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NUMERICAL RESULTS FOR SOBOLEV'S FUNCTION Q OF RADIATIVE TRANSFER

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The function Q is the solution of the Fredholm integral equation

$$Q(t, x) = 1 + \frac{\lambda}{2}\int_0^x E_1(|t-y|)\, Q(y, x)\, dy, \qquad 0 < t < x, \quad 0 < \lambda \leqslant 1.$$

Sobolev has emphasized the importance of this function in the theory of radiative transfer and multiple scattering. In this paper the method of invariant imbedding is used to obtain numerical values of the function Q.

1. *Introduction.* In a series of papers V. V. Sobolev [1—4] has discussed the importance of the function Q in radiative transfer. This function is the solution of the Fredholm integral equation

$$Q(t, x) = 1 + \frac{\lambda}{2}\int_0^x E_1(|t-y|)\, Q(y, x)\, dy.$$

$$0 \leqslant t \leqslant x, \quad 0 \leqslant \lambda \leqslant 1,$$

where the kernel E_1 is given by the integral

$$E_1(r) = \int_0^1 e^{-r/z}\, dz/z, \qquad r > 0.$$

Physically Q may be viewed as the average number of scatterings which a photon undergoes when it is isotropically emitted at altitude t in a slab of thickness x; the parameter λ is the albedo for single scattering, and each scattering is isotropic. This function has also been discussed by M. Heaslet and R. Warming [5].

In this paper the invariant imbedding [6] approach is used to transform the integral equation for the function Q into a Cauchy problem which is readily solved via modern digital or analog computing machines. Numerical results are presented in the form of tables and graphs, and a discussion of the accuracy attained is presented.

The discussion is self-contained, and no previous contact with invariant imbedding is assumed.

2. *Derivation of Cauchy System.* Consider the integral equation

$$Q(t, x) = 1 + \frac{\lambda}{2} \int_0^x E_1(|t-y|)\, Q(y, x)\, dy, \tag{1}$$
$$0 \leqslant t \leqslant x, \quad 0 \leqslant \lambda \leqslant 1,$$

where

$$E_1(r) = \int_0^1 e^{-r/z} z^{-1}\, dz, \quad r > 0, \tag{2}$$

Differentiate with respect to x to obtain the equation

$$\begin{aligned} Q_x(t, x) &= (\lambda/2)\, E_1(x-t)\, Q(x, x) + \\ &+ (\lambda/2) \int_0^x E_1(|t-y|)\, Q_x(y, x)\, dy, \quad 0 \leqslant t \leqslant x. \end{aligned} \tag{3}$$

Next intrcduce Sobolev's function Φ as the solution of the integra equation

$$\Phi(t, x) = (\lambda/2)\, E_1(x-t) + (\lambda/2) \int_0^1 E_1(|t-y|)\, \Phi(y, x)\, dy, \tag{4}$$
$$0 \leqslant t \leqslant x.$$

By regarding *Eq.* (3) as an integral equation for the function Q_x and keeping in mind the integral equation above for the function Φ, it is seen that

$$Q_x(t, x) = \Phi(t, x)\, Q(x, x), \quad x > t. \tag{5}$$

This is the basic differential equation for the function $Q(t, x)$. We shall first consider the function $\Phi(t, x)$ and then the function $Q(x, x)$.

Introduce the source function J as the solution of the integral equation

$$J(t, x, z) = \frac{\lambda}{4} e^{-(x-t)/z} + \frac{\lambda}{2} \int_0^x E_1(|t-y|)\, J(y, x, z)\, dy, \tag{6}$$

$$0 \leqslant t \leqslant x, \quad 0 \leqslant z \leqslant 1, \quad 0 \leqslant \lambda \leqslant 1.$$

In view of Eqs. (2) and (4) we see that the function Φ may be expressed simply in terms of the function J. The exact relation is

$$\Phi(t, x) = 2 \int_0^1 J(t, x, z')\, dz'/z', \qquad 0 \leqslant t \leqslant x. \tag{7}$$

