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The FRLW-FRLW metric combination as an attractor solution in bigravity

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Abstract. The theory of massive bigravity is an interesting alternative to the standard ΛCDM cosmology. However, despite its phenomenological success at the background level, the theory in its original formulation suffers from different instabilities at the perturbation level. One of the suggestions to cure the theory from these unwilling instabilities was to discuss the possibility of other geometrical forms for both metrics different from previously only studied FRLW-FRLW case. In this work, with the help of a simple example, we show that even if initially we chose a metric combination different from FRLW-FRLW case, in the late times of the cosmic evolution metrics asymptotically approach to the FRLW-FRLW structure. This therefore means that at late times we will have again the same instability issues similar to the previously studied cases of FRLW-FRLW metric combination.

Keywords: modified gravity, bigravity, dark energy, background cosmology.

1. Introduction

The late time accelerated expansion of the Universe is one of the puzzles in modern science [1-6]. The ΛCDM model is presumably one of the simplest models, which can address the problem of the late time acceleration of the Universe. In this model Λ stands for the vacuum energy of the Universe and CDM corresponds to the cold dark matter component. From the phenomenological point of view, this simple model shows the best consistency with current observational data. Despite its remarkable properties on the phenomenological level, the ΛCDM suffers from the lack of fundamental understanding. Namely, the value of Λ receives uncompensated quantum corrections at quantum level rendering it technically unnatural. This means that in order to preserve the value of Λ needed to explain the observations, an extreme fine-tuning has to be made at each order of loop expansion. This fact has provoked studies of other models, which can also address the problem of the late time acceleration. Among these alternatives, the theories of massive gravity [7-9] and bigravity [12, 14, 15, 22] have received large interest in the last years (see also Refs. [25, 26, 27] for other formulations of bimetric gravity theories). The block stone of the fundamental formulation of these theories is the assumption that gravitons are massive as opposed to what GR tells us. This is achieved by introducing a second metric also referred to as reference metric. This metric couple to the standard background metric through a particular interaction term, which is constructed in such way to preserve theories from ghost degrees of freedom. It has been shown that massive bigravity theory can lead to a consistent background evolution providing a self-accelerating solution for the late time cosmology [3, 19, 24]. Unfortunately, at the perturbation level different instabilities enter into the theory making it theoretically nonviable in its standard form [4-6, 17, 18, 20,].

Since then several mechanisms have been suggested, which could possibly cure the theory from those instabilities [2, 16, 21]. In Ref. [21] we have argued that such possibility can choose different metric combinations. Indeed, previously it has always been assumed that both metrics of the theory are of the same FRLW type. For the case of the background metric which is directly coupled to the matter sector, the above mentioned choice makes sense as it is in an agreement with the observational evidence that the Universe at large scales is homogeneous and isotropic. However, for the reference metric, which does not couple to the matter sector directly there is no any fundamental reason why it also should be of the FRLW type. Motivated by this fact in Ref. [21], we have studied the possibility of different metric combinations in bigravity. We have found that only for limited choices of the metric combination the theory is mathematically consistent.

In this work, we go a step further and study the physical evolution of the theory when the two metrics belong to different classes. In particular, we will assume that the two metrics have FRLW structure but in different coordinate systems. As we will see, this choice will correspond to a FRLW-Lamaître type combination in the same coordinate system.

Throughout the paper, we will work in flat space and natural units, i.e. units such that $c=\hbar=1$. Furthermore, we will denote with a "dot" derivative with respect to the cosmic time and with a "prime" derivative with respect to the radial coordinate r.

