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Resistive Magnetohydrodynamic Model for Harmonic Expansion of Cylindrical Plasma

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Abstract. Analytical model for resistive plasma expansion in a static uniform magnetic field in the regime in which the magnetic field does not perturb the plasma motion is proposed. The model is based on a class of exact solutions for the purely radial expansion of the plasma in the absence of a magnetic field. This approximation permits the reduction of the electromagnetic problem for consideration of a diffusion equation at the external magnetic field. Explicit solutions are derived for resistive cylindrical plasma expanding into a uniform ambient magnetic field. Some numerical examples related to the laser-produced plasma experiments are presented.

Keywords: magnetized plasma, dipole magnetic field, vortices, shock waves, numerical simulations

1. Introduction

The problem of hot plasma expansion into a vacuum or into a background plasma in the presence of an external magnetic field has been discussed in the analysis of many astrophysical and laboratory applications (see, e.g., Refs. [1-5] and references therein). In particular, such process is a topic of intense interest across a wide variety of disciplines, with applications to solar[6] and magnetospheric [7,8] physics, astrophysics [7,8,9] and pellet injection for tokomak refueling [10].

In this paper, we consider analytically resistive cylindrical plasma expanding into vacuum in the presence of an external magnetic field. Similar problem has been treated previously in Refs. [11] and [12] but for a spherical plasma expansion. The vast literature on the theory of plasma expanding into a vacuum or into a background plasma, Refs. [11-12] (see also references therein) illustrate various aspects and approaches. In Refs. [13-18] plasma has been considered as a highly conducting medium with zero magnetic field inside. From the point of view of electrodynamics, it is similar to the expansion of a superconductor in a magnetic field. An exact analytic solution for a uniformly expanding, highly conducting plasma sphere in an external uniform and constant magnetic field has been obtained in Ref. [13]. The nonrelativistic limit of this theory has been used by Raizer [14] to analyze the energy balance (energy emission and transformation) during plasma expansion. A similar problem has been considered in Ref. [15] in a one-dimensional (1D) geometry for a plasma layer. In our recent papers, we obtained an exact analytic solution for the uniform relativistic expansion of a highly conducting plasma sphere [16,17] or cylinder [18] in the presence of a dipole or homogeneous magnetic field, respectively.

The mentioned treatments [13-18] were obtained assuming a somewhat idealized situation: uniform expansion, infinite electrical conductivity of a plasma, etc. More realistic models for plasma expansion taking into account the deceleration (or acceleration) of the plasma boundary have been developed, for instance, in Refs. [19-22] (see also references therein) for spherical [19], planar (1D) [20,21] and cylindrical [22] expansions employing ideal magnetohydrodynamic (MHD) equations. However, it should be noted that the ideal MHD may not be justified in some experimental situations where the typical parameters are such that the plasma resistivity is not negligible.[1,3,11,12] In this case, the coupling of the magnetic field with the plasma motion, determined by the magnetic Reynolds number, should result in a distortion and diffusion of the field across the expanding plasma[11,12]. We present here calculations of the electromagnetic field configuration in the stages preceding significant deceleration of the plasma and in the regime in which the magnetic field does not perturb the purely radial motion of the cylindrical plasma. The latter assumption is valid at large initial ratios of plasma energy density to magnetic field energy density.

2. Resistive MHD model

Usually, the motion of the expanding plasma boundary is approximated as the motion with constant velocity (uniform expansion). In the present study, a quantitative analysis of plasma dynamics is developed based on a cylindrical model. Within the scope of this analysis, the nonuniform plasma expansion process is examined. We consider resistive collision-dominated magnetized plasma expanding into vacuum in the presence of a uniform and constant magnetic field. The relevant equations governing the expansion are those of resistive MHD [23], assuming that the characteristic length scales for plasma flow are much larger than the Debye length and Larmor radius of the ions. The set of equations describing the dynamics of this process is:

$$\begin{aligned} &\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0,\\ &\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]+\nabla p=\frac{1}{4\pi}[\nabla\times\mathbf{H}]\times\mathbf{H},\\ &\frac{\partial \mathbf{H}}{\partial t}=\nabla\times[\mathbf{u}\times\mathbf{H}]+D\nabla^2\mathbf{H}, \end{aligned} \tag{1}$$

with $\nabla\cdot\mathbf{H}=0$, where $\mathbf{H}$ is the magnetic field strength, $D=c^2/4\pi\sigma$ is the diffusion coefficient, ρ , $\mathbf{u}$, p and σ are the mass density, the velocity, the pressure and the electrical conductivity of the plasma, respectively. In this paper we assume an isotropic and homogeneous electrical conductivity (and hence the diffusion coefficient $D\frac{1}{2}$) of the plasma σ. The equations above must be accompanied by the equation of state and the equation for entropy. Using the thermodynamic relation between entropy, pressure, and internal energy as well as Eq. (1) the equation for pressure reads [23]

$$\frac{\partial p}{\partial t}+(\mathbf{u}\cdot\nabla)p+\gamma p(\nabla\cdot\mathbf{u})=(\gamma-1)\frac{j^2}{\sigma}. \tag{2}$$

Here γ is the ratio of the specific heats, $\mathbf{j}$ is the current density in a plasma which after the elimination of some unimportant terms from the generalized Ohm's law [23] is reduced to the form

$$\mathbf{j}=\sigma\left(\mathbf{E}+\frac{1}{c}[\mathbf{u}\times\mathbf{H}]\right), \tag{3}$$

where $\mathbf{E}$ is the electric field strength. Here it is convenient to introduce the vector potential $\mathbf{A}$. Within the scope of the present study, the free charge density is absent and a suitable gauge $\nabla\cdot\mathbf{A}=0$ allows the electric and magnetic fields to be determined from the vector potential $\mathbf{A}$,

$$\mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t},\quad \mathbf{H}=\nabla\times\mathbf{A}. \tag{4}$$

