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Two dissipative two-state level-crossing models conditionally integrable in terms of the Kummer functions

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Abstract. We present two dissipative level-crossing models of quantum time-dependent two-state problem. These are conditionally integrable models belonging to the single confluent Heun class. Each of the models is given by an exponentially varying Rabi frequency and a level-crossing detuning that starts from the exact resonance and exponentially diverges at the infinity. The models include irreversible losses from the excited state (without relaxation to the ground state). We show that the solution of the problem is written in terms of irreducible linear combinations of two Kummer confluent hypergeometric functions. Using this exact solution, we discuss the dynamics of states' populations under different interaction regimes.

Keywords: Quantum two-state problem, laser excitation, irreversible losses, analytic solutions, confluent Heun equation, Kummer confluent hypergeometric function

1. Introduction

We discuss the semi-classical two-state problem [1-2] involving irreversible losses from the excited state. Transforming both dependent and independent variables, we reduce the governing time-dependent Schrödinger equations to the single confluent Heun equation which is a linear second-order ordinary differential equation widely encountered in many branches of contemporary physics and mathematics [3-5]. The field-configurations we derive are given by exponentially varying Rabi frequencies and a level-crossing detuning that starts from the exact resonance and exponentially diverges at the infinity. The models belong to two distinct confluent Heun classes recently presented in [6]. Because a parameter of the field-configuration is fixed to a constant, the models are conditionally integrable.

To treat the solution, we expand the confluent Heun function in terms of the Kummer confluent hypergeometric functions [7]. The coefficients of the expansion obey a three-term recurrence relation between successive coefficients. Discussing the conditions for termination of the series, we reveal that the solution for the field-configurations we treat is written as an irreducible linear combination of two Kummer functions [8]. We note that this solution has an alternative representation [9] in terms of the Goursat generalized hypergeometric series [10,11]. Using the explicit solution, we discuss both weak and strong interaction regimes.

2. Solutions of the two-state problem in terms of the confluent Heun functions

Let the probability amplitudes of the ground and excited states of a dissipative two-state quantum system subject to excitation by a field of optical laser radiation are $a_1(t)$ and $b_2(t)$. If the decay of the excited state is supposed to be to a third state out of the system (that is, if the spontaneous relaxation to the ground state is neglected), the time-dependent Schrödinger equations governing the interaction process are written as [1]

$$i\frac{da_1}{dt}=U(t)\,b_2\,, \tag{1}$$

$$i\frac{db_2}{dt}=U(t)\,a_1+\left(\Delta(t)-i\Gamma\right)b_2\,, \tag{2}$$

where the Rabi frequency $U(t)$ and the detuning $\Delta(t)$ of the transition frequency from the field frequency are arbitrary real functions of time (with the proviso that $U(t)>0$), and the parameter Γ defines the rate of the losses from the excited state. Applying the unitary transformation $b_2=a_2(t)e^{-i\omega(t)}$ and further eliminating the probability amplitude a_1, this system is reduced to the following second-order linear differential equation for a_2:

$$\frac{d^2a_2}{dt^2}+\left(-i\omega_t-\frac{U_t}{U}\right)\frac{da_2}{dt}+U^2a_2=0\,, \tag{3}$$

where (and hereafter) the lowercase alphabetical index denotes differentiation with respect to the corresponding variable and $\omega_t=\Delta-i\Gamma$. We now discuss, following the lines of [6], the reduction of this equation to the confluent Heun equation [3-5]

$$u_{zz}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\varepsilon\right)u_z+\frac{\alpha z-q}{z(z-1)}u=0\,. \tag{4}$$

A useful point for this reduction is the class property of the integrable models of the two-state problem [12]. According to this property, if the function $a_2^*(z)$ is a solution of equation (3) rewritten for an auxiliary argument z and for some functions $U^*(z)$ and $\omega^*(z)$ then the function $a_2(t)=a_2^*(z(t))$ is the solution of equation (3) for the field-configuration

$$U(t)=U^*(z)\frac{dz(t)}{dt}\,, \tag{5}$$

$$\omega_t(t)=\omega_z^*(z)\frac{dz(t)}{dt}\,, \tag{6}$$

where $z(t)$ is an arbitrary complex-valued function. The pair of functions $U^*(z)$ and $\omega^*(z)$ is conventionally referred to as a basic integrable model.

