Armenian Journal of Physics, 2017, vol. 10, issue 4, pp. 154-162

Fermionic modes in locally anti-de Sitter spacetime with a compact subspace

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Received 27 September 2017

Abstract. We present a complete set of positive and negative energy fermionic modes in a $D+1$-dimensional locally anti-de Sitter (AdS) spacetime with a part of spatial dimensions compactified on a torus. Two geometries are considered: the boundary-free geometry and the geometry with an additional brane parallel to the AdS horizon. On the brane, the field operator obeys the bag boundary condition. Both regions between the brane and the horizon and between the brane and AdS boundary are considered.

Keywords: fermionic modes, cosmological constant, gravitational field, radion stabilization.

1. Introduction

The anti-de Sitter (AdS) spacetime for the corresponding geometry and coordinate systems [1] is the maximally symmetric solution of the Einstein equations with a negative cosmological constant as the only source of the gravitational field. Its popularity as a curved background in quantum field theory is motivated by several reasons. First, due to the high symmetry, numerous problems become exactly solvable on the AdS bulk. This allows to shed light on the influence of the gravitational field on quantum matter in more complicated geometries. The AdS spacetime generically arises as a ground state in extended supergravity and string theories, which is again potentially most important. Recent increase of interest to the AdS geometry is related to its crucial role in two exciting developments of the past decade such as the AdS/CFT correspondence [2, 3] and the brane world scenario with large extra dimensions [4].

Motivated by the problem of radion stabilization, the quantum effects in brane world scenarios with AdS bulk have been investigated in a large number of papers (see, for instance, references given in [5]). In the present paper, we consider the fermionic modes in locally AdS spacetime with a compact subspace (for quantum effects in brane world models with compact subspaces in the case of a scalar field see [6-13]). Two different problems will be considered in the absence and in the presence of a brane parallel to the AdS boundary. The complete set of fermionic modes is required in the canonical quantization procedure of the Dirac field on the background under consideration (for the quantization on general curved backgrounds see, for example, [14-16]). Having these modes, one can evaluate the vacuum expectation values (VEVs) of physical observables quadratic in the field operator. Note that the VEV of the current density for a charged scalar field in locally AdS spacetime with toroidally compactified spatial dimensions has been investigated in [17] for the boundary-free geometry, and in [18, 19] for geometries with a single and two parallel branes. The fermionic current in the boundary-free geometry has been recently discussed in [20].

The paper is organized as follows. In the next section, we present the geometry of the problem, the field equation and the periodicity conditions. In section 3, a complete set of solutions to the Dirac equation is given for the problem at the absence of the brane. The fermionic modes in the presence of a brane parallel to the AdS boundary are presented in section 4. Both regions between the AdS boundary and the brane and AdS horizon and the brane are considered. The main results are summarized in section 5.

2. Geometry of the problem

We consider a $(D+1)$-dimensional locally AdS spacetime described in Poincaré coordinates. The latter are the most frequently used coordinates in braneworld scenarios and in the discussion of AdS/CFT correspondence. The metric tensor is given by the line element

$$ds^2 = \left(\frac{a}{z}\right)^2 (\eta_{ik} dx^i dx^k - dz^2), \tag{1}$$

where a is the curvature radius of the spacetime, $0 \le z < \infty; i,k = 0,...,D-1$, and $\eta_{ik} = diag(1,-1,...,-1)$. In this form, the part of the AdS spactime covered by the Poincaré coordinates is conformally related to the Minkowski spacetime. The hypersurfaces $z=0$ and $z=\infty$ present the AdS boundary and the horizon, respectively. Even though the local geometry that we are going to consider here is an AdS spacetime, the global properties will be different. Namely, we assume that a part of the spatial dimensions, the coordinates $(x^{p+1},...,x^{D-1})$, is compactified to a q-dimensional torus $(S^1)^q; q = D-p-1$. Denoting the corresponding coordinate lengths by L_l, one has $0 \le x^l \le L_l; l = p+1,...,D-1$. Here, p is the number of uncompact dimensions and for them one has $-\infty < x^l < +\infty; l = 1,..., p$. Hence, the Minkowski spacetime, to which the geometry we consider is conformally related, has a spatial topology $R^p \times (S^1)^q$.