Then from the last expression in Eq. (1) one can derive a similar equation for the vector potential $\mathbf{A}$

$$\frac{\partial \mathbf{A}}{\partial t}=\mathbf{u}\times[\nabla\times\mathbf{A}]+D\nabla^2\mathbf{A}. \tag{5}$$

In deriving Eq. (5) we have neglected the displacement current which is justified for the nonrelativistic expansion of the plasma. More specifically this approximation is valid at $4\pi\sigma\tau_{\mathrm{D}}\gg 1$, where $\tau_{\mathrm{D}}=4\pi\sigma R^2/c^2$ is the characteristic diffusion time of the magnetic field and R is the characteristic size of the system, taken here as the radius of the cylindrical plasma. Alternatively, this inequality implies that $c\tau_{\mathrm{D}}$ is much larger than the plasma radius, $c\tau_{\mathrm{D}}\gg R$, which is well justified for the nonrelativistic expansion velocities.

The neglect of the Hall current in Eq. (3) and the assumption of a scalar electrical conductivity for the magnetized plasma is justified when the characteristic time for the Coulomb collisions ν_c^{-1} (where ν_c is the collision frequency) is much less than the cyclotron period of the electron.

Therefore, in this highly collisional regime the conductivity σ is essentially a function of plasma temperature alone [24]. Hereafter it is assumed that the plasma temperature and the conductivity $\sigma(t)$ are uniform and are the functions only of time.

The system of Eqs. (1)-(5) can be used for determination of the dynamics of the expanding plasma as well as for the investigation of the evolution of the induced electromagnetic fields. For further simplification of this system, we note that as long as the Lorentz force density $\frac{1}{c}[\mathbf{j}\times\mathbf{H}]$ and the Joule dissipation j^2/σ terms are negligible compared with the hydrodynamic terms in the left hand side of Eqs. (1) and (2), respectively, the plasma motion is a free radial expansion. For many realistic situations with laser-produced plasmas the free expansion is realized when the kinetic energy density of the expanding plasma is greater than the magnetic field energy density [1] (high-beta plasma), $\rho u^2/2 > H_0^2/8\pi$, where $\mathbf{H}_0$ is the strength of the initial unperturbed magnetic field. This is a necessary condition that the expansion remains radial and cylindrical. It has been shown [19] that the system of Eqs. (1) and (2) allows in this case the self-similar solutions for the quantities ρ , $\mathbf{u}$, and p which are realized under specified initial conditions. These solutions are characterized by a radial velocity distribution linearly dependent on the radial coordinate r . At $r \le R(t)$, $u_r(r,t) = r[\dot{R}(t)/R(t)]$, where $R(t)$ and $\dot{R}(t)$ are the radius and the velocity of the plasma boundary. In addition, the velocity $u_r(r,t)$ vanishes at $r > R(t)$, $u_r(r,t) = 0$. The self-similar solutions for the density ρ and the pressure p as well as the criterion of the violation of the free expansion solutions are discussed in Ref. [19]. However, we would like to emphasize that since the hydrodynamic terms are reduced rapidly as the plasma expands and the Joule heating increases the plasma temperature and the electrical conductivity the electromagnetic terms in the right hand side of Eqs. (1) and (2) will no longer be negligible at the final stage of the plasma expansion when plasma may fully be stopped and deformed by the magnetic field pressure. As mentioned above, the average plasma pressure $\bar{p}$ is strongly reduced compared to the magnetic pressure and the model of the purely radial expansion clearly becomes invalid in this case. Nevertheless, if the critical time interval Δt , where $\bar{p} < H^2/8\pi$, is much smaller than the typical time scale of the plasma flow (up to the full stop), the contribution of this interval to the overall plasma dynamics is negligible and use of the radial expansion model is justified.

In the next sections, we will use the profile of the plasma radial flow velocity $u_r(r,t) = r[\dot{R}(t)/R(t)]$ together with Eq. (5) to investigate the electromagnetic field configuration generated by the expanding cylindrical plasma.

3. Solution of the moving boundary and initial value problem

In this section, we consider the moving boundary problem of the plasma cylinder expansion in the vacuum in the presence of the constant and homogeneous magnetic field $\mathbf{H}_0$. Consider a cylindrical region of space with radius $r = R(t)$ at the time t containing neutral plasma which has expanded at $t = 0$ (with $R(0) = R_0$) to its present state from a cylindrical source with radius R_0 located around $r = 0$. We assume that at any time t the plasma cylinder is unbounded in z direction (i.e. the plasma cylinder is located at $-\infty < z < \infty$). To solve the boundary problem we introduce the cylindrical coordinate system (r , ϕ , z) with the z -axis along the plasma cylinder symmetry axis and the azimuthal angle ϕ is counted from the plane (xz -plane) containing the vector of the unperturbed magnetic field $\mathbf{H}_0$. The angle θ between the vector $\mathbf{H}_0$ and the z -axis is arbitrary.