The variable change $a_2=\varphi(z)\,u(z)$, $z=z(t)$ together with (5),(6) reduces equation (3) to the equation

$$u_{zz}+\left(2\frac{\varphi_z}{\varphi}-i\omega_z^*-\frac{U_z^*}{U^*}\right)u_z+\left(\frac{\varphi_{zz}}{\varphi}+\left(-i\omega_z^*-\frac{U_z^*}{U^*}\right)\frac{\varphi_z}{\varphi}+U^{*2}\right)u=0\,. \tag{7}$$

This equation becomes the confluent Heun equation (4)

if
$$\frac{\gamma}{z}+\frac{\delta}{z-1}+\varepsilon=2\frac{\varphi_z}{\varphi}-i\omega_z^*-\frac{U_z^*}{U^*} \tag{8}$$

and
$$\frac{\alpha z-q}{z(z-1)}=\frac{\varphi_{zz}}{\varphi}+\left(-i\omega_z^*-\frac{U_z^*}{U^*}\right)\frac{\varphi_z}{\varphi}+U^{*2}\,. \tag{9}$$

Though the general solution of this system is not known, however, many particular solutions can be derived by applying the ansatz

$$\varphi=e^{\alpha_0 z}z^{\alpha_1}(z-1)^{\alpha_2}\,,\quad U^*=U_0^* z^{k_1}(z-1)^{k_2}\,,\quad \omega_z^*=\delta_0+\frac{\delta_1}{z}+\frac{\delta_2}{z-1}\,. \tag{10}$$

This leads to 15 possible cases of integer or half-integer pairs $\{k_1,k_2\}$, obeying the inequalities $-1\le k_{1,2}\cup k_1+k_2\le 0$ [6]. With this and equations (5),(6), the physical field configurations for the confluent Heun models are given as

$$U(t)=U_0^* z^{k_1}(z-1)^{k_2}\frac{dz}{dt}\,, \tag{11}$$

$$\omega_t(t)=\left(\delta_0+\frac{\delta_1}{z}+\frac{\delta_2}{z-1}\right)\frac{dz}{dt} \tag{12}$$

Note that here the parameters U_0^* and $\delta_{0,1,2}$ are complex constants which should be chosen so that the Rabi frequency $U(t)$ and detuning $\Delta(t)$ are real for the chosen complex-valued $z(t)$. Since these parameters are arbitrary, all the derived classes are four-parametric in general.

The solution of the starting two-state problem is explicitly written as [6]

$$b_2=e^{-i\omega(t)}\,e^{\alpha_0 z}z^{\alpha_1}(z-1)^{\alpha_2}H_C(\gamma,\delta,\varepsilon;\alpha,q;z)\,, \tag{13}$$

where the parameters of the confluent Heun function $\gamma,\delta,\varepsilon,\alpha,q$ are given as

$$\gamma=2\alpha_1-i\delta_1-k_1\,,\quad \delta=2\alpha_2-i\delta_2-k_2\,,\quad \varepsilon=2\alpha_0-i\delta_0 \tag{14}$$

$$\alpha=-i\delta_0(\alpha_1+\alpha_2-\alpha_0)+\alpha_0(\gamma+\delta-\varepsilon)+Q^{(3)}(0)/6\,, \tag{15}$$

$$q=\alpha_0(\alpha_0-i\delta_0-k_1-i\delta_1)+\alpha_2(1-\alpha_2+k_1+i\delta_1+k_2+i\delta_2)+ \\ \alpha_1(1-\gamma-\delta+\varepsilon+\alpha_1)-Q''(0)/2-Q'''(0)/6 \tag{16}$$

with $Q(z)=U_0^{*2}z^{2k_1+2}(z-1)^{2k_2+2}$ and

$$\alpha_0^2-i\alpha_0\delta_0+Q^{(4)}(1)/4!=0\,, \tag{17}$$

$$\alpha_1^2-\alpha_1(1+k_1+i\delta_1)+Q(0)=0\,, \tag{18}$$

$$\alpha_2^2-\alpha_2(1+k_2+i\delta_2)+Q(1)=0\,. \tag{19}$$

The solution of the confluent Heun equation (4) can be expanded in terms of the Kummer confluent hypergeometric functions [7]:

$$H_C(\gamma,\delta,\varepsilon;\alpha,q;z)=\sum_{n=0}^{\infty}c_n\cdot {}_1F_1(\alpha/\varepsilon+n;\gamma;-\varepsilon z)\,. \tag{20}$$