We want to find a complete set of modes for a fermionic field $\psi(x)$ obeying the Dirac equation

$$i\gamma^\mu \nabla_\mu \psi - m\psi = 0, \tag{2}$$

where $\nabla_\mu = \partial_\mu + \Gamma_\mu$ is the covariant derivative and Γ_μ is the spin connection. For the geometry at hand, the curved spacetime Dirac matrices γ^μ can be taken as $\gamma^\mu = \frac{a}{z}\gamma^{(\mu)}$ where $\gamma^{(\mu)}$ are the corresponding flat spacetime matrices. For a fermionic field realizing the irreducible representation of the Clifford algebra, the latter are $N \times N$ matrices, where $N = 2^{[(D+1)/2]}$ with $[x]$ being the integer part of x. A possible representation of the flat spacetime gamma matrices that allows the separation of the equations for the upper and lower components of the spinor $\psi(x)$, has been discussed in [5]. Here we consider an alternative representation with a simpler structure for the fermionic mode functions.

The background geometry has a nontrivial topology and, in addition to the field equation, the periodicity conditions must be specified in the compact subspace. We will consider quasi-periodicity conditions

$$\psi(t, x^1, ..., x^l + L_l, ..., z) = e^{i\alpha_l}\psi(t, x^1, ..., x^l, ..., z), \tag{3}$$

with constant phases α_l. The special cases most frequently discussed in the literature correspond to untwisted $(\alpha_l = 0)$ and twisted $(\alpha_l = \pi)$ fields.

For the flat spacetime gamma matrices we take the representation

$$\gamma^{(0)} = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0^\dagger & 0 \end{pmatrix}, \quad \gamma^{(l)} = \begin{pmatrix} 0 & \sigma_l \\ -\sigma_l^\dagger & 0 \end{pmatrix}, \quad \gamma^{(0)} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

where $l = 1, 2, ..., D-1$, and the dagger represents the hermitian conjugate. From the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, for the matrices $\sigma_0; \sigma_1$ we find the relations

$$\sigma_l\sigma_n^\dagger + \sigma_n\sigma_l^\dagger = 2\delta_{nl}; \sigma_l^\dagger\sigma_n + \sigma_n^\dagger\sigma_l = 2\delta_{nl}, \tag{5}$$

and

$$\sigma_0\sigma_1^\dagger = \sigma_1\sigma_0^\dagger; \sigma_0^\dagger\sigma_1 = \sigma_1^\dagger\sigma_0; \sigma_0^\dagger\sigma_0 = 1, \tag{6}$$

for $l, n = 1, 2, ..., D-1$. The term in the Dirac equation (2) related to the spin connection is presented as $\gamma^\mu\Gamma_\mu = -D\gamma^{(D)}/(2a)$.

3. Fermionic Modes

By taking into account the planar symmetry along the directions parallel to the AdS boundary, the positive energy mode functions can be taken in the form

$$\psi_\beta^{(+)}(x) = \varphi(z)e^{ikx-iEt}, \tag{7}$$

where $kx = \sum_{l-1}^{D-1} k_l x^l$, $\varphi(z)$ is a N- component spinor and the collective index β stands for a set of quantum numbers specifying the solution (see below). We will split the momentum k into two parts, $k=(k_p,k_q)$, where $k_p=(k_1,...,k_p)$ and $k_q=(k_{p+1},...,k_{D-1})$ are the momenta in the uncompact and compact subspaces, respectively. For the components along uncompact dimensions one has $-\infty<k_1<+\infty; l=1,...,p.$ The components along the compact dimensions are quantized by the periodicity conditions (3):

$$k_l=\frac{2\pi n_l+\alpha_l}{L_l}; l=p+1,...,D-1, \tag{8}$$

where $n_l = 0, \pm 1, \pm 2, \ldots$

Decomposing the spinor $\varphi(z)$ into the upper and lower components, $\varphi=(\varphi_+,\varphi_-)^T$, from the equation (2) we get

$$\left(\partial_z-\frac{D}{2z}+\frac{ma}{z}\right)\varphi_+-\left(E\sigma_0-\mathbf{k}\sigma\right)\varphi_-=0;\left(\partial_z-\frac{D}{2z}+\frac{ma}{z}\right)\varphi_-+\left(E\sigma_0+\mathbf{k}\sigma\right)\varphi_+=0, \tag{9}$$

where $k\sigma=\sum_{l=1}^{D-1}k^l\sigma_l$. By using the relation

$$\left(E\sigma_0^{\dagger}+k\sigma^{\dagger}\right)\left(E\sigma_0-k\sigma\right)=\lambda^2, \tag{10}$$

with $\lambda=\sqrt{E^2-k^2}$, we obtain separate equations for the upper and lower components

$$\left[\partial_z^2-\frac{D}{z}\partial_z+\lambda^2+\frac{\left(D+1\right)^2-\left(2ma\pm 1\right)^2}{4z^2}\right]\varphi_\pm=0. \tag{11}$$

