corresponding point $z_k^*(a_\nu)$ with $|z_k^*(a_\nu)| > H$ such that

$$|\phi_0(z_k^*(a_\nu), w(z_k^*(a_\nu))| \geq \frac{1}{2}\Phi_\Lambda|(z_k^*(a_\nu)|^T > \frac{1}{2}\Phi_\Lambda H^T.$$

Making use of (4.6) and (4.7) applied at the same point (where $|z_k^*(a_\nu)| > H$) we have

$$|w'(z_k^*(a_\nu))| \leq \max_{1\leq i\leq m}\left[\frac{2mn(i)\Phi_i}{\Phi_\Lambda H^T}\max_{\mu(i)} M_i^{\mu(i)}(r)\right]^{1/i}.$$

Then applying (4.3) we get

$$A(r) \leq \frac{1}{K^2}r^2\max_{1\leq i\leq m}\left[\max_{\mu(i)} M_i^{\mu(i)}(r)\frac{2mn(i)\Phi_i}{\Phi_\Lambda H^T}\right]^{2/i}, \text{ for } r\to\infty,\ r\notin E,$$

so that obtain Theorem 2.1 with

$$K_1 \leq \frac{1}{K^2}\max_{1\leq i\leq m}\left[\frac{2mn(i)\Phi_i}{\Phi_\Lambda H^T}\right]^{2/i}.$$

In the second case, $\phi_0(z, w)$ become function in w merely, namely become function $\phi_0(w)$ in Proposition 4.1. Due to item 2 in Proposition 4.1, function $\phi_0(w)$ do not vanish at any point $w = a \in (a_1, \dots, a_5)$, respectively, we have $\Phi_0 = \min_{1 \le \nu \le q} |\phi_0(a_\nu)| > 0$. Repeating the above arguments we get Theorem 2.1 with

$$K_1 \le \frac{1}{K^2} \max_{1 \le i \le m} \left[\frac{2mn(i)\Phi_i}{\Phi_0} \right]^{2/i}.$$

In the third case, $\phi_0(z, w)$ is simply a constant: $\phi_0(z, w) = c_0 = const$ which should be non-zero, otherwise the degree m in our equation reduces. With the same arguments we obtain Theorem 2.1 with

$$K_1 \le \frac{1}{K^2} \max_{1 \le i \le m} \left[\frac{2mn(i)\Phi_i}{|c_0|} \right]^{2/i}.$$

The discussed three cases exclude each other so that in any given case we have Theorem 2.1 with one of the mentioned K_1.

4.2. **Proof of Theorem 3.1.** Assume that $e_k(\rho_k, a_k)$, $k = 1, 2, \dots, n$, are some disjoint domains in D which w maps one-to-one onto $\Delta(\rho_k, a_k)$; note that for different (even all) $e_k(\rho_k, a_k)$ the values of a_k and/or ρ_k may coincide. Any set of domains $e_k(\rho_k, a_k)$ contains a subdomain $e_k\left(\frac{\rho_k}{2}, a_k\right)$ such that $w\left(e_k\left(\frac{\rho_k}{2}, a_k\right)\right)$ coincides with $\Delta(\frac{\rho_k}{2}, a_k)$. Clearly, each $e_k\left(\frac{\rho_k}{2}, a_k\right)$ is contained in a domain $e_k(\rho_k, a_k)$.

The diameters $d\left(e_k\left(\frac{\rho_k}{2}, a_k\right)\right)$ of the domains $e_k\left(\frac{\rho_k}{2}, a_k\right)$ were first given in [2] and were applied to CDE. Later on similar applications were given in [6] and [5] based on the following inequality

$$\sum_{k=1}^{n} d\left(e_k\left(\frac{\rho_k}{2}, a_k\right)\right) \le \sqrt{\frac{3\pi}{2}} \sqrt{S(D)} \sqrt{n},$$

where $S(D)$ is the Euclidean area of D. We need a slightly more sharp inequality established in [[3], inequality (6′)]:

$$\sum_{k=1}^{n} d\left(e_k\left(\frac{\rho_k}{2}, a_k\right)\right) \le \sqrt{\frac{3\pi}{2}} \sum_{k=1}^{n} S^{1/2}(e_k(\rho_k, a_k)). \tag{4.8}$$

In addition we have also the following.