It has been shown that the coefficients of the expansion obey the recurrence relation

$$R_nc_n+Q_{n-1}c_{n-1}+P_{n-2}c_{n-2}=0 \tag{21}$$

with
$$R_n=n(n-\gamma+\alpha/\varepsilon)\,, \tag{22}$$

$$Q_n=-q+n\gamma+(\varepsilon-\delta-2n)(n+\alpha/\varepsilon)\,, \tag{23}$$

$$P_n=(n+\delta)(n+\alpha/\varepsilon)\,. \tag{24}$$

The series terminates if two successive coefficients, say c_{N+1} and c_{N+2}, vanish for some $N=0,1,2,...$. Equation $c_{N+2}=0$ is then satisfied if $P_N=0$. This is equivalent to the conditions $\alpha/\varepsilon=-N$ or $\delta=-N$. It is immediately seen from the form of the Kummer functions involved in expansion (20) that the choice $\alpha/\varepsilon=-N$ results in polynomial solutions. For this reason, below we examine the more advanced case $\delta=-N$ of non-polynomial solutions. We will see that this condition can indeed be satisfied for certain confluent Heun models. We present two such examples. Note finally that the equation $c_{N+1}=0$ leads to a polynomial equation of the degree $N+1$ for the accessory parameter q.

Concluding this section, we thus state that the solution of the confluent Heun equation can be written as a sum of a finite number of the (irreducible to polynomials) Kummer confluent hypergeometric functions of the form applied in expansion (20) only if two conditions (that is equation $\delta=-N$ and the polynomial equation for q) are satisfied. These two equations generally impose additional restrictions on the parameters of the two-state models under consideration. For this reason, in general the confluent Heun models integrable in terms of the confluent hypergeometric functions are conditionally integrable (the latter term indicates that the physical parameters of the system are not varied independently or a parameter ix fixed to a constant value). In the next section we discuss two such conditionally integrable models which turn to be dissipative.

3. Two conditionally integrable confluent Heun dissipative models

Consider if the series (20) can be terminated on the second term ($N=1$) so that the solution of the two-state problem can be written as a sum of just two Kummer confluent hypergeometric functions. For this to occur it should be

$$\delta=-1 \tag{25}$$

and the accessory parameter q should obey the quadratic equation [7]

$$q^2-q(2\ \alpha+\gamma+\varepsilon-1)+\alpha(\alpha+\gamma+\varepsilon). \tag{26}$$

Consider the classes with $(k_1,k_2)=(-1,1/2)$ and $(k_1,k_2)=(-1/2,1/2)$ for which the Rabi frequency is written as (see equation (11))

$$U(t)=U_0^*\frac{\sqrt{z-1}}{z}\frac{dz}{dt} \tag{27}$$

and

$$U(t)=U_0^*\sqrt{\frac{z-1}{z}\frac{dz}{dt}}, \tag{28}$$

respectively. Checking equation (25), we find that for both cases the parameter δ_2 should be chosen as

$$\delta_2=-i/2. \tag{29}$$

With this, it is readily checked that equation (26) for q is identically satisfied for any U_0^*, δ_0 and δ_1. Thus, we obtain two integrable confluent Heun models. For both models, the frequency detuning is given as (see equation (12)):

$$\omega_t(t)=\left(-\frac{i}{2(z-1)}+\delta_0+\frac{\delta_1}{z}\right)\frac{dz}{dt}. \tag{30}$$

As it is seen, here the strength of the first term in the brackets is fixed to a constant value so that the models are conditionally integrable. Below we show that because of this term the models turn to be dissipative.

According to expansion (20), a fundamental solution of the initial two-state problem (1) is written through a linear combination of two Kummer functions as

$$b_2^F(t) = e^{-i\omega(t)}\, e^{\alpha_0 z} z^{\alpha_1} (z-1)^{\alpha_2} \left(c_0 \cdot {}_1F_1(\alpha/\varepsilon;\gamma;-\varepsilon z) + c_1 \cdot {}_1F_1(1+\alpha/\varepsilon;\gamma;-\varepsilon z) \right), \tag{31}$$

with
$$\omega(t) = -\frac{i}{2}\ln(z-1) + \delta_0 z + \delta_1 \ln z \tag{32}$$

and
$$c_1 = \frac{\alpha - q}{q-\alpha-\alpha} c_0 , \tag{33}$$

where the expansion coefficient c_0 is an arbitrary constant.