The solutions of these equations corresponding to the modes regular on the AdS boundary are presented as

$$\varphi_\pm=z^{(D+1)/2}J_{ma\pm 1/2}(\lambda z)\chi_\pm^{(\sigma)}, \tag{12}$$

where $J_v(x)$ is the Bessel function and $\chi_\pm^{(\sigma)}$ are coordinate independent one column matrices with $N/2$ rows. One has $N/2$ linearly independent matrices numbered by $\sigma=1,2,...,N/2$.

The relation between $\chi_+^{(\sigma)}$ and $\chi_-^{(\sigma)}$ is found from (9):

$$\chi_-^{(\sigma)}=\frac{E\sigma_0^{\dagger}+\mathbf{k}\sigma^{\dagger}}{\lambda}\chi_+^{(\sigma)}. \tag{13}$$

Hence, the fermionic modes are specified by a set of quantum numbers $\beta = (\lambda, k, \sigma)$ and the corresponding positive energy mode functions are given by

$$\psi_{\beta}^{(+)}(x) = z^{\frac{(D+1)}{2}} e^{i\mathbf{k}x - iEt} \begin{pmatrix} J_{ma+1/2}(\lambda z)\chi_{+}^{(\sigma)} \\ \dfrac{(E\sigma_0^{\dagger} + \mathbf{k}\sigma^{\dagger})}{\lambda} J_{ma-1/2}(\lambda z)\chi_{+}^{(\sigma)} \end{pmatrix}. \tag{14}$$

The normalization condition for these functions has the form

$$\int d^D x \left(\frac{a}{z}\right)^D \psi_{\beta'}^{(+)\dagger}\psi_{\beta}^{+} = \delta_{\sigma\sigma'}\delta_{k_q k_{q'}}\delta(\lambda - \lambda')\delta(k_p - k_{p'}). \tag{15}$$

By using the result

$$\int_0^{\infty} dzz J_{\nu}(\lambda z) J_{\nu}(\lambda' z) = \frac{1}{\lambda}\delta(\lambda - \lambda'), \tag{16}$$

for the modes (14) from (15) one finds

$$\chi_{+}^{(\sigma)\dagger}\left[\left(E + k\sigma\sigma_0^{\dagger}\right)^2 + \lambda^2\right]\chi_{+}^{(\sigma')} = \lambda^2 C_0^2 \delta_{\sigma\sigma'}, \tag{17}$$

where

$$C_0^2 = \frac{\lambda}{(2\pi)^p V_q a^D}, \tag{18}$$

and $V_q = L_{p+1} \cdots L_{D-1}$ is the volume of the compact subspace.

Let us introduce the one column matrices $w^{(\sigma)}, \sigma = 1, 2, ..., N/2$, in accordance with

$$w^{(\sigma)} = \frac{E + \mathbf{k}\sigma\sigma_0^{\dagger} - i\lambda}{\lambda C_0}\chi_{+}^{(\sigma)}, \tag{19}$$

or inverting

$$\chi_{+}^{(\sigma)} = iC_0 \frac{E + \mathbf{k}\sigma\sigma_0^{\dagger} - i\lambda}{2E} w^{(\sigma)}. \tag{20}$$

In terms of $w^{(\sigma)}$, the normalization condition (17) is written as $w^{(\sigma)\dagger}w^{(\sigma')} = \delta_{\sigma\sigma'}$. From here it follows that as $w^{(\sigma)}$ we can take one-column matrices having $N/2$ rows with the elements $w_1^{(\sigma)} = \delta_{1\sigma}$.

