VEV of *Q***-operator in U(1) linear quiver 4d gauge theories**

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Abstract: Linear quiver N = 2 4d gauge theory in Ω background is considered. It is proved that the partition function in a simple way is related to the expectation values of Baxter's Q operator (at specific discrete values of the spectral parameter) in the gauge theory with the first node removed. Explicit expressions for the VEV of the Q operator in terms of generalized Apell's functions are found.

Keywords: N=2 supersymmetric gauge theory, Deformed Seiberg-Witten equation, Baxter's Q operator.

1. Introduction

Embedding N = 2 gauge theory in Ω -background was instrumental in all developments related to the instanton counting with the help of equivariant localization technics. $\epsilon_{1,2}$ are the Ω background parameters. Furthermore, sending both Ω -background parameters $\epsilon_{\!_{1,2}}$ to 0 , one gets the standard Seiberg-Witten theory [1,2]. It is interesting that even the case of U(1) gauge group, in contrast to the case without Ω -background, the theory is non-trivial. A characteristic feature of this case is that the instanton sums become tractable, and for Nekrasov partition function, one obtains closed formulae. In this paper it is shown that not only the partition function, but also a more refined quantity, namely the expectation value of the Q -observable can be computed in closed form. It was shown in [3] that the analog of Baxters Q operator in purely gauge theory context naturally emerges in Necrasov-Shatashvili limit ($\epsilon_2 = 0$) [4] as an entire function whose zeros are given in terms of an array of "critical" Young diagrams, namely those, that determine the most important instanton configuration contributing to the partition function. This observable encodes perfectly not only information about partition function (which is simply related to the total sum of column lengths of Young diagrams) but also the entire chiral ring [5] constructed from $\langle tr \Phi^J \rangle$, J = 0, 1, 2, ... (Φ is the scalar of vector multiplet). In present paper, simpler U(1) case in 4d setting is analyzed. The corresponding expectation values of Q in closed form are found. The solution is expressed in terms of a generalization of Appel's function. The rest of material is organized as follows.

Chapter 2 is a short review of 4d linear quiver gauge theory: the Nekrasov partition function and important observable Q are introduced. An extended quiver with specific parameters at the extra node is introduced and its relation to the Q- observable is analyzed. Chapter 3 specializes to the case of $U(1)^r$ theory. Explicit expressions for the Q observable in terms of generalized Appel and hypergeometric functions are found. Chapter 4 is the conclusion.

2. General setting

The instanton partition function of the 4d, A_{r+1} linear quiver theory with gauge group U(n) is given by (see Fig.1 for the setup)

$$\mathcal{Z} = \sum_{(\vec{Y}_1, \dots, \vec{Y}_r)} Z_Y q_1^{|\vec{Y}_r|} \dots q_r^{|\vec{Y}_r|}$$
(2.1)

The sum in (2.1) is over all possible r-tuples of arrays of n Young diagrams. $|\vec{Y}_k|$ is the total number of boxes in the k-th array of n Young diagrams and Z_Y is defined as:

$$Z_{\mathbf{Y}} = Z_{\vec{Y}_{1,...,\vec{Y}_{r}}}(\vec{a}_{0},\vec{a}_{1},...,\vec{a}_{r+1}) = \prod_{u,v=1}^{n} \frac{Z_{bf}(\emptyset, a_{0,u} \mid Y_{1,v}, a_{1,v}) Z_{bf}(Y_{1,u}, a_{1,u} \mid Y_{2,v}, a_{2,v}) \dots Z_{bf}(Y_{r,u}, a_{r,u} \mid \emptyset, a_{r+1,v})}{Z_{bf}(Y_{1,u}, a_{1,u} \mid Y_{1,v}, a_{1,v}) \dots Z_{bf}(Y_{r,u}, a_{r,u} \mid Y_{r,v}, a_{r,v})} , \quad (2.2)$$

For a pair of Young diagrams λ , μ the bifundamental contribution is given by [6,7]:

$$Z_{bf}(\lambda, a \mid \mu, b) = \prod_{s \in \lambda} (a - b - L_{\mu}(s)\epsilon_{1} + (1 + A_{\lambda}(s)\epsilon_{2}) \prod_{s \in \mu} (a - b + (1 + L_{\lambda}(s))\epsilon_{1} - A_{\mu}(s)\epsilon_{2}) .$$
(2.3)

