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On a Validity Criterion for the Born Approximation

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Abstract: In this presentation a validity criterion for the Born approximation is examined for elastic scattering on a finite range radially symmetric potential in multidimensional space. Our analysis utilizes the transformation properties of the radial Schrödinger equation for *S* waves. The analytic structure of criterion is found to yield the corresponding results for low dimensional cases in a rather natural way. Some peculiarities of perturbative scattering in 2+Ω dimensions are brought out.

Keywords: Born approximation, validity criterion, multidimensional space.

1. Introduction

The Born approximation [1] is a valuable computational scheme often used in studies of forward and inverse scattering problems in quantum mechanics [2-4] and cross-disciplinary fields (see, e. g., [5-8]). For the Schrödinger collision problem involving a radially symmetric potential $[U(\mathbf{r})=U(r)]$ in three space dimensions ($D=3$), a widely accepted criterion for the validity of the Born approximation is described by the expression [3, 4]

$$\gamma_B(k,D=3)=\left|\frac{2m}{\hbar^2}\int_0^\infty re^{ikr}\left(\frac{\sin kr}{kr}\right)U(r)dr\right|<<1\,, \tag{1}$$

where m is the mass of the particle propagating with the de Broglie wavelength $\lambda=2\pi/k$ and collision energy $E=\hbar^2k^2/2m$. In the short-wavelength limit,

$$\Theta\sim\frac{\lambda}{a}<<1\,,$$

when the scattered intensity is essentially concentrated within a narrow forward cone [2] of angular width Θ, the inequality (1) produces the restrictive relation [2, 3],

$$1>>\gamma_B(k,D=3)=\frac{I}{ka}\sim I\Theta\propto k^{-1},\qquad\qquad ka>>1\,, \tag{2}$$

$$I\sim\frac{ma^2}{\hbar^2}|U_0|,$$

in which U_0 is the characteristic strength of the potential with influence range a. A look at the scaling relation in (2) shows that at high incident energies (fast collisions) the leading term of γ_B is controlled by the forward cone angle $\Theta\propto\lambda$.

In this contribution, a procedure is described for examining the progenitor of the validity criterion (1) in D spatial dimensions [9-16]. Our treatment proceeds along heuristic lines and utilizes the D-dependent transformation properties of the radial Schrödinger equation. Since the information about the orbital angular momentum of the scattered particle is not represented in Eq. (1), we will specifically deal here with the case of collisions in the S wave channel.

2. Heuristic considerations

We begin our analysis by noting that for an S wave particle interacting with a central force field in D space dimensions the radial wave function, $\Phi(r)$, satisfies the differential equation [10-13,15,16],

$$\left[\frac{d^2}{dr^2}+\frac{D-1}{r}\frac{d}{dr}-q(r)+k^2\right]\Phi(r)=0, \tag{3}$$

in which

$$q(r)=\frac{2m}{\hbar^2}U(r)$$

is the reduced potential. One may reorganize Eq. (3) to make it look like a radial wave equation for a particle in three space dimensions by eliminating the first-order derivative term through the ansatz [12, 13,15,16],

$$\Phi(r)=r^{-(D-1)/2}\phi(r). \tag{4}$$

Insertion of (4) into (3) leads to Sturm-Liouville equation,

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-q(r)+k^2\right]\phi(r)=0, \tag{5}$$

where the D-dependent object, $l=l(D)$, determines the properties of the centrifugal barrier (or centripetal potential well) and is given by

$$l=\frac{D-3}{2}. \tag{6}$$

According to Eqs. (3)–(6), the S wave problem in continuous radial D dimensions [12, 13] becomes equivalent [12, 15] to familiar l-wave problem [2–4] in the three-dimensional configuration space.

Taking advantage of this equivalence, we may proceed further by using the fact that in the l-th partial wave channel the radial (outgoing wave) Green's function associated with (5) behaves as [17],

$$G_0^+(r,r',l,k)=\frac{i\pi}{2}\sqrt{rr'}\begin{cases}J_{l+1/2}(kr)H^{(1)}_{l+1/2}(kr'), & r\le r',\\ H^{(1)}_{l+1/2}(kr)J_{l+1/2}(kr'), & r'\le r,\end{cases} \tag{7}$$

where $J_\nu(z)$ is the Bessel function and $H_\nu^{(1)}(z)$ is the Hankel function of the first kind. A careful examination of Eq. (7) reveals that the fundamental structure of the integrand appearing in Eq. (1) can be interpreted in the following physically transparent way,

$$re^{ikr}\left(\frac{\sin kr}{kr}\right)=G_0^+(r,r'=r,l=0,k)\,, \tag{8a}$$

with important supplementary information provided by

$$r=G_0^+(r,r'=r,l=0,k=0)\,, \tag{8b}$$

$$\frac{r}{2l+1}=G_0^+(r,r'=r,l,k=0)\,, \qquad l>-1/2\,. \tag{8c}$$

By relying on such an *interpretation* and employing Eq. (6), we arrive at the conclusion that the multidimensional analogue of the validity criterion (1) is given by inequality

$$\gamma_B(k,D)=\left|\int_0^\infty G_0^+(r,r'=r,l=l(D),k)q(r)dr\right|<<1\,. \tag{9}$$