As the cylindrical plasma expands, it both perturbs the external magnetic field and generates an electric field. We shall obtain an analytic solution of the electromagnetic field configuration. We consider the case of the purely radial expansion of the plasma cylinder with an arbitrary (but

nonrelativistic) expansion velocity $\dot{R}(t)$. Having in mind the symmetry of the unperturbed magnetic field and the fact that the electromagnetic fields do not depend on the coordinate z it is sufficient to choose the vector potential in the form $A_r = 0$,

$$H_{0\|}W(r,t) = \frac{\partial}{\partial r}(rA_\phi), \quad A_z = H_{0\perp}\Psi(r,t)\sin\phi, \tag{6}$$

where $W(r,t)$ and $\Psi(r,t)$ are some unknown functions. From symmetry considerations, the functions $W(r,t)$ and $\Psi(r,t)$ are independent on the cylindrical coordinate ϕ . Here $H_{0\perp}$ and $H_{0\|}$ are the components of the unperturbed magnetic field $\mathbf{H}_0$ transverse and parallel to the z -axis, respectively. The components of the electromagnetic field are expressed by these functions as $H_z = H_{0\|}(W/r)$,

$$H_r = H_{0\perp}\frac{\Psi}{r}\cos\phi, \quad H_\phi = -H_{0\perp}\frac{\partial\Psi}{\partial r}\sin\phi, \tag{7}$$

$$E_\phi = -\frac{1}{c}\frac{\partial A_\phi}{\partial t}, \quad E_z = -\frac{1}{c}H_{0\perp}\frac{\partial\Psi}{\partial t}\sin\phi, \tag{8}$$

and $E_r = 0$. The equation for the vector potential $\mathbf{A}(\mathbf{r},t)$ inside the plasma cylinder ($r \le R(t)$) is obtained from the MHD equation for the magnetic field diffusion, Eq.(5) , which for the unknown functions $\Psi(r,t)$ and $W(r,t)$ yields a system of equations

$$\frac{\partial\Psi}{\partial t} + u_r\frac{\partial\Psi}{\partial r} = D\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} - \frac{\Psi}{r^2}\right), \tag{9}$$

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial r}(u_r W) = D\left(\frac{\partial^2 W}{\partial r^2} - \frac{1}{r}\frac{\partial W}{\partial r} + \frac{W}{r^2}\right). \tag{10}$$

Here $u_r(r,t) = r[\dot{R}(t)/R(t)]$. In the vacuum surrounding the plasma cylinder ($r \ge R(t)$), the magnetic field is determined by the Maxwell equation $\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{j}$ with $\mathbf{j} = 0$ which for the functions $\Psi(r,t)$ and $W(r,t)$ become

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} - \frac{\Psi}{r^2} = 0, \tag{11}$$

$$\frac{\partial^2 W}{\partial r^2} - \frac{1}{r}\frac{\partial W}{\partial r} + \frac{W}{r^2} = 0. \tag{12}$$

In the plasma, the magnetic field is the solution of the diffusion equation with a diffusion coefficient $D(t) = c^2/4\pi\sigma(t)$. The plasma conductivity $\sigma(t)$ is essentially a function of plasma temperature alone. If it is assumed that the plasma temperature is uniform, this coefficient is a function only of time and Eqs. (9) - (12) may be solved by the method of separation of variables.

The system of equations (9)-(12) is to be solved in the internal ($r \le R(t)$) and external ($r \ge R(t)$) regions subject to the boundary and initial conditions. Since the plasma under consideration has finite electrical conductivity, there are no surface currents at the plasma-vacuum boundary and $\mathbf{H}$ must be continuous at the moving boundary. The continuity of the magnetic field $\mathbf{H}$ requires that $\Psi(r,t)$, $W(r,t)$ and the radial derivative $\partial\Psi/\partial r$ to be continuous at the expanding plasma surface. In addition, we require that as $r\to\infty$, the magnetic field $\mathbf{H}(r,\phi,t)$ being time-independent and asymptotically approach the uniform magnetic field $\mathbf{H}_0$. This is

equivalent to the boundary conditions $\Psi(r,t)=W(r,t)=r$ at $r\to\infty$.

The initial conditions are at $t=0$. In principle two distinct sets of initial conditions could be considered [11]. (i) In the case of the poorly conducting plasma the initial conditions are imposed for arbitrary r :

$$\Psi(r,0)=W(r,0)=r. \tag{13}$$

(ii) In the case of the perfectly conducting plasma the initial conditions are imposed separately for the domains inside ($r\le R_0$ with $R_0=R(0)$) and outside ($r\ge R_0$) the plasma cylinder [18]:

$$\Psi(r,0)=W(r,0)=0,\quad r\le R_0, \tag{14}$$

$$\Psi(r,0)=r-\frac{R_0^2}{r},\quad W(r,0)=r,\quad r\ge R_0.$$

In the first case, Eq.(13) , the plasma is initially cold and poorly conducting so that the external magnetic field is completely penetrated inside the plasma. At the other extreme case (ii), the initial conditions (14) imply that initially the plasma is highly conducting so that the magnetic field is completely excluded from the initial plasma volume. We consider below the initial value problem (ii). The extension of the obtained solution to the case (i) is straightforward.

At $r\ge R(t)$ we look for the solutions of Eqs. (11) and (12) for the functions $\Psi(r,t)$ and $W(r,t)$ in the form $\sim r^{\alpha}$, where α is some numerical constant. Therefore, taking into account the boundary condition at $r\to\infty$ the full solution in the domain outside the plasma cylinder is given by

$$\Psi(r,t)=r-C(t)\frac{R_0^2}{r}, \tag{15}$$

$$W(r,t)=r\left[1+C_1(t)\ln\frac{r}{R(t)}\right], \tag{16}$$

where $C(t)$ and $C_1(t)$ are the arbitrary functions of time with the initial conditions $C(0)=1$ and $C_1(0)=0$. However, since the magnetic field should be finite at $r\to\infty$ we set $C_1(t)=0$. At $r\ge R(t)$ this gives the final solution $W(r,t)=r$ for the function $W(r,t)$.