Also, A_{λ} and L_{λ} known as the arm and leg lengths respectively, are defined as: if *s* is a box with coordinates (*i*,*j*) and λ_i (λ_j) is the length of *i*-th (*j*-th) column (row), then:

$$L_{\lambda}(s) = \lambda_{j'} - i, \quad A_{\lambda}(s) = \lambda_{j} - j \tag{2.4}$$



Figure 1. The linear quiver U(n) gauge theory: r circles stand for gauge multiplets; two squares represent n anti-fundamental (on the left edge) and n fundamental (the right edge) matter multiplets while the line segments connecting adjacent circles represent the bi-fundamentals. q_1, \ldots, q_r are the exponentiated gauge couplings, the n-dimensional vectors $\vec{a}_0, \ldots, \vec{a}_{r+1}$ encode respective (exponentiated) masses/VEV's and $\vec{Y}_0, \ldots, \vec{Y}_{r+1}$ are n-tuples of young diagrams specifying fixed (ideal) instanton configurations.

The Q observable plays an important role, in Nekrasov-Shatashvili limit $\epsilon_2 \rightarrow 0$ this observable satisfies Baxter's T-Q equation [3], and is defined as:

$$\mathbf{Q}(x,\lambda) = \prod_{x \in \lambda} \frac{x - i\epsilon_1 - (j-1)\epsilon_2}{x - (i-1)\epsilon_1 - (j-1)\epsilon_2}$$
(2.5)

Generalization for the case of generic Ω background (in both 4d and 5d cases) is due to [8]. The expectation value of the Q -operator associated to the first node, by definition is

$$Q(x) = Z^{-1} \sum_{(\vec{I}_1, \dots, \vec{I}_r)} \prod_{u=1}^n \mathbf{Q} \left(x - a_{1,u}, Y_{1,u} \right) Z_Y q_1^{|\vec{Y}_1|} \dots q_r^{|\vec{Y}_r|}$$
(2.6)

It was noticed in [9] that such insertion of the operator Q is equivalent to adding an extra node with specific expectation values. Let's look at a quiver with r+1 nodes with expectation values at the additional node (denoted as $\tilde{0}$) specified as (see Fig.2):

$$a_{\tilde{0},u} = a_{0,u} - \epsilon_1 \delta_{1,u}.$$
 (2.7)

Due to the specific choice of $\vec{a_0}$, in order to give a nonzero contribution, the array of n diagrams associated with the special node $\tilde{0}$ has to be severely restricted. Namely, the diagram $Y_{\tilde{0},1}$ should consist of a single column and the remaining n-1 diagrams $Y_{\tilde{0},2},...,Y_{\tilde{0},n-1}$ must be empty. There is a close relation between the Nekrasov partition function associated to above described specific length r+1 quiver and the expectation value of a particular Q operator in a generic quiver with r nodes. This relation is a consequence of the identity:



Figure 2. The quiver diagram with an extra node, labeled by $\hat{0}$.

$$Z_{\vec{Y}_{0},\vec{Y}_{1},\ldots,\vec{Y}_{r}}(\vec{a}_{0},\vec{a}_{0},\vec{a}_{1},\ldots,\vec{a}_{r+1})q_{0}^{l}q_{1}^{|\vec{Y}_{1}|}q_{2}^{|\vec{Y}_{2}|}\ldots q_{r}^{|\vec{Y}_{r}|} =$$

$$=\prod_{u=1}^{n} \left(\mathbf{Q}(a_{0,1}-a_{1,u}+l\epsilon_{2},Y_{1}) \frac{\left(\frac{a_{0,u}-a_{1,u}+\epsilon_{2}}{\epsilon_{2}}\right)_{l}}{\left(\frac{a_{0,1}-a_{0,u}+\epsilon_{2}}{\epsilon_{2}}\right)_{l}} \right) Z_{\vec{Y}_{1},\dots,\vec{Y}_{r}}(\vec{a}_{0},\vec{a}_{1},\dots,\vec{a}_{r+1})q_{0}^{l}q_{1}^{|\vec{Y}_{1}|}\dots q_{r}^{|\vec{Y}_{r}|}$$
(2.8)