Lemma 4.1 ([3, Lemma 2]). *Let z_k be the point in $e_k(\rho_k, a_k)$ which w maps onto a_k, i.e. $w(z_k) = a_k$. Then*

$$|w'(z_k)| \ge \frac{\rho_k}{2d\left(e_k\left(\frac{\rho_k}{2}, a_k\right)\right)}, \quad k = 1, 2, \dots, n. \tag{4.9}$$

Now we take q pairwise different values a_ν, $q = 1, 2, \dots, q$, and consider as $\cup e_k\left(\frac{\rho_k}{2}, a_k\right)$ the union of all domains $e_{\nu,t}$, $\nu = 1, 2, \dots, q$, $t = 1, 2, \dots, n(a_\nu)$, which w maps one-to-one and onto the disk $\Delta(\rho_\nu, a_\nu)$. (Important remark: in this case the disk $\Delta(\rho_\nu, a_\nu)$ remains the same for all $t = 1, 2, \dots, n(a_\nu)$). In other words, function w maps each domain $e_{\nu,t}$ onto an Ahlfors simple island over $\Delta(\rho_\nu, a_\nu)$ so that $n(a_\nu)$ becomes (in this case) the usual number $n(D, \Delta(\rho_\nu, a_\nu), w)$ of simple islands over $\Delta(\rho_\nu, a_\nu)$.

Thus we can rewrite (4.8) and (4.9) as

$$\sum_{\nu=1}^{q} \sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} d\left(e_{\nu,t}\right) \leq \sqrt{\frac{3\pi}{2}} \sum_{\nu=1}^{q} \sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S^{1/2}\left(e_{\nu,t}\right) \tag{4.10}$$

and

$$|w'(z_t(a_\nu))| \geq \frac{\rho_\nu}{2d\left(e_{\nu,t}\right)},\ \nu = 1, 2, \ldots, q,\ t = 1, 2, \ldots, n(D, \Delta(\rho_\nu, a_\nu), w), \tag{4.11}$$

where $z_t(a_\nu) \in e_{\nu,t}$ and satisfies $w(z_t(a_\nu)) = a_\nu$.

Denote $N = \sum_{\nu=1}^{q} n(D, \Delta(\rho_\nu, a_\nu), w)$. Due to (4.10) we conclude that the set of all domains $e_{\nu,t}$ implies some domains $\tilde{e}_s$, $s = 1, 2, \ldots, \tilde{n} = \left[\frac{1}{2}N + 1\right]'$, (here $[x]'$ means entire part of x), which satisfy

$$d\left(\tilde{e}_s\right) \leq \frac{1}{\tilde{n}}\sqrt{\frac{3\pi}{2}} \sum_{\nu=1}^{q} \sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S^{1/2}\left(e_{\nu,t}\right);$$

indeed assuming contrary we come to contradiction with inequality (4.8). Since $(N/\tilde{n}) \leq 2$, we have for any $s = 1, 2, \ldots, \tilde{n} = \left[\frac{1}{2}N + 1\right]'$,

$$d\left(\tilde{e}_s\right) \leq \sqrt{6\pi}\frac{1}{N} \sum_{\nu=1}^{q} \sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S^{1/2}\left(e_{\nu,t}\right). \tag{4.12}$$

Since $\tilde{e}_s$ coincides with one of $e_{\nu,t}$, we conclude $\tilde{e}_s$ implies an a_ν-point $z_t(a_\nu)$; to stress that this is namely a point lying in $\tilde{e}_s$ (which satisfies (4.12)) we denote it by $\tilde{z}_t(a_\nu)$. This means that (4.10) is valid also for any given $\tilde{e}_s$ with corresponding point $\tilde{z}_t(a_\nu)$. Now (4.10) and (4.12) yield

$$N \leq \frac{\sqrt{24\pi}}{\rho_\nu} |w'(\tilde{z}_t(a_\nu))| \sum_{\nu=1}^{q} \sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S^{1/2}\left(e_{\nu,t}\right). \tag{4.13}$$