For the class of separable solutions the motion is such that, for a given element of plasma, the quantity $\xi=r/R(t)$ is a constant of the motion. Solutions of Eqs. (9) and (10) inside the plasma cylinder (i.e. at $r\le R(t)$) are then facilitated by the representation of the functions $\Psi(r,t)$ and $W(r,t)$ in the form

$$\Psi(r,t)=r+\sum_{n=1}^{\infty}a_nT_n(t)\Phi_n(\xi), \tag{17}$$

$$W(r,t)=r+\frac{1}{R(t)}\sum_{n=1}^{\infty}b_nU_n(t)\Theta_n(\xi), \tag{18}$$

where $T_n(t)$, $\Phi_n(\xi)$, $U_n(t)$, and $\Theta_n(\xi)$ are some unknown functions, a_n and b_n are the unknown expansion coefficients. Next, inserting these expansions into Eqs. (9) and (10) for $\Phi_n(\xi)$ and $\Theta_n(\xi)$ one arrives at the ordinary differential equations for the cylindrical functions [25]. We choose only the regular solutions of the obtained equations finite at the origin $r=0$ (or at $\xi=0$). Thus,

$$\Phi_n(\xi)=A_nJ_1(\lambda_n\xi),\quad \Theta_n(\xi)=B_n\xi J_0(\kappa_n\xi). \tag{19}$$

Here A_n and B_n are the integration constants, λ_n and κ_n are some arbitrary parameters (depending only on n), arising due to the separation of the variables, and J_0 and J_1 are the Bessel functions of the first kind. In the same way, for the time-dependent functions $T_n(t)$ and $U_n(t)$ we obtain the following set of the ordinary differential equations:

$$\dot{T}_n(t)+\lambda_n^2\dot{\vartheta}(t)T_n(t)=\dot{R}(t), \tag{20}$$

$$\dot{U}_n(t)+\kappa_n^2\dot{\vartheta}(t)U_n(t)=2R(t)\dot{R}(t), \tag{21}$$

where

$$\vartheta(t)=\int_0^t\frac{D(\tau)}{R^2(\tau)}d\tau. \tag{22}$$

The general solutions of the first order differential equations (20) and (21) can be represented in the form

$$T_n(t)=e^{-\lambda_n^2\vartheta(t)}\left[t_0+\int_0^t e^{\lambda_n^2\vartheta(\tau)}\dot{R}(\tau)d\tau\right], \tag{23}$$

$$U_n(t)=e^{-\kappa_n^2\vartheta(t)}\left[u_0+2\int_0^t e^{\kappa_n^2\vartheta(\tau)}R(\tau)\dot{R}(\tau)d\tau\right]. \tag{24}$$

Here $t_0=T_n(0)$ and $u_0=U_n(0)$ are the initial values of $T_n(t)$ and $U_n(t)$, respectively, to be determined by imposing the initial conditions. In addition, it should be emphasized that the expansions given by Eqs. (9), (17) and (18) are the solutions of the system of Eqs. (9) and (10) only if the functions $\Phi_n(\xi)$ and $\Theta_n(\xi)$ satisfy at arbitrary $0\le\xi\le1$ the equations

$$\sum_{n=1}^{\infty}a_n\Phi_n(\xi)=\sum_{n=1}^{\infty}b_n\Theta_n(\xi)=-\xi. \tag{25}$$

These relations impose some additional constrains on the expansion coefficients a_n and b_n.

Now let us consider the boundary condition at $r=R(t)$ for the functions $W(r,t)$, $\Psi(r,t)$, and $\frac{\partial}{\partial r}\Psi(r,t)$. For the function $W(r,t)$ this boundary condition yields $J_0(\kappa_n)=0$, i.e. the quantities κ_n (with $n=1,2,...$) must be the positive zeros of the Bessel function $J_0(z)$. Later on, we will assume that the zeros κ_n are arranged in ascending order of magnitude. The same boundary condition for the quantity $\Psi(r,t)$ determines the unknown function $C(t)$ in Eq.(15),

$$C(t)=-\frac{R(t)}{R_0^2}\sum_{n=1}^{\infty}a_nA_nT_n(t)J_1(\lambda_n). \tag{26}$$

Finally, the boundary condition at $r=R(t)$ for $\frac{\partial}{\partial r}\Psi(r,t)$ yields another relation for the function $C(t)$ which should be consistent with Eq.(26). This is only possible if $J_0(\lambda_n)=0$, i.e. $\lambda_n=\kappa_n$ are also the zeros of the Bessel function.

Let us now turn to the determination of the expansion coefficients a_n and b_n using the constrains in Eq.(25). Inserting Eq.(19) into Eq.(25) and using the summation formulas (56) and (54) one can easily prove that the equations in Eq. (25) are satisfied if $a_nA_n=-4/[\lambda_n^2J_1(\lambda_n)]$ and $b_nB_n=-2/[\lambda_nJ_1(\lambda_n)]$. Then, taking into account the summation formulas (57) and (54) the initial conditions $C(0)=1$ and $W(r,0)=0$ (at $r\le R_0$) for the functions $C(t)$ and $W(r,t)$ imply that $T_n(0)=R_0$ and $U_n(0)=R_0^2$, respectively. Moreover, having in mind the relations (56) and (57) the initial conditions for the quantity $\Psi(r,0)$ inside ($r\le R_0$) and outside ($r\ge R_0$) the plasma cylinder (see Eq.(14)) are then satisfied automatically.