We shall now derive a Cauchy system for the function J. Differentiation of Eq. (6) with respect to x yields the equation

$$\begin{aligned} J_x(t, x, z) = {} & -z^{-1}(\lambda/4)\, e^{-(x-t)/z} + (\lambda/2)\, E_1(x-t)\, J(x, x, z) + \\ & + (\lambda/2) \int_0^x E_1(|t-y|)\, J_x(y, x, z)\, dy. \end{aligned} \tag{8}$$

By regarding the last equation as an integral equation for the function J_x with two inhomogeneous terms, we find that its solution is

$$J_x(t, x, z) = -z^{-1} J(t, x, z) + \Phi(t, x)\, J(x, x, z). \tag{9}$$

$$0 \geqslant t.$$

The function $J(x, x, z)$ must now be considered. The integral equation (6) at $t = x$ yields the representation

$$J(x, x, z) = \frac{\lambda}{4} + \frac{\lambda}{2} \int_0^x E_1(x-y)\, J(y, x, z)\, dy. \tag{10}$$

Using the definition of the function E_1 we may write

$$J(x, x, z) = \frac{\lambda}{4} + \frac{\lambda}{2}\int_0^x\int_0^1 e^{-(x-y)/z'}(dz'/z')J(y, x, z)\,dy =$$
$$= \frac{\lambda}{4} + \frac{\lambda}{2}\int_0^1 (dz'/z')\int_0^x e^{-(x-y)/z'}J(y, x, z)\,dy. \tag{11}$$

Finally we introduce the reflection function R through the definition

$$R(v, z, x) = 4\int_0^x e^{-(x-y)/v}J(y, x, z)\,dy, \quad 0 \leqslant v, \quad z \leqslant 1, \quad 0 \leqslant x. \tag{12}$$

It follows that

$$J(x, x, z) = \frac{\lambda}{4} + \frac{\lambda}{8}\int_0^1 R(z', z, x)\,dz'/z'. \tag{13}$$

We shall now obtain the Cauchy system for the function R.

Differentiate both sides of Eq. (12) with respect to x. The result is

$$R_x(v, z, x) = 4\Big\{J(x, x, z) - v^{-1}\int_0^x e^{-(x-y)/v}J(y, x, z)\,dy +$$
$$+ \int_0^x e^{-(x-y)/v}[-z^{-1}J(y, x, z) + \Phi(y, x)J(x, x, z)]\,dy\Big\}. \tag{14}$$

A slight rearrangement yields the equation

$$R_x(v, z, x) = -\left(\frac{1}{v} + \frac{1}{z}\right)R(v, z, x) +$$
$$+ 4J(x, x, z)\left\{1 + \int_0^x e^{-(x-y)/v}\Phi(y, x)\,dy\right\}. \tag{15}$$

The integral in the last equation may be evaluated by observing that

$$\int_0^1 e^{-(x-y)/v}\Phi(y, x)\,dy = \int_0^x e^{-(x-y)/v}2\int_0^1 J(y, x, z')(dz'/z')\,dy =$$
$$= 2\int_0^1 (dz'/z')\int_0^x e^{-(x-y)/v}J(y, x, z')\,dy = \frac{1}{2}\int_0^1 R(v, z', x)\,dz'/z'. \tag{16}$$

In view of this result and Eq. (13), Eq. (15) may be rewritten as

$$R_x(v, z, x) = -\left(\frac{1}{v} + \frac{1}{z}\right) R(v, z, x) +$$

$$+\lambda\left[1 + \frac{1}{2}\int_0^1 R(z', z, x)\,dz'/z'\right]\left[1 + \frac{1}{2}\int_0^1 R(v, z', x)\,dz'/z'\right], \qquad (17)$$

$$x > 0.$$

We may now pass to a consideration of the second factor on the right side of Eq. (5), $Q(x, x)$. From Eq. (1) we see that

$$Q(x, x) = 1 + (\lambda/2)\int_0^x E_1(x-y)\,Q(y, x)\,dy =$$

$$= 1 + (\lambda/2)\int_0^x\int_0^1 e^{-(x-y)/z}\,dz/z\,Q(y, x)\,dy = 1 + (\lambda/2)\int_0^1 e(z, x)\,dz/z, \qquad (18)$$

where we have introduced the new emergence function e,

$$e(v, x) = \int_0^x e^{-(x-y)/v}\,Q(y, x)\,dy, \quad x \geqslant 0, \quad 0 \leqslant v \leqslant 1. \qquad (19)$$

A Cauchy system for the function e will now be obtained.