2. The theory of massive bigravity

The Hassan-Rosen theory of ghost-free, massive bigravity is characterized by the action [13]

$$S=-\frac{M_g^2}{2}\int d^4x\sqrt{-\det g}R_g-\frac{M_f^2}{2}\int d^4x\sqrt{-\det f}R_f+ \\ +m^2M_g^2\int d^4x\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n(\sqrt{g^{-1}f})+\int d^4x\sqrt{-\det g}L_m(g,\Phi). \tag{1}$$

where M_g and M_f are Planck masses and R_g and R_f are the Ricci scalars for the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. The $g_{\mu\nu}$ is the background metric coupled to the matter sector through the Lagrangian L_m. The metric $f_{\mu\nu}$ is the reference metric, which does not interact directly with the matter sector and affects it only indirectly. The interaction term which connects the two metrics together consists of elementary symmetric polynomials e_n (one can find the forms of these polynomials in, e.g., Ref. [13]). These are functions of the eigenvalues of the matrix $\sqrt{g^{-1}f}$. This matrix is defined such that the condition $\sqrt{g^{-1}f}\sqrt{g^{-1}f}\equiv g^{\mu\nu}f_{\mu\nu}$ is satisfied. The quantities $\beta_n(n\in\{0,4\})$ together with the mass parameter m represent a full set of free parameters for this theory. As is argued in Ref. [17] one can express the masses in the units of M_g and absorb m^2 into the parameters β_n. Doing so, the action (1) simplifies to

$$S=-\frac{1}{2}\int d^4x\sqrt{-\det g}R_g-\frac{M_f^2}{2}\int d^4x\sqrt{-\det f}R_f+ \\ +\int d^4x\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n(\sqrt{g^{-1}f})+\int d^4x\sqrt{-\det g}L_m(g,\Phi). \tag{2}$$

By varying the action (2) with respect to (w.r.t.) $g_{\mu\nu}$, we obtain the following generalized Einstein equation for the g-metric

$$G^g_{\mu\nu}+I^g_{\mu\nu}=T_{\mu\nu}, \tag{3}$$

with the g-metric interaction term $I^g_{\mu\nu}$ given by

$$I^g_{\mu\nu}\equiv\frac{1}{2}g_{\mu\nu}R_g+\sum_{n=0}^{3}(-1)^n\beta_n g_{\mu\lambda}Y^{\lambda}_{(n)\nu}(\sqrt{g^{-1}f}). \tag{4}$$

In these equations $G^g_{\mu\nu}$ is the g-metric Einstein tensor $\left(G^g_{\mu\nu}\equiv R^g_{\mu\nu}-1/2g_{\mu\nu}R^g\right)$, and the matrices $Y_{(n)}(X)$ are defined as [13]

$$Y_{(0)}(X)\equiv\mathbf{I},Y_{(1)}(X)\equiv X-\mathbf{I}[X],Y_{(2)}(X)\equiv X^2-X[X]+\frac{1}{2}\mathbf{I}\left([X]^2-[X^2]\right), \tag{5}$$

$$Y_{(3)}(X)\equiv X^3-X^2[X]+\frac{1}{2}X\left([X]^2-[X^2]\right)-\frac{1}{6}\mathbf{I}\left([X]^3-3[X][X^2]+2[X^3]\right), \tag{6}$$

where $X\equiv\left(\sqrt{g^{-1}f}\right)$, $\mathbf{I}$ is the identity matrix, and $[\ldots]$ denotes the trace operator. In Eq. (3) the tensor $T_{\mu\nu}$ stands for a perfect fluid energy momentum tensor defined as

$$T^{\mu}_{\ \nu}\equiv diag\{-\rho,P,P,P\}, \tag{7}$$

where ρ and P are the energy density and the pressure of the fluid, respectively.