Therefore, the complete solution of Eqs. (9) -(12) for the initial and boundary conditions inside ($r\le R(t)$) and outside ($r\ge R(t)$) the plasma cylinder, respectively, may be represented as

$$\Psi(r,t)=r-4\sum_{n=1}^{\infty}T_n(t)\frac{J_1(\lambda_n\xi)}{\lambda_n^2J_1(\lambda_n)}, \tag{27}$$

$$W(r,t)=r\left[1-\frac{2}{R^2(t)}\sum_{n=1}^{\infty}U_n(t)\frac{J_0(\lambda_n\xi)}{\lambda_nJ_1(\lambda_n)}\right], \tag{28}$$

and $W(r,t)=r$,

$$\Psi(r,t) = r - \frac{4R(t)}{r}\mathbf{T}(t) \tag{29}$$

with

$$\mathbf{T}(t) = \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}T_n(t), \quad \mathbf{U}(t) = \sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}U_n(t). \tag{30}$$

Note that $\Psi(r,t) = W(r,t) = r$ at $r \to \infty$ as expected by the boundary conditions at the infinity. The e -component of the vector potential is determined by the first relation in Eq.(6) . The straightforward integrations in Eq.(28) with respect to the radial coordinate r result in [26]

$$A_\phi(r,t) = \frac{H_{0\|}}{2}\left[r - \frac{4}{R(t)}\sum_{n=1}^{\infty}U_n(t)\frac{J_1(\lambda_n\xi)}{\lambda_n^2 J_1(\lambda_n)}\right], \tag{31}$$

$$A_\phi(r,t) = \frac{H_{0\|}}{2}\left[r - \frac{G(t)}{r}\right] \tag{32}$$

inside ($r \le R(t)$) and outside ($r \ge R(t)$) the plasma cylinder, respectively. Here $G(t)$ is an arbitrary function of time to be determined by the boundary condition at $r = R(t)$. From this condition, one obtains $G(t) = 4\mathbf{U}(t)$. Equations (27) - (32) represent the complete solution of the problem and determine the structure of the electromagnetic fields both inside and outside the expanding plasma cylinder. Expressions for the components of the electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ may now be obtained by use of Eqs. (7) and (8). These components are, for $r \le R(t)$,

$$H_r = H_{0\perp}\cos\phi\left[1 - \frac{4}{r}\sum_{n=1}^{\infty}T_n(t)\frac{J_1(\lambda_n\xi)}{\lambda_n^2 J_1(\lambda_n)}\right], \tag{33}$$

$$H_\phi = -H_{0\perp}\sin\phi\left[1 - \frac{4}{R}\sum_{n=1}^{\infty}T_n(t)\frac{J_1'(\lambda_n\xi)}{\lambda_n J_1(\lambda_n)}\right], \tag{34}$$

$$H_z = H_{0\|}\left[1 - \frac{2}{R^2}\sum_{n=1}^{\infty}U_n(t)\frac{J_0(\lambda_n\xi)}{\lambda_n J_1(\lambda_n)}\right], \tag{35}$$

$$E_\phi = \frac{2H_{0\|}}{cR}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2 J_1(\lambda_n)} \times\left[\dot{U}_n(t)J_1(\lambda_n\xi) - \frac{\dot{R}}{R}U_n(t)\lambda_n\xi J_0(\lambda_n\xi)\right], \tag{36}$$

$$E_z = \frac{4H_{0\perp}}{c}\sin\phi\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2 J_1(\lambda_n)} \times\left[\dot{T}_n(t)J_1(\lambda_n\xi) - \frac{\dot{R}}{R}T_n(t)\lambda_n\xi J_1'(\lambda_n\xi)\right], \tag{37}$$

and, for $r \ge R(t)$, $H_z = H_{0\|}$,

$$H_r = H_{0\perp}\cos\phi\left[1 - \frac{4R}{r^2}\mathbf{T}(t)\right], \tag{38}$$

$$H_\phi = -H_{0\perp}\sin\phi\left[1 + \frac{4R}{r^2}\mathbf{T}(t)\right], \tag{39}$$

$$E_\phi = \frac{H_{0\|}}{cr}\left[R\dot{R} - \frac{2D}{R^2}\sum_{n=1}^{\infty}U_n(t)\right], \tag{40}$$

$$E_z = \frac{H_{0\perp}}{cr}\sin\phi\left\{\dot{R}[R+4\mathbf{T}(t)]-\frac{4D}{R}\sum_{n=1}^{\infty}T_n(t)\right\}. \tag{41}$$

In the latter expressions for the components of the electric field the time-derivatives $\dot{U}_n(t)$ and $\dot{T}_n(t)$ have been excluded by means of Eqs. (20) and (21) . Also in Eqs. (33) -(41) the prime indicates the derivative of the Bessel function with respect to the argument.

The induced current density $\mathbf{j}$ is defined only inside the plasma cylinder (i.e., for $r \le R(t)$) by the relation (3) (or using the Maxwell equation $\mathbf{j}=\frac{c}{4\pi}\nabla\times\mathbf{H}$) and has the following components: $j_r = 0$, and

$$j_\phi = -\frac{cH_{0\|}}{2\pi R^3(t)}\sum_{n=1}^{\infty}U_n(t)\frac{J_1(\lambda_n\xi)}{J_1(\lambda_n)}, \tag{42}$$

$$j_z = -\frac{cH_{0\perp}}{\pi R^2(t)}\sin\phi\sum_{n=1}^{\infty}T_n(t)\frac{J_1(\lambda_n\xi)}{J_1(\lambda_n)}. \tag{43}$$