Hence, for the normalized positive-energy fermionic modes, up to a constant phase, one gets

$$\psi_{\beta}^{(+)}(x)=\frac{C_0}{2E}z^{\frac{(D+1)}{2}}e^{i\mathbf{k}x-iEt}\begin{pmatrix}(E-\mathbf{k}\sigma\sigma_0^{\dagger}-i\lambda)w^{(\sigma)}J_{ma+1/2}(\lambda z)\\ -i\sigma_0^{\dagger}(E+\mathbf{k}\sigma\sigma_0^{\dagger}+i\lambda)w^{(\sigma)}J_{ma-1/2}(\lambda z)\end{pmatrix}. \tag{21}$$

In a similar way, for the negative-energy modes one gets

$$\psi_{\beta}^{(-)}(x)=\frac{C_0}{2E}z^{\frac{(D+1)}{2}}e^{i\mathbf{k}x+iEt}\begin{pmatrix}-i\sigma_0(E+\mathbf{k}\sigma^{\dagger}\sigma_0-i\lambda)w^{(\sigma)}J_{ma+1/2}(\lambda z)\\ (E-\mathbf{k}\sigma^{\dagger}\sigma_0+i\lambda)w^{(\sigma)}J_{ma-1/2}(\lambda z)\end{pmatrix}. \tag{22}$$

It can be checked that the modes (21) and (22) are orthogonal.

4. Fermionic modes in the presence of a brane

In this section we consider the background geometry described in section 2 with an additional brane parallel to the AdS boundary and located at $z=z_0$. On the brane, the field operator obeys the bag boundary condition

$$(1+i\gamma^{\mu}n_{\mu})\psi=0, \tag{23}$$

where n_{μ} is the normal to the brane. One has $n_{\mu}=\delta_{\mu}^{D}a/z$ for the region $z\le z_0$ and $n_{\mu}=-\delta_{\mu}^{D}a/z$ for the region $z\ge z_0$. With the boundary condition (23), the fermionic current density through the brane vanishes.

First we consider the region $z\le z_0$. Similar to the problem discussed in the previous sections, the positive energy modes have the form

$$\psi_{\beta}^{(+)}(x)=\frac{C_1}{2E}z^{\frac{(D+1)}{2}}e^{i\mathbf{k}x-iEt}\begin{pmatrix}(E-\mathbf{k}\sigma\sigma_0^{\dagger}-i\lambda)w^{(\sigma)}J_{ma+1/2}(\lambda z)\\ -i\sigma_0^{\dagger}(E+\mathbf{k}\sigma\sigma_0^{\dagger}+i\lambda)w^{(\sigma)}J_{ma-1/2}(\lambda z)\end{pmatrix}. \tag{24}$$

From the boundary condition (23) follows that the eigenvalues of the quantum number λ are zeros of the function $J_{ma-1/2}(\lambda z_0)$:

$$J_{ma-1/2}(\lambda z_0)=0. \tag{25}$$

Denoting the zeros of the function $J_{ma-1/2}(x)$ by $\lambda_n, n=1,2,...$ for the eigenvalues of λ one finds $\lambda=\lambda_n/z_0$. Hence, in the region between the brane and AdS boundary we have a discrete set of eigenvalues for λ.

The normalization condition for the mode functions (24) is similar to (15), with the difference that now the integration over z goes in the region $[0, z_0]$ and instead of the delta function $\delta(\lambda - \lambda')$ one has $\delta_{\lambda_n \lambda_{n'}}$. From that condition, up to a phase, we find

$$C_1^{-2} = \frac{1}{2}(2\pi)^p V_q a^D z_0^2 J_{ma+1/2}^2(\lambda_n). \tag{26}$$

The negative energy modes are found by a similar scheme and have the form

$$\psi_\beta^{(-)}(x) = \frac{C_1}{2E} z^{\frac{(D+1)}{2}} e^{i\mathbf{k}x+iEt} \begin{pmatrix} -i\sigma_0 (E + \mathbf{k}\sigma^\dagger \sigma_0 - i\lambda) w^{(\sigma)} J_{ma+1/2}(\lambda z) \\ (E - \mathbf{k}\sigma^\dagger \sigma_0 + i\lambda) w^{(\sigma)} J_{ma-1/2}(\lambda z) \end{pmatrix}. \tag{27}$$

with the same normalization coefficient as in (24).

Now let us turn to the region $z \geq z_0$. In this region, the solution of the equation (11) is expressed in terms of the linear combination of the functions $J_v(x)$ and $Y_v(x)$, where $Y_v(x)$ is the Neumann function. The positive energy mode functions are presented as

$$\psi_\beta^{(+)}(x) = \frac{C_2}{2E} z^{\frac{(D+1)}{2}} e^{i\mathbf{k}x-iEt} \begin{pmatrix} (E - \mathbf{k}\sigma\sigma_0^\dagger - i\lambda) w^{(\sigma)} Z_{ma+1/2}(\lambda z) \\ -i\sigma_0^\dagger (E + \mathbf{k}\sigma\sigma_0^\dagger + i\lambda) w^{(\sigma)} Z_{ma-1/2}(\lambda z) \end{pmatrix}. \tag{28}$$

where

$$Z_{ma\pm1/2}(\lambda z) = J_{ma\pm1/2}(\lambda z) + B_\beta Y_{ma\pm1/2}(\lambda z). \tag{29}$$