Next, by varying the action (2) with respect to the reference metric $f_{\mu\nu}$ we obtain

$$G^f_{\mu\nu}+I^f_{\mu\nu}=0, \tag{8}$$

where $G^f_{\mu\nu}$ is the f-metric Einstein tensor and $I^f_{\mu\nu}$ is the f-metric interaction tensor defined as

$$I^f_{\mu\nu} \equiv \frac{1}{M_f^2}\sum_{n=0}^{3}(-1)^n \beta_{4-n} f_{\mu\lambda} Y^{\lambda}_{(n)\nu}(\sqrt{g^{-1}f}). \tag{9}$$

Performing the rescaling $f_{\mu\nu} \to M_f^{-2} f_{\mu\nu}$ one finds that the Ricci scalar R_f transforms as $R_f \to M_f^2 R_f$, which thus leads to

$$\sqrt{-\det f}\, R_f \to M_f^{-2}\sqrt{-\det f}\, R_f. \tag{10}$$

Under this condition the interaction terms in the action (2) transform as

$$\sum_{n=0}^{4}\beta_n e_n(\sqrt{g^{-1}f}) \to \sum_{n=0}^{4}\beta_n e_n(M_f^{-1}\sqrt{g^{-1}f}). \tag{11}$$

Now, using the fact that the elementary symmetric polynomials $e_n(X)$ are of the order X^n, it is easy to see that the rescaling of $f_{\mu\nu}$ by a constant factor M_f^{-2} corresponds to a redefinition of the coupling constants $\beta_n \to M_f^n \beta_n$, and hence we are allowed to assume $M_f = 1$ [17] (see, however, Ref. [2] for caveats associated with this rescaling)

In addition to the equations of motion (3) and (8) for the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, the viable solutions of the bigravity have to satisfy the Bianchi constraints [21]. These constraints are the direct consequence of simultaneous realization of Bianchi identities and covariant conservation of the energy momentum tensor. Indeed, by demanding the consistency of the Bianchi identities $\left(\nabla^{\mu}G_{\mu\nu} = 0\right)$ as well as the covariant conservation of the energy momentum tensor $\left(\nabla^{\mu}T_{\mu\nu} = 0\right)$ from Eq. (3) we will get the following constraint equations also referred to as Bianchi constraints

$$\nabla_g^{\mu} I^g_{\mu\nu} = \frac{1}{2}\nabla_g^{\mu}\sum_{n=0}^{3}(-1)^n \beta_n g_{\mu\lambda} Y^{\lambda}_{(n)\nu}(\sqrt{g^{-1}f}) = 0, \tag{12}$$

where ∇_g is the g-metric covariant derivative.

3. A FRLW - FRLW combination in different coordinate systems

As it was mentioned in the introduction, the main goal of this paper is to understand the behavior of the theory in cases when the two metrics have the same form in different coordinate systems K and $\tilde{K}$, respectively. For this purpose, the simplest and most realistic choice is a FRLW-FRLW metric combination. In this case, line elements for the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ in spherical coordinates $\{r,\theta,\phi\}$ can be written as

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -dt^2 + a^2(t)\left[\frac{dr^2}{1-k_g r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{13}$$

$$f_{\mu\nu}dx^{\mu}dx^{\nu} = -\tilde{N}^2(\tilde{t})d\tilde{t}^2 + \tilde{b}^2(\tilde{t})\left[\frac{d\tilde{r}^2}{1-k_f \tilde{r}^2} + \tilde{r}^2\left(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2\right)\right], \tag{14}$$

where k_g and k_f are the curvature constants for the metrics g and f, respectively. For simplicity reasons from now on we will assume that $k_f = k_g = 0$. In Eqs. (13) and (14) a is the scale factor of the metric g, whereas $\tilde{N}$ and $\tilde{b}$ are the laps and scale factors for the metric f, respectively.

Since our metrics are coupled to each other through the interaction term, in order to write down equations of motion we need to represent them in the same coordinate system. To do this we perform the following coordinate transformation

$$\tilde{t} = V(t,r); \tilde{r} = U(t,r); \tilde{\theta} = \theta; \tilde{\phi} = \phi. \tag{15}$$

Implementing this transformation we get the following expressions for the line elements (13)-(14) represented in the same coordinate system

$$d\tilde{s}_g^2 = -dt^2 + a^2 dr^2 + a^2 r^2 d\Omega^2, \tag{16}$$

$$\begin{aligned} d\tilde{s}_f^2 = &-\left(N^2\dot{V}^2 - b^2\dot{U}^2\right)dt^2 + 2\left(b^2U'\dot{U} - N^2V'\dot{V}\right)dtdr + \\ &+\left(b^2U'^2 - N^2V'^2\right)dr^2 + b^2U^2d\Omega^2. \end{aligned} \tag{17}$$