Applying Cauchy-Schwarz inequality to the last double sum we have

$$\sum_{\nu=1}^{q} \sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S^{1/2}\left(e_{\nu,t}\right) \leq N^{1/2} \left(\sum_{\nu=1}^{q} \sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S\left(e_{\nu,t}\right) \right)^{1/2},$$

so that (4.13) yields

$$N \leq \frac{24\pi}{\rho_\nu^2} \left|w'(\tilde{z}_t(a_\nu))\right|^2 \sum_{\nu=1}^{q} \sum_{k=1}^{n(D,\Delta(\rho_\nu,a_\nu),w)} S\left(e_{\nu,t}\right),$$

and taking into account that the last sum dominated by the area $S(D)$ we obtain

$$\sum_{\nu=1}^{q} n(D, \Delta(\rho_\nu, a_\nu), w) \leq \frac{24\pi}{\rho_\nu^2} \left|w'(\tilde{z}_t(a_\nu))\right|^2 S(D). \tag{4.14}$$

Assume now that $w(z)$ is a solution of $(F^{r,r}(D))$ in Theorem 3.1. Considering the equation $(F^{r,r}(D))$ at this point $\tilde{z}_t(a_\nu)$, we notice that all coefficients in $(F^{r,r}(D))$ are defined at this point since we assumed in Theorem 3.1 that $(a_1,\dots,a_q)\in w(\bar{D})$. Arguing as in (4.6) we obtain

$$|w'(\tilde{z}_t(a_\nu))| \le \max_{1\le i\le m}\left[m\frac{\phi_i(\tilde{z}_t(a_\nu),w(\tilde{z}_t(a_\nu)))}{\phi_0(\tilde{z}_t(a_\nu),w(\tilde{z}_t(a_\nu)))}\right]^{1/i}. \tag{4.15}$$

Since the values $a_1,\dots,a_q$ are fixed in Theorem 3.1 and functions $\chi_{i,\mu(i)}(w)$ are regular (so that all $\chi_{i,\mu(i)}(a_\nu)$ are finite for any $\nu=1,2,\dots,q$) we have

$$\Phi_i = \max_{1\le\nu\le q}\sum_{\mu(i)=1}^{n(i)}|\chi_{i,\mu(i)}(a_\nu)|<\infty;$$

note that Φ_i depend only on functions $\chi_{i,\mu(i)}(w)$ and values $a_1,\dots,a_q$.

Applying this to (4.15) we get

$$\begin{aligned}|w'(\tilde{z}_t(a_\nu))| &\le \max_{1\le i\le m}\left[mn(i)\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\sum_{\mu(i)=1}^{n(i)}|\chi_{i,\mu(i)}(a_\nu)|\right]^{1/i}\\ &\le \max_{1\le i\le m}\left[mn(i)\Phi_i\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\right]^{1/i}\\ &\le \max_{1\le i\le m}\left[mn(i)\Phi_i\right]^{1/i}\max_{1\le i\le m}\left[\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\right]^{1/i}.\end{aligned}$$

In turn applying the last inequality to (4.14) we obtain the following inequality

$$\sum_{\nu=1}^{q}n(D,\Delta(\rho_\nu,a_\nu),w)\le\frac{24\pi}{\rho^2}\max_{1\le i\le m}\left[mn(i)\Phi_i\right]^{2/i}\max_{1\le i\le m}\left[\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\right]^{2/i}S(D),$$

i.e. we obtain Theorem 3.1 with

$$K_2=\frac{24\pi}{\rho^2}\max_{1\le i\le m}\left[mn(i)\Phi_i\right]^{2/i}\max_{1\le i\le m}\left[\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\right]^{2/i}.$$

4.3. **Proof of Theorem 3.2.** Let $w(z)$ be a meromorphic function in $\bar{D}$ solving equation $(F^{r,m}(D))$. Clearly we should assume that all $\chi_{i,\mu(i)}(w)$ defined on $w(\bar{D})$.

Remember that $D(\varrho)\subset w(\bar{D})$ so that all coefficients $\chi_{i,\mu(i)}(a_\nu)$ defined at any point $w=a\in D(\varrho)$.