From the boundary condition (23) one finds

$$B_\beta = -\frac{J_{ma+1/2}(\lambda z_0)}{Y_{ma+1/2}(\lambda z_0)}, \tag{30}$$

for both the upper and lower signs in (29). Similar to the case of locally AdS geometry without the brane, the eigenvalues of λ are continuous, $0 \leq \lambda < \infty$.

The normalization condition is given by (15) with the integration over z in the range $[z_0, \infty)$. From that condition we get

$$C_2^{-2} = (2\pi)^p V_q a^D \frac{1 + B_\beta^2}{\lambda}. \tag{31}$$

The negative energy modes in the region $z \geq z_0$ are given by

$$\psi_{\beta}^{(-)}(x)=\frac{C_2}{2E}z^{\frac{(D+1)}{2}}e^{i\mathbf{k}x+iEt}\begin{pmatrix}-i\sigma_0(E+\mathbf{k}\sigma^{\dagger}\sigma_0-i\lambda)w^{(\sigma)}J_{ma+1/2}(\lambda z)\\ (E-\mathbf{k}\sigma^{\dagger}\sigma_0+i\lambda)w^{(\sigma)}J_{ma-1/2}(\lambda z)\end{pmatrix}. \tag{32}$$

with the same normalization coefficient as in (24). In the limit when the brane tends to the AdS boundary, $z_0\to 0$, one has $B_\beta\to 0$ and the modes (28), (32) are reduced to the modes at the absence of the brane, discussed in section 3.

Having the complete set of fermionic modes, we can evaluate the VEVs of physical quantities bilinear in the field operator, $F=F(\bar{\psi},\psi)$, with $\bar{\psi}=\psi^{\dagger}\gamma^0$ being the Dirac adjoint. Examples are the fermionic condensate, $\bar{\psi}\psi$, the current density, $j^{\mu}=\bar{\psi}\gamma^{\mu}\psi$, and the energy-momentum tensor

$$T_{\mu\nu}=\frac{i}{2}\left[\bar{\psi}\gamma_{(\mu}\nabla_{\nu)}\psi-(\nabla_{(\mu}\bar{\psi})\gamma_{\nu)}\psi\right], \tag{33}$$

where the brackets enclosing the indices in (33) mean the symmetrization. Expanding the field operator in terms of the complete set of mode functions and using the anticommutation relations for the creation and annihilation operators, for the VEV one gets

$$\left\langle 0\left|F(\bar{\psi},\psi)\right|0\right\rangle=\frac{1}{2}\sum_{\beta}\left[F(\bar{\psi}_{\beta}^{(-)},\psi_{\beta}^{(-)})-F(\bar{\psi}_{\beta}^{(+)},\psi_{\beta}^{(+)})\right], \tag{34}$$

where the sum symbol over β is understood as summation for discrete subset of quantum numbers and integration over the continues ones.

5. Conclusion

Thus, we have considered the complete set of fermionic modes in locally AdS spacetime with toroidally compactified subspace in Poincaré coordinates. Along the compact dimension, the field operator obeys quasiperiodicity conditions (3) with general phases. These conditions lead to the quantization of the momentum components with the eigenvalues (8). In the boundary-free problem, a complete set of solutions to the Dirac equation is given by (21) and (22) for the positive and negative energy modes, respectively.

At the presence of a brane, an additional boundary condition is imposed on the field operator. Here we have discussed the most popular reflective boundary condition for spinor fields, namely, the bag boundary condition (23). In the region between the brane and AdS boundary, $0\le z\le z_0$, the eigenvalues of the quantum number λ are quantized by the boundary condition on the brane. These eigenvalues are expressed in terms of the zeros of the Bessel function $J_{ma-1/2}(x)$. The corresponding positive and negative energy fermionic mode functions are presented as (24) and (27) with the normalization coefficient defined by (26). In the region between the brane and the AdS horizon, the eigenvalues of λ are continuous and the complete set of modes is given by (28) and (32) and the normalization coefficient is defined as (31). Given the complete set of the mode functions, the VEVs of physical quantities quadratic in the field operator are evaluated by using the mode-sum formula (34).

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