In the derivation of these expressions, we have used that

$$d\tilde{t}^2 = \dot{V}^2 dt^2 + 2V'\dot{V}dtdr + V'^2 dr^2, \tag{18}$$

$$d\tilde{r}^2 = \dot{U}^2 dt^2 + 2U'\dot{U}dtdr + U'^2 dr^2, \tag{19}$$

and introduced new functions N and b according to

$$\tilde{N}(\tilde{t}) \to N(t,r); \tilde{b}(\tilde{t}) \to b(t,r). \tag{20}$$

For our discussion, it will be very useful to introduce the following functional notations

$$\begin{aligned} A(t,r) &\equiv N^2V^2 - b^2U^2, \\ C(t,r) &\equiv b^2U'U - N^2V'V, \\ Y(t,r) &\equiv b^2U'^2 - N^2V'V, \\ B(t,r)^2 &\equiv b^2U^2. \end{aligned} \tag{21}$$

After performing all above-mentioned simplifications for Eqs. (16) and (17), we finally arrive at

$$d\tilde{s}_g^2 = -dt^2 + a^2(t)dr^2 + a^2(t)r^2 d\Omega^2, \tag{22}$$

$$d\tilde{s}_f^2 = -A(t,r)dt^2 + 2C(t,r)dtdr + Y(t,r)dr^2 + B(t,r)^2 d\Omega^2, \tag{23}$$

where $d\Omega$ stands for the differential solid angel defined as

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2. \tag{24}$$

Before explicitly writing down generalized Einstein equations (3) and (8) in the metric representation (22) and (23), an important point still requires clarification. Namely, we need to understand how the tensorial function $X \equiv \sqrt{g^{-1}f}$ will be expressed in this metric representation. In order to calculate the elements of the $\sqrt{g^{-1}f}$ we will use the method implemented in Refs. [10, 11]. This method is based on the Cayley-Hamilton theorem in the linear algebra, according to which a matrix solves its own characteristic polynomial, i.e. for a 2×2 matrix A one has

$$[A]A = A^2 + (\det A)\mathbf{I}_2, \tag{25}$$

where $\mathbf{I}_2$ is the 2×2 unity matrix. Implementing this theorem for our case and remembering the relation $\det A^n = (\det A)^n$, we get the following expressions for traces of X^n

$$[X] = \sqrt{Z} + \frac{2B}{ar}, \tag{26}$$

$$[X^2] = Z - 2W + \frac{2B^2}{a^2r^2}, \tag{27}$$

$$[X^3] = Z^{3/2} - 3W\sqrt{Z} + \frac{2B^3}{a^3r^3}, \tag{28}$$

$$[X^4] = Z^2 - 2W(2X - W) + \frac{2B^4}{a^4r^4}, \tag{29}$$

with functions Z and W defined as

$$
\begin{aligned}
Z &= 2\left(\frac{C^2+AY}{a^2}\right)^{1/2} + \frac{Y}{a^2} + A, \\
W &= \left(\frac{C^2+AY}{a^2}\right)^{1/2}.
\end{aligned}
\tag{30}
$$

It is also useful to present functions of W and X through transformation functions V and U. This can be done by using Eq. (21). After some simple algebraic manipulations, we get

$$
W = \frac{Nb}{a}\left|\dot{V}U' - \dot{U}V'\right|,
\tag{31}
$$

$$
Z = \left(N\dot{V} + \mu\frac{b}{a}U'\right)^2 - \left(\frac{N}{a}V' + \mu b\dot{U}\right)^2,
\tag{32}
$$