Until now, we have considered the initial value problem (ii) assuming that initially the plasma conductivity is so high that the magnetic field is completely excluded from the initial volume of plasma. The extension of the obtained solution to the case of the initial value problem (i) (with highly resistive plasma at $t = 0$) is straightforward. From the consideration above it follows that the solution of the boundary and initial value problem (i) is again determined by Eqs. (27)-(43), where, however, the initial conditions $T_n(0) = R_0$ and $U_n(0) = R_0^2$ for the functions $T_n(t)$ and $U_n(t)$ (see Eqs.(23) and (24)) should be replaced by the zero initial conditions, $T_n(0) = U_n(0) = 0$. Comparison of the complete solutions obtained by the initial value problems (i) and (ii) shows that the electromagnetic fields and the induced current for the two distinct cases differ only in the terms containing $T_n(0)$ and $U_n(0)$ in Eqs. (23) and (24) . These terms force the matching of the solution to the initial condition of the magnetic field completely excluded initially from the plasma volume $r \le R_0$. Because of their exponential dependence on $\vartheta(t)$ (and hence on the time t) these terms become negligible compared to the other terms in the electromagnetic fields in the time required for $\vartheta(t)$ to become of the order $\vartheta(t) \sim 1$. This time interval is the characteristic time for diffusion of the magnetic field into a stationary plasma of radius R and conductivity σ , i.e. $\tau_{\mathbf{D}} \simeq R^2 / D = 4\pi\sigma R^2 / c^2$. Thus, the initial conditions for the initially perfectly conducting plasma are forgotten by the plasma at $t \ge \tau_{\mathbf{D}}$.

In the context of the two distinct initial value problems (i) and (ii) it should be also emphasized that the initial value of the plasma conductivity $\sigma(0)$ should be consistent with the chosen physical model. Indeed, the cases (i) and (ii) imply vanishing ($\sigma(0) \to 0$) and very large ($\sigma(0) \to \infty$) initial conductivities of the expanding plasma, respectively. As a demonstration of the importance of the initial value $\sigma(0)$ consider, for instance, Eqs.(36), (37), (40) and (41) for the generated electric field. Using Eqs. (20) and (21) as well as the summation formulas of Appendix sec: app1 it is straightforward to show that at $t = 0$ the electric field inside (Eqs. (36) and (37)) the plasma cylinder is given by

$$E_\phi(0) = r\frac{H_{0\|}}{c}\frac{\dot{R}_0}{R_0}\left(1-\frac{u_0}{R_0^2}\right), \tag{44}$$

$$E_z(0) = r\frac{H_{0\perp}}{c}\frac{\dot{R}_0}{R_0}\sin\phi\left(1-\frac{t_0}{R_0}\right), \tag{44}$$

where $\dot{R}_0 = \dot{R}(0)$. Thus, the initial electric field in the plasma volume vanishes or is finite in the cases (ii) (with $u_0 = R_0^2$, $t_0 = R_0$) and (i) (with $u_0 = t_0 = 0$), respectively. The initial electric field

in a vacuum is determined by Eqs. (40) and (41) at $t = 0$. It is seen that at $t \to 0$ the last terms in these expressions proportional to the diffusion coefficient $D(t)$ vanish for the initial condition (i) while diverging as $\sim[\vartheta(t)]^{-1/2}$ in the case (ii). In the latter case assuming, however, perfectly conducting (in fact infinitely conducting) plasma at $t = 0$ one must consider the limit $D(0) \to 0$ in the last terms of Eqs. (40) and (41) which vanish eventually at $t \to 0$. Finally, we note that at $t \to 0$ the nonvanishing terms in Eqs.(40) , (41), (44) and (45) are proportional to the initial expansion velocity $\dot{R}_0$ of the plasma. Therefore, it is not surprising that at $\dot{R}_0 \neq 0$ the plasma expansion builds up instantly an initial electric field, although the induced magnetic field is zero.

At the end of this section, consider briefly the nondiffusive limit of the obtained solutions, Eqs.(33) - (43), when the diffusion coefficient vanishes, $D \to 0$. This limit can be obtained using at $D \to 0$ the expressions (22) - (24) which yield $T_n(t) = R(t)$ and $U_n(t) = R^2(t)$. Therefore, having in mind the summation formula (57) from Eq. (30) one finds $\mathbf{T}(t) = R(t)/4$ and $\mathbf{U}(t) = R^2(t)/4$. Using these results and the summation formulas derived in Appendix sec: app1 it is straightforward to show that $j_\phi = j_z = 0$ and the electromagnetic fields are, for $r \le R(t)$, $\mathbf{H}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) = 0$, and, for $r \ge R(t)$, $H_z = H_{0\|}$, $H_r = H_{0\perp}\cos\phi(1 - R^2/r^2)$, $H_\phi = -H_{0\perp}\sin\phi(1 + R^2/r^2)$, $E_\phi = \beta H_{0\|}(R/r)$, $E_z = 2\beta H_{0\perp}(R/r)\sin\phi$, where $\beta = \dot{R}/c$. These expressions have been derived previously in Ref. [18] for the cylindrical plasma expansion neglecting the diffusion of the magnetic field.

4. Energy balance

Previously significant attention has been paid [4,5,14-18] to the question of what fraction of energy is emitted and lost in the form of electromagnetic pulse propagating outward of the expanding plasma. In this section, we consider the energy balance during the plasma cylinder expansion in the presence of the homogeneous magnetic field.