Differentiate Eq. (19) with respect to x, which yields the relations

$$e_x(v, x) = Q(x, x) - v^{-1}e(v, x) + \int_0^x e^{-(x-y)/v}\,\Phi(y, x)\,Q(x, x)\,dy =$$

$$= -v^{-1}e(v, x) + Q(x, x)\left[1 + \int_0^x e^{-(x-y)/v}\,\Phi(y, x)\,dy\right]. \qquad (20)$$

By using Eqs. (18) and (16) we find that the function e satisfies the differential equation

$$e_x(v, x) = -v^{-1}e(v, x) +$$

$$+\left[1 + (\lambda/2)\int_0^1 e(z, x)\,dz/z\right]\left[1 + (1/2)\int_0^1 R(v, z', x)\,dz'/z'\right]. \qquad (21)$$

Let us now summarize the basic differential equations and initial conditions of the Cauchy system. The functions R and e satisfy the differential equations

$$R_x(v, z, x) = -\left(\frac{1}{v} + \frac{1}{z}\right) R(v, z, x) +$$

$$+\lambda \left[1 + \frac{1}{2}\int_0^1 R(z', z, x)\, dz'/z'\right]\left[1 + \frac{1}{2}\int_0^1 R(v, z', x)\, dz'/z'\right],$$

$$e_x(v, x) = -v^{-1} e(v, x) +$$

$$+\left[1 + \frac{1}{2}\int_0^1 e(z', x)\, dz'/z'\right]\left[1 + \frac{1}{2}\int_0^1 R(v, z', x)\, dz'/z'\right],$$

$$x \geqslant 0, \quad 0 \leqslant v, \quad z \leqslant 1.$$

The initial conditions at $x = 0$ are

$$R(v, z, 0) = 0, \quad 0 \leqslant v, \ z \leqslant 1, \tag{22}$$

$$e(v, 0) = 0, \quad 0 \leqslant v \leqslant 1. \tag{23}$$

At $x = t$ the functions J and Q satisfy the initial conditions

$$J(t, t, z) = \frac{\lambda}{4}\left[1 + \frac{1}{2}\int_0^1 R(z', z, x)\, dz'/z'\right], \quad 0 \leqslant z \leqslant 1, \tag{24}$$

$$Q(t, t) = 1 + \frac{\lambda}{2}\int_0^1 e(z', x)\, dz'/z'. \tag{25}$$

The differential equations for the functions J and Q, for $x > t$, are

$$J_x(t, x, z) = -z^{-1} J(t, x, z) +$$

$$+\frac{\lambda}{2}\left[1 + \frac{1}{2}\int_0^1 R(z', z, x)\, dz'/z'\right]\int_0^1 J(t, x, z')\, dz'/z', \tag{26}$$

$$x \geqslant t, \quad 0 \leqslant z \leqslant 1,$$

$$Q_x(t, x) = 2\left[1 + \frac{\lambda}{2}\int_0^1 e(z', x)\, dz'/z'\right]\int_0^1 J(t, x, z')\, dz'/z', \tag{27}$$

$$x > t.$$

3. *Numerical Method.* The Cauchy system can readily be solved numerically using Gaussian quadrature to approximate the integrals. This is the method of lines, and it reduces the original Cauchy system to a nonlinear system of ordinary differential equations with known initial conditions. Let $r_1, r_2, \dots, r_N$ be the N roots of the shifted Legendre polynomial $P_N(1-2z)$, and let $w_1, w_2, \dots, w_N$ be the corresponding Christoffel weights. These are tabulated in [7]. Also let

$$R(r_i, r_j, x) = R_{ij}(x), \quad i, j = 1, 2, \dots, N, \quad x > 0. \tag{28}$$