where μ denotes the sign of the functional combination $\left(\dot{V}U' - \dot{U}V'\right)$, i.e. $\mu = \operatorname{sgn}\left(\dot{V}U' - \dot{U}V'\right)$. Finally, by plugging Eqs. (26)-(29) into Eq. ([intg]) we obtain for diagonal elements of the interaction tensor $I^g_{\mu\nu}$

$$
I^g_{11} = -\frac{\left(W + \frac{Y}{a^2}\right)\left(\beta_1 + 2\beta_2\frac{B}{ra} + \beta_3\frac{B^2}{r^2a^2}\right) + \sqrt{Z}\left(\beta_0 + 2\beta_1\frac{B}{ra} + \beta_2\frac{B^2}{r^2a^2}\right)}{\sqrt{Z}}
\tag{33}
$$

$$
I^g_{22} = -\frac{a^2\left(W + A\right)\left(\beta_1 + 2\beta_2\frac{B}{ra} + \beta_3\frac{B^2}{r^2a^2}\right) + a^2\sqrt{Z}\left(\beta_0 + 2\beta_1\frac{B}{ra} + \beta_2\frac{B^2}{r^2a^2}\right)}{\sqrt{Z}}
\tag{34}
$$

and for the only non-vanishing off-diagonal term

$$
I^g_{12} = -\frac{C\left(\beta_1 + 2\beta_2\frac{B}{ra} + \beta_3\frac{B^2}{r^2a^2}\right)}{\sqrt{Z}}.
\tag{35}
$$

Coming back to our initial discussion it is important to mention that we want to find a solution of our system, which will preserve the flat FRLW form for our background g-metric. This argument is justified by the current observational data, which state that at large scales our Universe is homogenous and isotropic [1]. This assumption sets some constraints on the elements of the interaction tensor $I^g_{\mu\nu}$. Indeed, since the off-diagonal elements of the Einstein tensor $G^g_{\mu\nu}$ and the energy momentum tensor $T_{\mu\nu}$ are vanishing for the FRLW metric, the off-diagonal components of the interaction tensor should vanish according to Eq.(3). Now, by setting I^g_{12} to zero, from Eq. (35) we find two possible cases satisfying this conditions. Namely, we get that either $C=0$ or $\left(\beta_1+2\beta_2\frac{B}{ra}+\beta_3\frac{B^2}{r^2a^2}\right)=0$. Let us start our discussion from the second case, which corresponds to

$$\beta_1+2\beta_2\frac{B}{ra}+\beta_3\frac{B^2}{r^2a^2}=0. \tag{36}$$

This equation is a simple second order algebraic equation and has the following solutions for the variable B/ra

$$\frac{B}{ra}=Q_\pm=\frac{-\beta_2\pm\sqrt{\beta_2^2-\beta_1\beta_3}}{\beta_3}. \tag{37}$$

Using the algebraic condition (36) the interaction tensor $I^{g\mu}_{\nu}=g^{\mu\alpha}I^g_{\alpha\nu}$ boils down to

$$I^{g1}_1=I^{g2}_2=-\left(\beta_0+2\beta_1Q_\pm+\beta_2Q_\pm^2\right), \tag{38}$$

$$I^{g3}_3=I^{g4}_4=-\left(\beta_3Q_\pm W+\beta_2\left(Q_\pm\sqrt{Z}+W\right)+\beta_1\left(\sqrt{Z}+Q_\pm\right)+\beta_0\right). \tag{39}$$

Taking into account Eqs. (38)-(39), as well as Eq. (7) the Einstein's equations (3) for the g-metric can be written as

$$\frac{3\dot{a}^2}{a^2}=\rho+\beta_0+2\beta_1Q_\pm+\beta_2Q_\pm^2, \tag{40}$$

$$\frac{\dot{a}^2}{a^2}+2\frac{\ddot{a}}{a}=-P+\beta_0+2\beta_1Q_\pm+\beta_2Q_\pm^2, \tag{41}$$

$$\frac{\dot{a}^2}{a^2}+2\frac{\ddot{a}}{a}=-P+\beta_3Q_\pm W_g+\beta_2(Q_\pm\sqrt{Z}+W)+\beta_1(\sqrt{Z}+Q_\pm)+\beta_0. \tag{42}$$