A complementary observation concerning the derived fundamental solution is that the linear combination of the two confluent hypergeometric functions involved in equation (31) can be written in terms of a single Goursat generalized hypergeometric function [10]. Applying the result of [9], the solution is then rewritten in a more compact form:

$$b_2^F(t) = e^{-i\omega(t)}\, e^{\alpha_0 z} z^{\alpha_1} (z-1)^{\alpha_2}\, {}_2F_2(\alpha/\varepsilon, e+1;\gamma, e;-\varepsilon z) , \tag{34}$$

where the new parameter e is given as

$$e = \frac{\alpha}{q-\alpha}. \tag{35}$$

Discussing now the general solution of the problem, we note that the parameters of the pre-factor are readily calculated to be

$$k_{1,2} = (-1, 1/2): \quad \alpha_0 = \alpha_2 = 0 , \quad \alpha_1 = \frac{i}{2}\left(\delta_1 \pm \sqrt{\delta_1^2 - 4U_0^2} \right), \tag{36}$$

$$k_{1,2} = (-1/2, 1/2): \quad \alpha_0 = \frac{i}{2}\left(\delta_0 \pm \sqrt{\delta_0^2 + 4U_0^2} \right), \quad \alpha_1 = \alpha_2 = 0 . \tag{37}$$

Since the different signs in front of the squire roots in these equations produce linearly independent solutions (this can readily be verified by checking the Wronskian) the general solution can be constructed by taking the linear combination of these solutions. However, the general solution can more conveniently be written as follows:

$$b_2(t) = e^{-i\omega(t)}\, z^{\alpha_1} (z-1)^{\alpha_2} e^{i\delta(t)} \left(F + z \frac{q-\alpha}{\alpha} \frac{dF}{dz} \right), \tag{38}$$

where
$$F = C_1 \cdot {}_1F_1(\alpha/\varepsilon;\gamma;-\varepsilon z) + C_2 \cdot z^{1-\gamma}\, {}_1F_1(1-\gamma+\alpha/\varepsilon; 2-\gamma;-\varepsilon z) \tag{39}$$

with arbitrary constants $C_{1,2}$. Here one can choose any sign for the roots involved in $\alpha_{0,1}$.

4. Population dynamics

Consider now a physical interaction suggested by the presented models. For definiteness, consider the first class with $(k_1,k_2)=(-1,1/2)$. If the independent variable transformation is taken as the real-valued function

$$z=1+\exp(2\Gamma t) \tag{40}$$

we obtain the three-parametric field configuration

$$U(t)=\frac{2U_0 e^{3t}}{1+e^{2t}}, \tag{41}$$

$$\omega_t(t)\equiv\Delta(t)-i\Gamma=2e^{2t}\left(\delta_0+\frac{\delta_1}{1+e^{2t}}\right)-i\,. \tag{42}$$

where U_0, δ_0 and δ_2 are arbitrary real parameters. Here we have put $U_0^*=U_0$ and $\Gamma=1$. The last condition implies that the involved parameters are supposed dimensionless.

Equations (41),(42) define a field configuration with a detuning function $\Delta(t)$ describing an asymmetric-in-time level-crossing process (Fig.1). The crossing of the resonance occurs at the time point

$$t_0=\frac{1}{2}\ln\left(-\frac{\delta_0+\delta_1}{\delta_0}\right). \tag{43}$$

In the vicinity of this point the behavior of the detuning is approximately modeled by the linear-in-time crossing law of the Landau-Zener type [13,14]:

$$\Delta(t)\sim-\frac{4(\delta_0+\delta_1)^2}{\delta_1}(t-t_0),\quad U(t)\approx U(t_0)\,. \tag{44}$$

Accordingly, the effective Landau-Zener parameter is given as

$$\lambda=\frac{2U_0^2(\delta_0+\delta_1)}{\delta_0\delta_1}. \tag{45}$$

This is the physical parameter that characterizes the interaction. The strong interaction regime corresponds to large $\lambda\gg1$, while the weak interaction occurs at $\lambda\ll1$. The population dynamics for the case when the system starts from the ground state is shown in Fig.2-Fig.5. The first three figures present the occupation probabilities $p_1=|a_1(t)|^2$ and $p_2=|b_2(t)|^2$ of the ground and excited states. As expected, the dissipation results in the complete removal of the population from the excited state. The behavior of the ground state's population, however, depends on the interaction regime. The dynamics of the total population is shown in Fig.5.