Then for the function $R_{ij}(x)$ we may write the approximate relation

$$\begin{aligned} R'_{ij}(x) &= -\left(\frac{1}{r_i} + \frac{1}{r_j}\right) R_{ij}(x) + \\ &+ \lambda \left[1 + \frac{1}{2} \sum_{m=1}^{N} R_{mj}(x) \frac{w_m}{r_m}\right]\left[1 + \frac{1}{2} \sum_{m=1}^{N} R_{im}(x) \frac{w_m}{r_m}\right], \\ & x \geqslant 0, \end{aligned} \tag{29}$$

and the initial conditions

$$R_{ij}(0) = 0, \quad i, j = 1, 2, \dots, N, \tag{30}$$

where a prime represents differentiation with respect to x. Previous numerical experiments indicate that $N \cong 7$ is appropriate [7]. Observe that Eq. (29) represents N^2 ordinary differential equations. The fact that the function R is symmetric in its first two arguments allows us to reduce this number to $N(N+1)/2$, a significant saving when $N \cong 7$.

The system for function e is

$$e'_i(x) = r_i^{-1} e_i(x) + \left[1 + \frac{1}{2} \sum_{m=1}^{N} e_m(x) \frac{w_m}{r_m}\right]\left[1 + \frac{1}{2} \sum_{m=1}^{N} R_{im}(x) \frac{w_m}{r_m}\right], \quad x \geqslant 0, \tag{31}$$

$$e_i(0) = 0, \quad i = 1, 2, \dots, N. \tag{32}$$

The systems (29) and (31) are to be integrated simultaneously, a total of $[N(N+1)/2] + N$ ordinary differential equations.

Let t be a fixed positive number. At $x = t > 0$ we adjoin the system of ordinary differential equations

$$J'_i(x) = -r_i^{-1} J_i(x) + \frac{\lambda}{2}\left[1 + \frac{1}{2} \sum_{m=1}^{N} R_{mj}(x) \frac{w_m}{r_m}\right] \sum_{m=1}^{N} J_m(m) \frac{w_m}{r_m}, \tag{33}$$

$$i = 1, 2, \dots, N, \quad x > t,$$

$$Q'(x) = 2\left[1 + \frac{\lambda}{2}\sum_{m=1}^{N} e_m(x)\frac{w_m}{r_m}\right]\sum_{m=1}^{N} J_m(x)\frac{w_m}{r_m}, \qquad x > t. \tag{34}$$

In the above $N+1$ equations we employ the notation

$$J_i(x) = J(t, x, r_i), \quad i = 1, 2, \ldots, N, \tag{35}$$

$$Q(x) = Q(t, x), \quad x > t. \tag{36}$$

The initial conditions at $x = t$ are

$$J_i(t) = \frac{\lambda}{4}\left[1 + \sum_{m=1}^{N} R_{mi}(t)\frac{w_m}{r_m}\right], \quad i = 1, 2, \ldots, N, \tag{37}$$

and

$$Q(t) = 1 + \frac{\lambda}{2}\sum_{m=1}^{N} e_m(t)\frac{w_m}{r_m}. \tag{38}$$

On the interval $0 < x \leqslant t$ we integrate $N(N+1)/2 + N$ ordinary differential equations. On the interval $t \leqslant x \leqslant x_{max}$ we integrate $N(N+1)/2 + 2N + 1$ ordinary differential equations, each with a known initial condition.

As a rule we shall require the values of Q for a set of values of t. rather than a single one. Let these values be denoted $t_1, t_2, \ldots, t_M$, with

$$0 < t_1 < t_2 < \cdots < t_M < x_{max}. \tag{39}$$

In this case for $1 \leqslant x \leqslant t_1$ we integrate the system in Eqs. (29) and (31) subject to the initial conditions at $x = 0$ in Eqs. (30) and (32). At $x = t_1$ we adjoin $N+1$ differential equations of the form of Eqs. (33) and (34) for the functions $J_i(x) = J(t_1, x, r_i)$, $i = 1, 2, \ldots, N$, and $Q(t_1, x)$, $t_1 \leqslant x \leqslant t_2$. The initial conditions at $x = t_1$ are

$$J_i(t_1) = \frac{\lambda}{4}\left[1 + \sum_{m=1}^{N} R_{mi}(t_1)\frac{w_m}{r_m}\right] \tag{40}$$

and

$$Q(t_1) = 1 + \frac{\lambda}{2}\sum_{m=1}^{N} e_m(t_1)\frac{w_m}{r_m}. \tag{41}$$

The right sides are known numerically at this point. At $x = t_2$ we adjoin $N+1$ additional differential equations for the functions $J(t_2, x, r_1)$, $J(t_2, x, r_2), \ldots, J(t_2, x, r_N)$ and $Q(t_2, x)$ together with the appropriate initial conditions. On the interval $t_M \leqslant x \leqslant x_{max}$ the number of ordinary differential equations being integrated is $N(N+1)/2+M(N+1)$. For $N=7$ and $M=10$ we integrate numerically 108 simultaneous ordinary differential equations, a reasonable number in 1970.