From here, as one can easily notice, the left hand sides of Eqs. (41) and (42) are the same. Thus, by demanding also the equality of the right hand sides of those equations, we obtain

$$\beta_1Q_\pm+\beta_2Q_\pm^2=\left(\beta_3Q_\pm+\beta_2\right)W+\left(\beta_1+\beta_2Q_\pm\right)\sqrt{Z}. \tag{43}$$

This equation can be further simplified by inserting the value of β_1 from Eq. (36). As a result we find the following simple relation between W and $\sqrt{Z}$.

$$\sqrt{Z} = \frac{W}{Q_{\pm}} + Q_{\pm}. \tag{44}$$

In the theory of massive bigravity consistency of the Bianchi constraints and Einstein's equations is the main criteria for the mathematical validity of a particular solution. For our solution discussed above, we have seen that Einstein's equations are mathematically consistent and here we should check that the Bianchi constraints are also satisfied. To do this, we need to ensure that under the conditions (36) and (43) Bianchi constraints

$$\nabla_g^{\mu} \sum_{n=0}^{3} (-1)^n \beta_n g_{\mu\lambda} Y_{(n)\nu}^{\lambda} (\sqrt{g^{-1} f}) = 0, \tag{45}$$

are hold. For our case, after involved mathematical simplifications it turns out, that the constraints (45) are identically zero, which means that the solution under consideration is one of allowed branches of our model. This branch in the literature is referred to as the algebraic branch [13].

After a mathematical consistency check, let us now investigate the physical properties of our solutions. Here, one can immediately see that, if conditions (36) and (43) are satisfied we have a complete decoupling of g and f sectors, which means that they will have completely independent cosmological evolutions. Based on Eq. (41) for the metric g we will have a dynamics driven by the following cosmological constant term

$$\Lambda_g = \beta_0 + 2\beta_1 Q_{\pm} + \beta_2 Q_{\pm}^2. \tag{46}$$

Hence, the metric g at late times of its evolution will be of a *de Sitter* type. For the case of $f-$ metric, we first notice that its interaction tensor $I_{\nu}^{f\mu} = f^{\mu\alpha} I_{\alpha\nu}^{f}$ (9), under the condition (36), reduces to

$$I_{\nu}^{f\mu} = \frac{\beta_2 + 2\beta_3 Q_{\pm} + \beta_4 Q_{\pm}^2}{Q_{\pm}^2} \delta_{\nu}^{\mu} = \Lambda_f \delta_{\nu}^{\mu}, \tag{47}$$

where $\delta_{\mu\nu}$ is the Kronecker delta function and the f-metric cosmological constant term Λ_f is defined as

$$\Lambda_f \equiv \frac{\beta_2 + 2\beta_3 Q_{\pm} + \beta_4 Q_{\pm}^2}{Q_{\pm}^2}. \tag{48}$$

We have also checked that the f-metric Einstein tensor $G_{\mu\nu}^{f}$ is mathematically consistent but because the expressions are burdensome we do not present them here. Again using the fact that under the conditions (36) and (43) the Einstein equations for g and f-metrics are completely decoupled, we can make a variable redefinition only in the f sector without changing the equations for the g-

metric. In Ref. [23] it has been shown that after a convenient variable redefinition one finds from Eq. (8) that the f-metric is also of the de Sitter type with the cosmological constant Λ_f.