Arguing as in the Proposition 4.1 we can fix five points $a_\nu\in D(\varrho)$, $\nu=1,\dots,5$, with non-intersecting closures of $\Delta(\varrho/10,a_\nu)$, $\nu=1,\dots,5$, such that these points do not pass through zeros and poles of these coefficients. Respectively we have

$$\Phi_i=\max_{1\le i\le m}\sum_{\mu(i)=1}^{n(i)}|\chi_{i,\mu(i)}(a_\nu)|<\infty.$$

Repeating the proof of Theorem 3.2 with similar $a_1,\ldots,a_5$, we find first the point $\tilde{z}_t(a_\nu)$, (where $a_\nu \in (a_1,\ldots,a_5)$) and obtain (instead of (4.14)) the following inequality

$$\sum_{\nu=1}^{5} n\left(D, \Delta\left(1, a_\nu\right), w\right) \le \left(\frac{10}{\varrho}\right)^2 24\pi \left|w'(\tilde{z}_t(a_\nu))\right|^2 S(D).$$

Then we put this $\tilde{z}_t(a_\nu)$ into equation $(F^{r,m}(D))$ and arguing as above (after (4.15)), we get similarly

$$\begin{aligned} |w'(\tilde{z}_t(a_\nu))| &\le \max_{1\le i\le m}\left[mn(i)\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\sum_{\mu(i)=1}^{n(i)}|\chi_{i,\mu(i)}(a_\nu)|\right]^{1/i} \\ &\le \max_{1\le i\le m}\left[mn(i)\Phi_i\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\right]^{1/i}. \end{aligned}$$

The last two inequalities yield

$$\sum_{\nu=1}^{5} n\left(D, \Delta\left(, a_\nu\right), w\right) \le K_3 S(D), \tag{4.16}$$

where

$$K_3 = 24\pi\left(\frac{10}{\varrho}\right)^2 \max_{1\le i\le m}\left[mn(i)\Phi_i\right]^{2/i} \max_{1\le i\le m}\left[\max_{\mu(i)}\frac{M_i^{\mu(i)}(D)}{c(D)}\right]^{2/i};$$

so that K_3 depends only on equation $(F^{r,m}(D))$ and ϱ.

Finally we need the second fundamental theorem in Ahlfors theory of covering surfaces (see [1] and [9, Chapter 13]): *for any w in $\bar{D}$ and any set of pairwise different points a_ν, $\nu = 1, 2, \ldots, q$, $q > 4$, we have*

$$(q-4)A(D,w) \le \sum_{\nu=1}^{q} n(D, \Delta(\rho_\nu, a_\nu), w) + hL(D,w),$$

where $h < \infty$ is a constant depending on $\Delta(\rho_\nu, a_\nu)$, $\nu = 1, 2, \ldots, q$.

Applying this inequality with the above five values $a_1,\ldots,a_5$, we have

$$A(D,w) \le \sum_{\nu=1}^{5} n\left(D, \Delta\left(\frac{1}{4}, a_\nu\right), w\right) + h_3 L(D,w),$$

where h_3 depends on these values $a_1,\ldots,a_5$; in other words depends only on ϱ. From here taking into account (4.16) we obtain Theorem 3.2 with the above defined K_3.

Список литературы

[1] L. Ahlfors, "Zur Theorie der Überlagerungsflächen", Acta Soc. Sci. Fenn., 1/9, 1 – 40 (1930).

[2] G. Barsegian, "Estimates of derivatives of meromorphic functions on sets of a-points", Journal of London Math. Soc., **34**(2), 543 – 400 (1986).

[3] G. Barsegian, "A new principle for arbitrary meromorphic functions in a given domain", Georgian Math. Journal, **25**(2), 181 – 186 (2018).

[4] G. Barsegian, "Survey of Some General Properties of Meromorphic Functions in a Given Domain", 47 – 67, in book "Analysis as a Life. Dedicated to Heinrich Begehr on the Occasion of his 80th Birthday", Editors: Rogosin S., Celebi, O., Birkhauser (2019).

[5] G. Barsegian, I. Laine and D. T. Lê, "On topological behavior of solutions of some algebraic differential equations", Complex Variables and Elliptic Equations, **53**, 411 – 421 (2008).
[6] G. Barsegian and D. T. Lê, "On a topological description of solutions of complex differential equations", Complex Variables, **50**(5), 307 – 318 (2005).
[7] A. A. Gol'dberg, "On the single valued integrals of differential equations of the first order", Ukrainian Math. J., **8**, 254 – 261 (1956).
[8] I. Laine, "Nevanlinna theory and complex differential equations", Berlin: Walter de Gruyter (1993).
[9] R. Nevanlinna, Eindeutige Analytische Funktionen, Springer-Verlag, Berlin (1936).
[10] G. Polya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Zweiter Band, Springer-Verlag, Berlin-Göttingen-Heidelberg (1954).

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