Our starting point is the energy balance equation

$$\nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E} - \frac{\partial}{\partial t}\frac{H^2}{8\pi}, \tag{45}$$

where $\mathbf{S} = \frac{c}{4\pi}[\mathbf{E} \times \mathbf{H}]$ is the Poynting vector and $\mathbf{j}$ is the induced current. Note that the density of the electric field energy has been neglected in Eq (45) since $\dot{R} \ll c$ and $E \ll H$. The energy emitted to infinity is measured as a Poynting vector integrated over time and over the lateral surface S_c of the cylinder with radius r_c , length l_c and the volume Ω_c (control cylinder) enclosing the plasma cylinder ($r_c > R(t)$). Integrating over time and over the volume Ω_c , Eq. (45) can be represented as

$$W_{\mathbf{S}}(t) = W_{\mathbf{J}}(t) + \Delta W_{\mathbf{M}}(t), \tag{46}$$

where

$$W_{\mathbf{S}}(t) = r_c \int_0^t dt' \int_0^{2\pi} S_r \, d\phi, \tag{47}$$

$$W_{\mathbf{J}}(t) = -\frac{1}{l_c}\int_0^t dt' \int_{\Omega_c} \mathbf{j} \cdot \mathbf{E} \, d\mathbf{r}.$$

Here S_r is the radial component of the Poynting vector. Note that the total flux of the energy over the bases of the control cylinder determined by S_z vanishes due to the symmetry reason. $W_{\mathbf{M}}(t)$ and $\Delta W_{\mathbf{M}}(t) = W_{\mathbf{M}}(0) - W_{\mathbf{M}}(t)$ are the total magnetic energy and its change (with minus sign) per unit length in a volume Ω_c , respectively. $W_{\mathbf{J}}(t)$ is the energy (per unit length) transferred from plasma cylinder to the magnetic field and is the mechanical work with minus sign performed by the plasma on the external magnetic pressure. At $t = 0$ the magnetic fields are given

by $\mathbf{H}=\mathbf{H}_0$ in the model (i) and by Eq.(14) in the model (ii). Hence in (i) $W_{\mathbf{M}}(0)$ is the total magnetic energy per unit length in a volume Ω_c and is given by $W_{\mathbf{M}}(0)\equiv Q_c=\pi r_c^2(H_0^2/8\pi)$ while in (ii) $W_{\mathbf{M}}$ is the total magnetic energy in a volume Ω_c' of the control cylinder excluding the volume of the plasma cylinder (we take into account that $\mathbf{H}=0$ at $t=0$ in a plasma cylinder in the model (ii)) and $W_{\mathbf{M}}(0)=Q_c(1-\kappa^{-2}\cos^2\theta-\kappa^{-4}\sin^2\theta)$. Here $\kappa=r_c/R_0$ and θ is the angle between $\mathbf{H}_0$ and the symmetry axis of the plasma cylinder (z -axis). Then the change of the magnetic energy $\Delta W_{\mathbf{M}}(t)$ in a volume Ω_c can be evaluated as

$$\Delta W_{\mathbf{M}}(t)=W_{\mathbf{M}}(0)-\frac{1}{l_c}\int_{\Omega_c}\frac{H^2}{8\pi}d\mathbf{r}. \tag{48}$$

Hence the total energy flux $W_{\mathbf{S}}(t)$ given by Eq.(47) is calculated as a sum of the energy loss by plasma due to the external magnetic pressure and the decrease of the magnetic energy in a control volume Ω_c . For expansion of a one-dimensional plasma slab and for uniform external magnetic field $W_{\mathbf{S}}\simeq 2W_{\mathbf{J}}\simeq 2\Delta W_{\mathbf{M}}$, i.e. approximately the half of the outgoing energy is gained from the plasma, while the other half is gained from the magnetic energy [15]. In the case of expansion of highly conducting spherical plasma with radius R in the uniform magnetic field $\mathbf{H}_0$ the outgoing energy W_{S} is distributed between W_{J} and $\Delta W_{\mathbf{M}}$ according to $W_{\mathbf{J}}=1.5Q_0$ and $\Delta W_{\mathbf{M}}=0.5Q_0$ with $W_{\mathbf{S}}=2Q_0$, where $Q_0=H_0^2R^3/6$ is the magnetic energy escaped from the spherical plasma volume [14]. Therefore, in this case the released magnetic energy is mainly gained from the plasma.

Consider now each energy component $W_{\mathbf{S}}(t)$, $W_{\mathbf{J}}(t)$ and $\Delta W_{\mathbf{M}}(t)$ separately. $W_{\mathbf{S}}(t)$ is calculated from Eq.(47) using the expressions for the electromagnetic fields generated outside the plasma ($r\geq R(t)$). Recalling that $H_z=H_{0\|}$ at $r\geq R(t)$ from Eqs. (39)-(41) and (47) we obtain

$$W_{\mathbf{S}}(t)=H_{0\|}^2\left[\mathbf{U}(t)-\frac{u_0}{4}\right]+H_{0\perp}^2\left[R(t)\mathbf{T}(t)\right. \tag{49}$$
$$\left.+\frac{2R^2(t)}{R_0^2\kappa^2}\mathbf{T}^2(t)-R_0\frac{t_0}{4}-\frac{t_0^2}{8\kappa^2}\right].$$

Here $\mathbf{T}(t)$ and $\mathbf{U}(t)$ are given by Eq.(30) .

Next, we evaluate the energy loss $W_{\mathbf{J}}(t)$ by the plasma which is determined by the induced current density $\mathbf{j}$. This current has two azimuthal and axial components and has been determined in Sec.3, see Eqs. (42) and (43). Since the current is localized only within a plasma volume $\Omega_{\mathbf{R}}$, in Eq.(47) the volume Ω_c can be replaced by the plasma volume $\Omega_{\mathbf{R}}$. The total energy loss by the plasma cylinder is calculated as

$$W_{\mathbf{J}}(t)=\frac{H_{0\|}^2}{2}\left[\frac{\mathfrak{A}(t)}{R^2(t)}-\frac{u_0^2}{4R_0^2}\right]+H_{0\perp}^2\left[\mathfrak{I}(t)-\frac{t_0^2}{4}\right], \tag{50}$$

where

$$\mathfrak{I}(t)=\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}T_n^2(t),\quad \mathfrak{A}(t)=\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}U_n^2(t). \tag{51}$$