In the end of this section let us briefly discuss the behavior of the first solution of the condition $I^g_{12} = 0$, namely, the case when $C = 0$. For this case the diagonal components of the interaction tensor $I^g_{\mu\nu}$ will be not only functions of the time t but will also depend on the spherical coordinate r as can be seen from Eqs. (33) and (34). On the other hand, we know that the components of the Einstein tensor $G^g_{\mu\nu}$ for the homogeneous and isotropic FRLW metric will have only time dependence. Therefore, the only possibility to preserve the consistency of Eq. (3) is to assume that the energy momentum tensor (7), coupled to the background metric g, is also inhomogeneous. Namely, we can choose it to have a structure which will allow us to cancel the inhomogeneity of the interaction term $I^g_{\mu\nu}$, thus recovering the consistency of Eq. (3). Though being an interesting possibility to resolve the above-mentioned issue, the physical realization of the energy-momentum tensor with such properties is highly non-trivial and will not be discussed in this work.

4. Dynamics of U and V transformation functions

Having found the complete set of equations for the algebraic branch (36) at the background level, it is the time to investigate the evolution of different physical solutions. In particular, for the g-metric we have standard Einstein equations with a cosmological constant (46). The same holds also for the Einstein equations for the f-metric (47). In our case the f-metric (17) can be also written as

$$f_{\mu\nu}dx^{\mu}dx^{\nu} = -N^2dV^2 + b^2\left[dU^2 + U^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{49}$$

In this representation the transformation functions $V(t,r)$ and $U(t,r)$ are now playing the role of the time and spatial coordinates for the metric f, respectively. Here, as the two metrics are completely decoupled, without loss of generality we can take for the lapse function $N \equiv 1$. The Einstein equations (8) for the f-metric defined in Eq. (49) take the form corresponding to a de Sitter solution. By solving these equations we find for the evolution of the scale factor b of the f-metric

$$b(V) = e^{V\sqrt{\Lambda_f/3}}, \tag{50}$$

where Λ_f is given by Eq. (48). On the other hand, for the algebraic branch from Eqs. (37) and (21) we obtain that

$$\frac{bU}{ra} = Q_{\pm}. \tag{51}$$

Inserting into this equation the expression for b from Eq. (50) we obtain the following relation between the transformation functions U and V

$$U = Q_{\pm} are^{-V\sqrt{\Lambda_f/3}}. \tag{52}$$

In this work, we are interested in the behavior of the model at late times of its evolution. Namely, we will assume that the metric g is already at the de Sitter stage of its evolution and hence its scale factor a is given by

$$a = e^{t\sqrt{\Lambda_g/3}}. \tag{53}$$

Furthermore, by inserting the expression of the scale factor a from Eq. (53) into Eq. (52), we find for the transformation function U

$$U = Q_{\pm} re^{\left(t\sqrt{\Lambda_g/3} - f\sqrt{\Lambda_f/3}\right)}. \tag{54}$$

As the next step, we need to find the structure of the function V at the late stages of the cosmic evolution. To do this we will use the constraint equation (44). Indeed, with the use of Eqs. (50), (53) and (54), Eq. (44) can be written as

$$\begin{gathered} V'^2\left(-Q_{\pm}^2 r^2\frac{\Lambda_f}{3} + r^2\frac{\Lambda_g}{3} + e^{-2\sqrt{\frac{\Lambda_g}{3}}t}\right) + Q_{\pm}^2 r^2 e^{2\sqrt{\frac{\Lambda_g}{3}}t}\left(\sqrt{\frac{\Lambda_g}{3}} - \sqrt{\frac{\Lambda_f}{3}}\dot{V}\right)^2 + \\ +2rV'\left(Q_{\pm}^2\sqrt{\frac{\Lambda_f}{3}} - \sqrt{\frac{\Lambda_g}{3}}\dot{V}\right) = 0 \end{gathered} \tag{55}$$