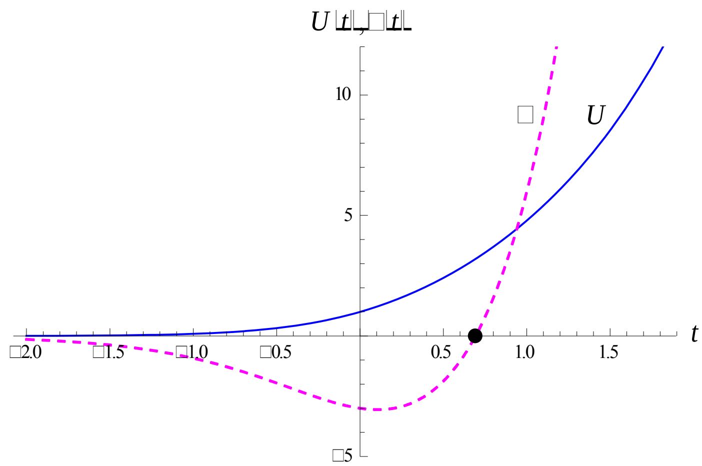

Fig. 1. Level-crossing model (41)-(42) with $U_0 = 1$ and $\delta_0 = 1$, $\delta_1 = -5$. The filled circle indicates the crossing time-point t_0.

It is understood that in the weak interaction regime the excited state (from which the losses occur) is only slightly populated for a relatively short time, hence, at the end of the process the ground state still possesses a remnant population (Fig.2). Thus, in this regime the total population of the two states is depleted only partially. As the field amplitude increases, the excitation of the system intensifies, that is the excited state becomes more populated and for a longer time, hence, the losses become more pronounced (Fig.3 and Fig.4). As a result, the population completely leaves the system (Fig.5). A complementary observation concerning the strong interaction regime is that at large λ the system exhibits strongly pronounced Rabi oscillations during the evolution (Fig.4).

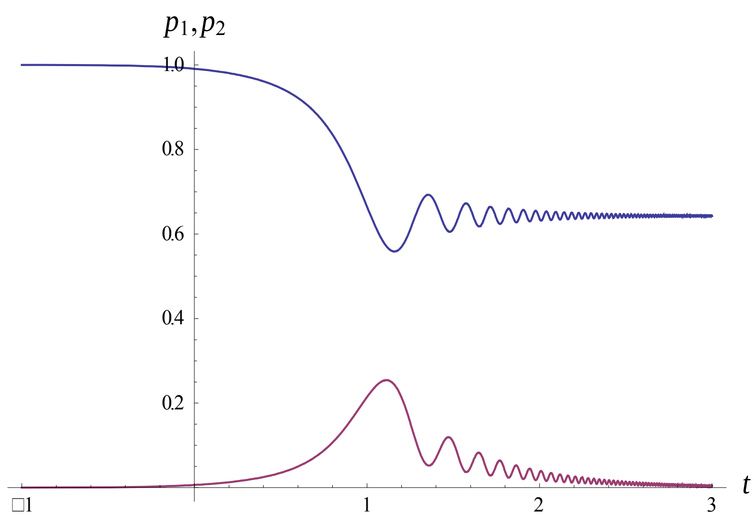

Fig. 2. Occupation probabilities for the ground (upper line) and excited (lower line) states for field parameters $U_0 = 0.3$ and $\delta_{0,1} = 1, -5$ ($\lambda = 0.144$).

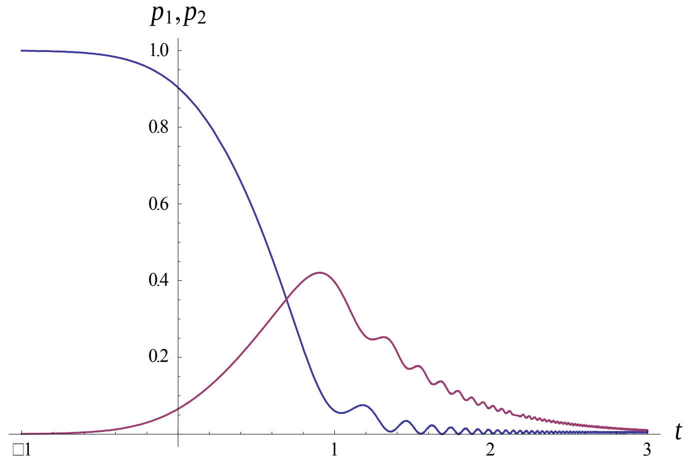

Fig. 3. Occupation probabilities for the ground and excited states (blue and magenta lines, respectively) for field parameters $U_0 = 1$ and $\delta_{0,1} = 1, -5$ ($\lambda = 1.6$).