4. *Numerical Results.* Tables 1 to 4 present calculated values of the function $Q(t, x)$ for $\lambda = 0.2$, 0.6, 0.9 and 1.0 respectively. Figure 1 to 4 are graphs of the function $Q(t, x)$ for $\lambda = 0.2$, 0.6, 0.9 and 1.0. Figure 5 gives graphs of the maximum value of $Q(t, x)$, which occurs at

Table 1

$Q(t, x)$ for $\lambda = 0.2$

t	x	$Q(t, x)$
.005	1	1.0989
.010		1.1013
.500		1.1543
1.000		1.0964
.500	5	1.2009
1.000		1.2256
1.500		1.2365
2.000		1.2413
2.500		1.2427
5.000		1.1178

Table 2

$Q(t, x)$ for $\lambda=0.6$

t	x	$Q(t, x)$
.1	1	1.5181
.2		1.5813
.3		1.6212
.4		1.6436
.5		1.6508
1.00		1.3962
2.00	5	2.3669
2.50		2.3813
3.00		2.3670
3.50		2.3198
5.00		1.5769

Table 3

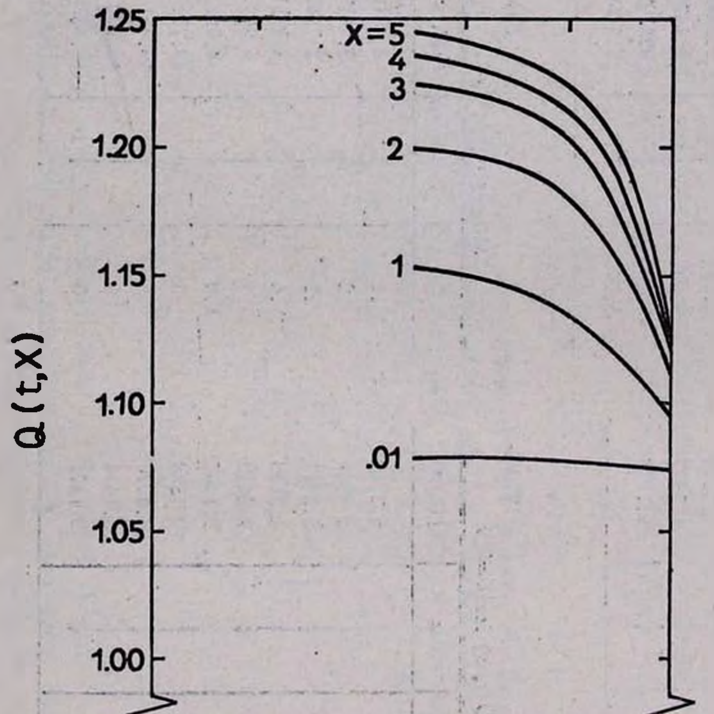
$Q(t, x)$ for $\lambda=0.9$

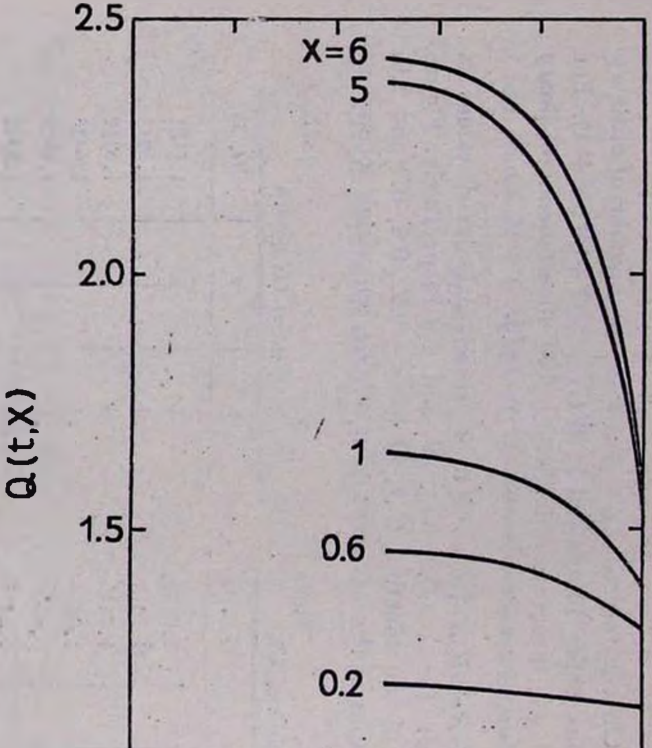
t	x	$Q(t, x)$
1.000	4	5.3710
2.000		5.9849
3.000		5.3712
3.500		4.5153
4.000		2.8412
.500	5	5.1407
5.000		9.1338
7.000		8.6070
9.500		5.1416
10.000		3.1477

Table 4

$Q(t, x)$ for $\lambda=1.0$

t	x	$Q(t, x)$
.1	1	2.4019
.2		2.5917
.3		2.7147
.4		2.7851
.5		2.8080
1.0		2.0673
3	10	43.771
5		49.781
6		48.282
10		9.8870


1.25
1.20
1.15
1.10
1.05
1.00
Q (t,X)
X=5
4
3
2
1
.01


2.5
2.0
1.5
Q(t,X)
X=6
5
1
0.6
0.2

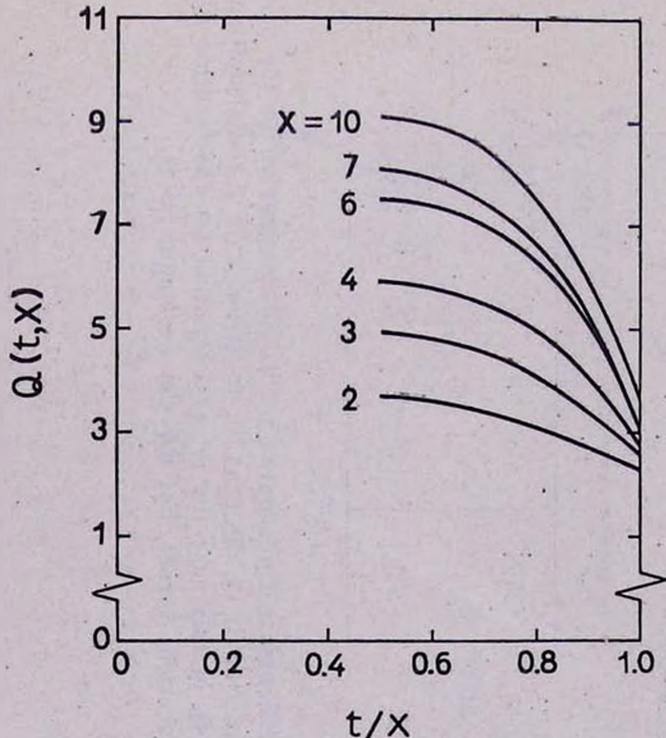


Fig. 3. Graphs of the function Q for $\lambda=0.9$.