This formula demonstrates explicitly how the k-dependent features of perturbative scattering in a central field of force become correlated with continuously changing radial dimensions. In particular, one may clearly see from Eqs. (6), (8c) and (9) that in the limit of vanishing incident energy [18] the object $\gamma_B(k\to 0,D)$ can exhibit non-divergent character when the condition $D>2$ is fulfilled. At this juncture, it is necessary to emphasize the following: the Born perturbation expansion [1] has been extensively studied for more than 90 years, and the bibliography accumulated in this area of research is vast. It is therefore not unreasonable to presume that our heuristically constructed D-sensitive form (9) might have been enunciated in some earlier publication(s). However, up to this moment we have not been able to spot an appropriate reference to this issue. Let us remark that it is also possible to incorporate particle's grand orbital angular momentum [10,15,16] into the mathematical structure of the inequality (9) via straightforward generalization [19] of considerations presented herein.

3. Different faces of $\gamma_B(k,D)$

It is a simple matter to verify that for the familiar three-dimensional case $(l=0)$ the expression (9) obviously agrees with Eq. (1), as it should. For the case of two-dimensional potential scattering $(l=-1/2)$, the construction (9) also automatically reproduces the analytic structure of the corresponding criterion [20],

$$\gamma_B(k,D=2)=\left|\frac{\pi m}{\hbar^2}\int_0^\infty rJ_0(kr)H_0^{(1)}(kr)U(r)dr\right|<<1\,.$$

In one dimension ($l=-1$, $r=|x|$, $U(-x)=U(x)$), Eq. (9) reorganizes into

$$\gamma_B(k, D=1) = \left| \frac{2m}{\hbar^2} \int_0^\infty r e^{ikr} \left(\frac{\cos kr}{kr} \right) U(r) dr \right| << 1\,, \tag{10}$$

and shows, in its own manner, how the Born approximation is destined to fail [2] in the long-wavelength limit $k \to 0$. We may also mention, for the sake of completeness, that if the one-dimensional potential is globally attractive and shallow [21],

$$\int_{-\infty}^{\infty} U(x) dx < 0\,, \qquad I << 1\,,$$

then the threshold condition,

$$\gamma_B(k = i\kappa, D=1) = 1\,, \qquad \kappa a << 1\,, \tag{11}$$

rigorously gives in an alternative and succinct way the asymptotic expression [2, 21] for the eigenenergy $E_0 < 0$ of the weakly bound state:

$$\kappa = \left| \frac{2m}{\hbar^2} \int_0^\infty U(r) dr \right| \sim \frac{I}{a}\,, \qquad |E_0| = \frac{\hbar^2 \kappa^2}{2m} = \frac{m}{2\hbar^2} \left(\int_{-\infty}^{\infty} U(x) dx \right)^2 << |U_0|\,.$$

An additional facet of the Born approximation scattering event [9, 11, 12, 14] can be unraveled by examining the γ_B–versus–D behavior in $D = 2 + \Omega$ dimensions. Working from Eq. (9), we obtain in the short-wavelength domain a restrictive relation,

$$1 >> \gamma_B = \frac{I}{\Omega}\,, \qquad \Omega >> ka >> 1\,, \tag{12}$$

which is clearly *anomalous* from the standpoint of the customary scaling relation (2). The principal inference to be drawn from (12) is that in extra [16, 22] dimensions $(1 >> \Theta >> \Omega^{-1})$ the leading term of γ_B is *no more* under control of such basic parameter [2] of wave-mechanical scattering theory as cone angle Θ. It is quite remarkable that the specific wave vector window originating in (12) becomes heavenly wide in the asymptotic limit of infinite $(\Omega \to \infty)$ dimensions [11,13]. One may notice, at the same time, that this peculiar window disappears from stage (becomes *invisible*) when a transition is made to the opposite extreme, $\Omega \to ka \to 1$ (i.e., $D \to 3$). Under such transition, γ_B of (12) descends to three-dimensional world and exhibits the familiar structure of Born's constraint for slow $(\Theta \sim 1)$ collisions [2, 3],

$$1 >> \gamma_B(k \sim a^{-1}, D=3) \sim I.$$

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