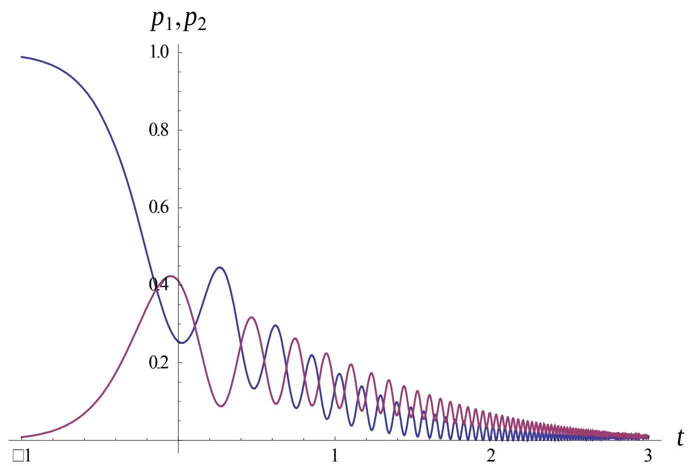

Fig. 4. Occupation probabilities for the ground and excited states (blue and magenta lines, respectively) for field parameters $U_0 = 4$ and $\delta_{0,1} = 1, -5$ ($\lambda = 25.6$).

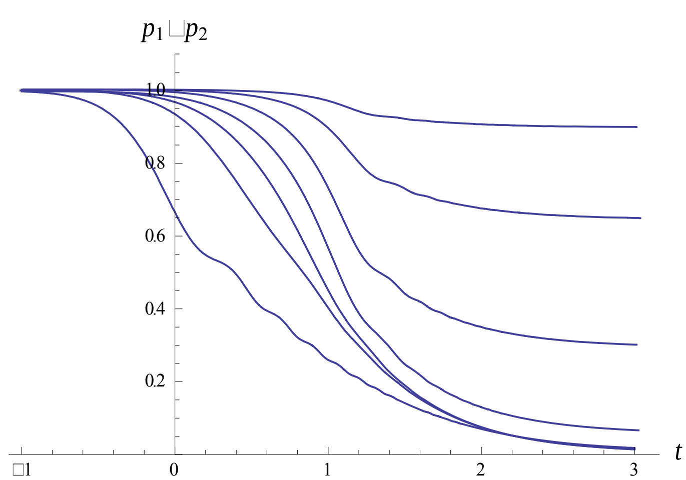

Fig. 5. Total occupation probability versus the Landu-Zener parameter. From top to bottom: $\lambda = 0.036,\ 0.144,\ 0.4,\ 0.9,\ 1.6,\ 3.6,\ 25.6$.

5. Discussion

Thus, we have presented two new conditionally integrable level-crossing models of semi-classical time-dependent two-state quantum problem. These are dissipative models involving irreversible losses (spontaneous emission, collision relaxation, etc.) from the excited state. It is supposed that the decay occurs to a third state out of the model system, that is, the decay relaxation to the ground state is neglected.

To derive the models, we have discussed the field-configurations for which the problem is reduced to the single confluent Heun equation which is a second-order ordinary linear differential equation having two regular singularities and an irregular singularity of rank 1. In order to manage the confluent Heun function involved in the solution, we have applied a series expansion in terms of the Kummer confluent hypergeometric functions of a specific form. As the next step, we have discussed the conditions for the series to terminate thus resulting in solutions involving a finite number of the Kummer functions. We have seen that this is possible only if two conditions are imposed on the parameters of the two-state models under consideration. Requiring the series to terminate on the second term, we have identified the two models presented here. The models turn to be conditionally integrable in the sense that a parameter (the decay rate) is fixed to a constant.

The analytic solutions of quantum two-state problem in terms of the functions of the hypergeometric class are rare (see [1] and references therein). Perhaps, the most known example is the Landau-Zener model with constant Rabi frequency and linearly-in-time varying detuning [13-16] (for the dissipative version of this model see [17]). The field-configurations we have presented are given by an exponentially diverging Rabi frequency and a level-crossing detuning that starts from the exact resonance and exponentially diverges at the infinity. Using the explicitly given two-term confluent hypergeometric solution (this solution can alternatively be written in terms of a single Goursat generalized confluent hypergeometric function), we have studied the population dynamics in different interaction regimes. The behavior of the system has much in common with that of the recently presented biconfluent Heun dissipative model [18] for which the solution of the problem is written in terms of two non-integer-order Hermite functions of a scaled and shifted argument.

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