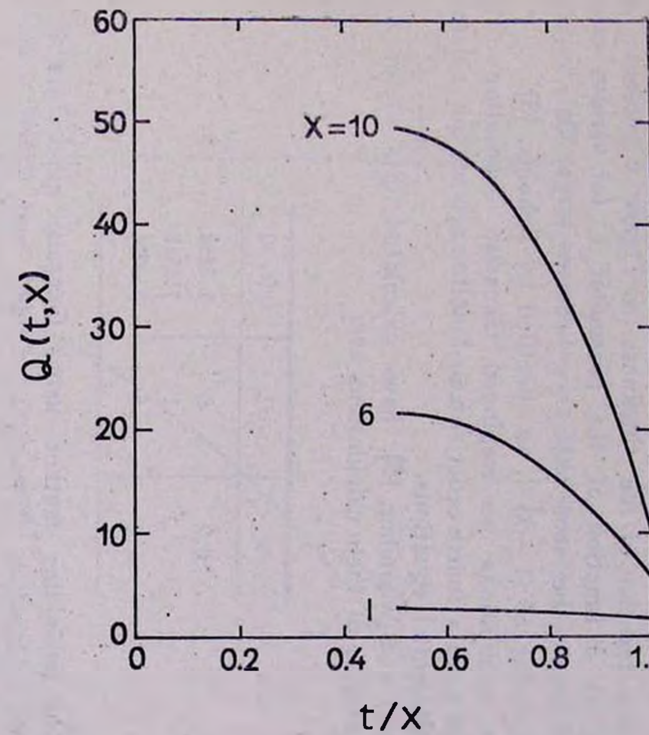


Fig. 4. Graphs of the function Q for $\lambda=10$.

$t/x = 0.5$, as a function of the thickness x. Figure 6 presents $Q_{max} = Q(x/2, x)$ as a function of the parameter λ, for various values of thickness x. It can be seen that as x becomes large, Q_{max} approaches the limiting value of $(1-\lambda)^{-1}$, as specified by Sobolev, [3].

In the calculations we employed Gaussian quadrature of orders 7 and 9 and used a fourth order Adams-Moulton scheme for solving the ordinary differential equations.

Heaslett and Warming, [5], have calculated $Q(t, x)$ for $\lambda = 1.0$. Numerical results of their calculations are:

x	t	$Q(t, x)$
0.2	0	1.2868
	.1	1.3768
1.0	.5	2.8084

The invariant imbedding method using Gaussian quadrature of orders 7 and 9, and integration step sizes of 0.005 and 0.0025 respectively gives, by comparison,

x	t	Order of Quadrature	$Q(t, x)$
0.2	0	7	1.2864
		9	1.2867
	.1	7	1.3776
		9	1.3761
1.0	.5	7	2.8084
		9	2.8080

5. *Discussion.* Subsequent work will be devoted to the calculation of the function Q in the case of radiation with a redistribution in frequency [4]. In this case the integral equation for the function Q retains its displacement kernel, but Eq. (2) no longer holds.