This equation is a second order non-linear partial differential equation and one can easily check that

$$V = \sqrt{\frac{\Lambda_g}{\Lambda_t}}t, \tag{56}$$

is a solution of it. Moreover, we will also argue that at late times of the cosmic evolution $(t \to \infty)$ Eq. (56) is the only solution of Eq. (55). Indeed, from Eq. (55) one can notice that its second term is multiplied by an exponential prefactor which gives a big contribution in the late times, for the fixed r, making this term the dominant one. Hence, if the second term in Eq. (55) dominates over the other terms, the only solution of Eq. (55) will be (56). Of course, one can argue that depending on the function V there can be situations where the first term in Eq. (55) becomes of the same order as the second term, thus leading to a solution other than (56) (it is easy to see that the third term in Eq. (55), for an arbitrary choice of the function V, will be always sub-dominate compare with other terms at late times). To see whether this can be the case or not we will follow some simple arguments. Let us assume that at late times our transformation function V can be decomposed as $V(t,r)=s(t)\,p(r)$. For fixed values of the radial coordinate r, it is clear that the function $s(t)$ will either grow or decrease in the late times. Let us first discuss the case of growing $s(t)$ when $t\to\infty$. In this case the first two terms in Eq. (55) will give equal contribution only when

$$s^2 \approx e^{2\sqrt{\frac{\Lambda_g}{\Lambda_t}}t}\dot{s}^2. \tag{57}$$

By solving this equation we will get for s

$$s = s_0 \exp\left[-\sqrt{\frac{3}{\Lambda_g}}\exp\left[-\sqrt{\frac{\Lambda_g}{3}}t\right]\right]. \tag{58}$$

As we can observe from Eq. (58) the function s decreases when $t\to\infty$, which is then in contradiction with the initial assumption that we have growing s. Let us now discuss the second possibility which corresponds to a decreasings. In this case from Eq. (55) we immediately notice that because of the exponential prefactor the second term in Eq. (55) becomes quickly dominant in the limit $t\to\infty$, leading to the solution (56).

To summarize, here we find that for the decomposed function of V the only solution is (56). Furthermore, evaluating Eq. (55) numerically for large range of parameters, we found that the solution (58) is indeed the only solution at late times. Finally, by plugging this form of V from Eq. (56) into Eq. (54) we find for the function U

$$U = Q_{\pm} r. \tag{59}$$

For these solutions the f -metric line-element (49) will be given by the following FRLW form

$$f_{\mu\nu}dx^\mu dx^\nu = -\frac{\Lambda_g}{\Lambda_t}dt^2 + Q_\pm^2 e^{2t\sqrt{\frac{\Lambda_g}{3}}}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{60}$$

Thus, from the discussion of this section we can state that for the algebraic branch (36), the FRLW -FRLW metric combination in the same coordinate system is an attractor solution. This conclusion can be also generalized for cases of non-vanishing metric curvature constants, i.e. $k_f = k_g \neq 0$. Indeed, during the late time evolution, the cosmological constant becomes the dominant source and we can neglect the role of curvatures and therefore the previous result will hold also in this case.

5. Conclusions

Thus, we have studied the late time cosmological evolution of a bigravity model where the two metrics do not have the same form. In particular we have assumed that both metrics have FLRW structure but in different coordinate systems. To investigate the physical properties of the model we made a coordinate transformation, which allows to represent the two metrics in the same coordinate system. After this transformation, the background metric has again FLRW structure whereas the reference metric becomes of a Lameître type. From the Bianchi constraints for this particular metric combination, we have found that the model has two possible branches of solutions. One of these branches to be realized we need to assume a particular highly inhomogeneous distribution of the matter content of the Universe, realization of which is physically highly non-trivial. On the other hand for the second branch, which is also referred to as an algebraic branch, the EoMs for two metrics decouple from each other and can be solved independently in a self-consistent way. By solving these background equations, we find that even though initially two metrics have different structures at late times of the cosmic evolution they will be attracted towards the same FLRW structure. If this conclusion based on our simple example will be extended also for other metric combinations, it will mean that FLRW-FLRW metric combination is not just initial choice but there is some fundamental reason behind. From the phenomenological point of view, the generalization of our result would mean that the choice of different metric combinations could not cure the bigravity theory from existing instabilities. This is simply because the FLRW-FRLW metric combination is an attractor solution and hence at late times we will have again the same instabilities existing in the initial studies, where this combination was chosen already from the beginning.

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