Where $\vec{Y}_{0,u}$ for u = 1 is a one column diagram with length *l* and the rest are empty diagrams (see figure 2). The Pochhammer's symbol is defined as:

$$(a)_{l} = (a)(a+1)\cdots(a+l-1)$$
 (2.9)

3. *Q* observable for U(1) quiver theory

From now on we'll restrict ourselves to the simplest case of the quiver of U(1)'s. Nekrasov partition function of such linear quiver can be found for example in [10] :

$$Z = \prod_{i=1}^{r} \prod_{j=i}^{r} \left(1 - \frac{p_i}{p_j} \right)^{-\frac{(a_i - a_{i-1})(a_{j+1} - a_j - \epsilon_1 - \epsilon_2)}{\epsilon_1 \epsilon_2}}$$
(3.1)

Where:

$$p_i = \prod_{j=1}^i q_j \tag{3.2}$$

Expanding (2.8) in powers of $q_{\tilde{0}}$ and taking into account (3.1), we'll get the form of $Q(x_l)$ for $l \in \mathbb{Z}$. Where

$$x_l = a_0 - l\epsilon_2 \tag{3.3}$$

Since Q(x) and hence the entire LHS of the eq. (4.5) restricted up to an arbitrary instanton order is a rational function of x, the form must be valid also for generic values of x. We find:

$$Q(x) = C \sum_{m_1,...,m_r \ge 0} \frac{(\frac{a_0 - x}{\epsilon_2})_{m_1 + m_2 + ... + m_r}}{(\frac{a_1 - a_2 + \epsilon_1 + \epsilon_2}{\epsilon_2})_{m_1} ... (\frac{a_r - a_{r+1} + \epsilon_1 + \epsilon_2}{\epsilon_2})_{m_k}}{(\frac{a_1 - x}{\epsilon_2})_{m_1 + m_2 + ... + m_r}} m_1! ... m_r!$$

$$= CF_1^{(r)} \Big(\frac{a_0 - x}{\epsilon_2}, \frac{a_1 - a_2 + \epsilon_1 + \epsilon_2}{\epsilon_2}, ..., \frac{a_r - a_{r+1} + \epsilon_1 + \epsilon_2}{\epsilon_2}; \frac{a_1 - x}{\epsilon_2}; p_1, ..., p_r\Big)$$
(3.4)

Where $F_1^{(r)}$ is the Apell's F_1 function generalized for an arbitrary number of variables and C is the normalization factor, fixed from the condition $\lim_{x \to \infty} Q(x) = 1$:

$$F_1^{(k)}(a, b_1, \dots, b_k; c; x_1, \dots, x_k) = \sum_{m_1, \dots, m_k \ge 0} \frac{(a)_{m_1 + \dots + m_r} (b_1)_{m_1} \dots (b_k)_{m_k}}{(c)_{m_1 + \dots + m_r} m_1! \dots m_k!} (x_1)^{m_1} \dots (x_k)^{m_k}$$
(3.5)

$$C = \prod_{i=1}^{r} \left(1 - p_i\right)^{\frac{a_i - a_{i+1} + \epsilon_1 + \epsilon_2}{\epsilon_2}}$$
(3.6)

For the simplest case, where r = 1 in (3.4) Q(x) becomes a hypergeometric function:

$$Q(x) = (1-q)^{\frac{a_1 - a_2 + \epsilon_1 + \epsilon_2}{\epsilon_2}} F_1\left(\frac{a_0 - x}{\epsilon_2}, \frac{a_1 - a_2 + \epsilon_1 + \epsilon_2}{\epsilon_2}; \frac{a_1 - x}{\epsilon_2}; q\right)$$
(3.7)

4. Conclusion

Thus, starting from a general setting in chapter 2 we illustrate the A_{r+1} linear quiver theory with gauge group U(n). In chapter 3 we explicitly represent the VEV of Baxter's Q operator (3.4) in terms of generalized Apell's functions, which is the main original result in this paper.

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