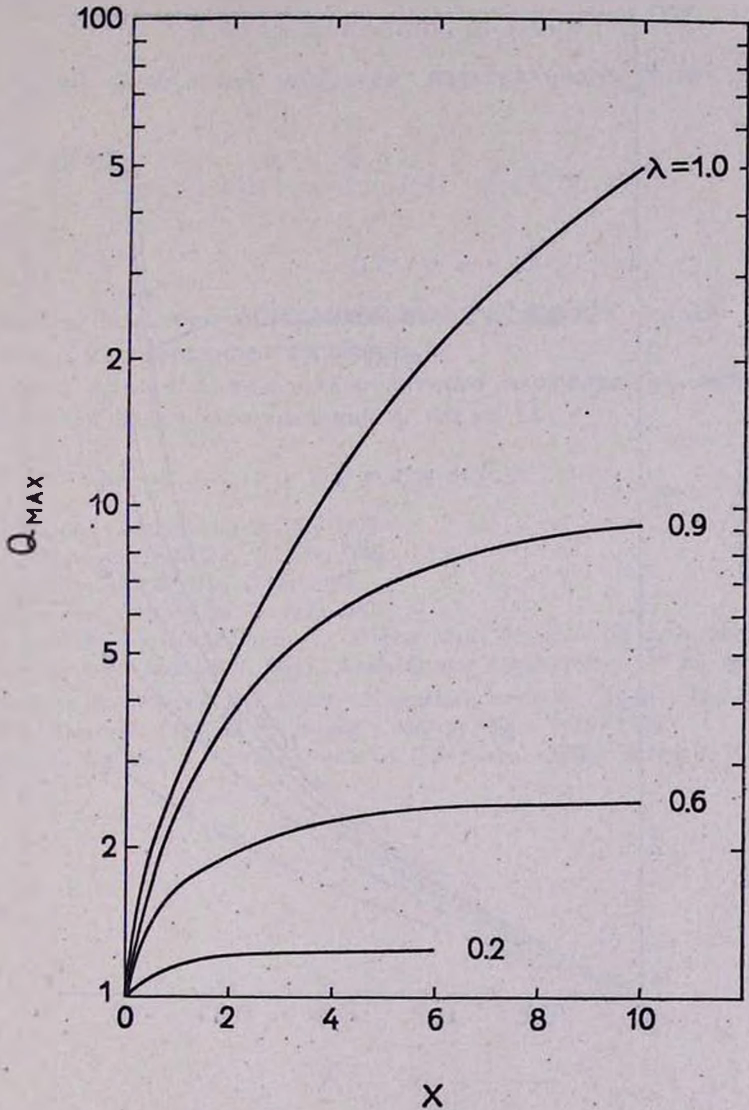


Fig. 5. Graphs of the function $Q_{\max}$ as a function of x.

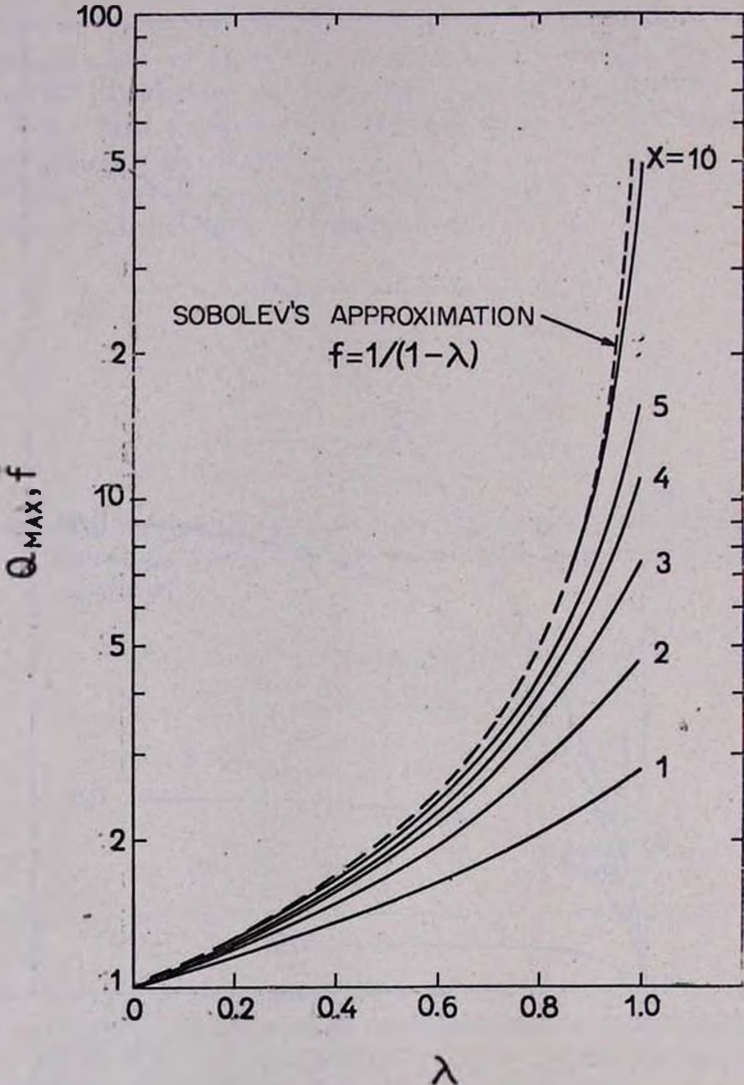


Fig. 6. Graphs of the function Q_{max} as a function of λ.

ЧИСЛЕННЫЕ РЕЗУЛЬТАТЫ ДЛЯ ФУНКЦИИ СОБОЛЕВА Q ПЕРЕНОСА ИЗЛУЧЕНИЯ

Ж. БЕЛЛ, Р. КАЛАБА, С. УЭНО

Функция Q является решением интегрального уравнения Фредгольма

$$Q(t, x) = 1 + \frac{\lambda}{2} \int_0^x E_1(|t-y|) Q(y, x)\, dy;$$

$$0 \leqslant t \ll x, \quad 0 \leqslant \lambda \leqslant 1$$

Соболев подчеркнул важность этой функции в теории переноса излучения и многократного рассеяния.

В этой статье метод инвариантного вложения был использован для получения численных значений функции Q.

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