The change of the magnetic energy in a control cylinder is calculated from Eq.(48) . For evaluation of the magnetic energy inside and outside the plasma volume, we use Eqs.(33)-(35), (38) and (39) , together with $H_z=H_{0\|}$, respectively. Thus, the change of the total magnetic energy in the control cylinder is represented as

$$\Delta W_{\mathbf{M}}(t)=\frac{H_{0\|}^2}{2}\left[2\mathbf{U}(t)-\frac{\mathfrak{A}(t)}{R^2(t)}-\frac{u_0}{4}\left(2-\frac{u_0}{R_0^2}\right)\right] \tag{52}$$

$$+H_{0\perp}^2\left\{R(t)\mathbf{T}(t)-\Im(t)+2\mathbf{T}^2(t)\frac{R^2(t)}{R_0^2\kappa^2}\right.$$

$$\left.-\frac{t_0}{8}\left[\frac{t_0}{\kappa^2}+2\left(R_0-t_0\right)\right]\right\}.$$

Comparing now Eqs. (49),(50) and (52) we conclude that $\Delta W_{\mathbf{M}}(t)+W_{\mathbf{J}}(t)=W_{\mathbf{S}}(t)$ as predicted by the energy balance equation (46). Let us recall that Eqs. (49), (50) and (52) are valid for both initial conditions (i) and (ii) choosing appropriate values for the quantities t_0 and u_0 (i.e., $t_0=u_0=0$ in (i) and $t_0=R_0$, $u_0=R_0^2$ in (ii)).

The energy components $W_{\mathbf{S}}(t)$ and $\Delta W_{\mathbf{M}}(t)$ depend naturally on the radius r_c of the control cylinder while $W_{\mathbf{J}}(t)$ is determined only by the volume of the plasma cylinder. At $r_c\to\infty$, Eqs. (49) and (52) are finite and in this case $W_{\mathbf{S}}(t)$ represents the electromagnetic energy emitted to infinity. It should be noted that at $r\to\infty$ the induced magnetic and electric fields behave as $H\sim r^{-2}$ and $E\sim r^{-1}$, respectively, see Eqs. (38)-(41). Consequently, at $r_c\to\infty$ the induced magnetic field does not contribute to the energy flux $W_{\mathbf{S}}(t)$ and the latter is determined by the electric field and the unperturbed magnetic field $\mathbf{H}_0$.

5. Numerical examples

In this section using theoretical findings of the preceding sections, we present the results of our model calculations for the electromagnetic fields generated due to the radial expansion of the magnetized cylindrical plasma into a vacuum. As an example we consider the radial expansion with $R(t)=R_0\left\{1+a[1-\cos(\omega t)]\right\}$, where v is the expansion velocity at $t\to\infty$, $a=D_0/vR_0$. Note that initially $\dot{R}_0=0$ in this case. The electrical conductivity is modeled by $\sigma(t)=\sigma_0(\tau/t)$, where τ is some characteristic decay time of $\sigma(t)$ and σ_0 is a constant. Accordingly, the diffusion coefficient is given by $D(t)=D_0(t/\tau)$ with $D_0=c^2/4\pi\sigma_0$. Thus, initially the conductivity of the plasma is very high which corresponds to the initial condition (ii). However, as plasma expands the plasma will be eventually cooled off and the conductivity decreases with time. For the chosen model

$$\vartheta(t)=\frac{2D}{\omega R_0^2}\int_0^{\omega t/2}\frac{d\tau}{(1+2a\sin^2\tau)^2}$$

$$=\frac{D}{\omega R_0^2}\frac{1}{1+2a}\left[\frac{a\sin(\omega t)}{1+2a\sin^2\dfrac{\omega t}{2}}+\frac{2(1+a)}{\sqrt{1+2a}}\arctan\left(\sqrt{1+2a}\tan\frac{\omega t}{2}\right)\right]$$

as it follows from Eq.(22) . In order to obtain a physical idea of the length and time scales involved, let us consider briefly a numerical example. Taking, for instance, $R_0=1$ mm and $v=10^6$ cm/s, one obtains $\tau_{\mathbf{R}}=R_0/v=0.1$ μs. For a laser generated plasma with a temperature $T\sim10^5$ K, one can take $\sigma_0\simeq5\times10^{14}$ s^{-1} and $D_0\simeq2\times10^5$ cm^2/s. Using these parameters the dimensionless diffusion coefficient $\delta=1$ implies the characteristic decay time $\tau\simeq0.18$ μs for the conductivity. Using it all in equations (23) and (24) we have some numerical results (see Fig. 1).

We consider the temporal and spatial distributions of the magnetic field for various values of θ ,

a and δ . Figure 1 demonstrates the absolute values of the magnetic field $H(r,t)$ (in units of H_0) on the symmetry axis of the plasma cylinder, $r=0$, as a function of time. As it follows from Eqs. (33)-(37), the electric field vanishes on this axis, $\mathbf{E}=0$, while $H_r = H_{0\perp}\mathbf{F}_1(t)\cos\phi$, $H_\phi = -H_{0\perp}\mathbf{F}_1(t)\sin\phi$, $H_z = H_{0\|}\mathbf{F}_2(t)$, where

$$\mathbf{F}_1(t) = 1 - \frac{2}{R(t)}\sum_{n=1}^{\infty}\frac{T_n(t)}{\lambda_n J_1(\lambda_n)}, \tag{53}$$

$$\mathbf{F}_2(t) = 1 - \frac{2}{R^2(t)}\sum_{n=1}^{\infty}\frac{U_n(t)}{\lambda_n J_1(\lambda_n)}. \tag{54}$$

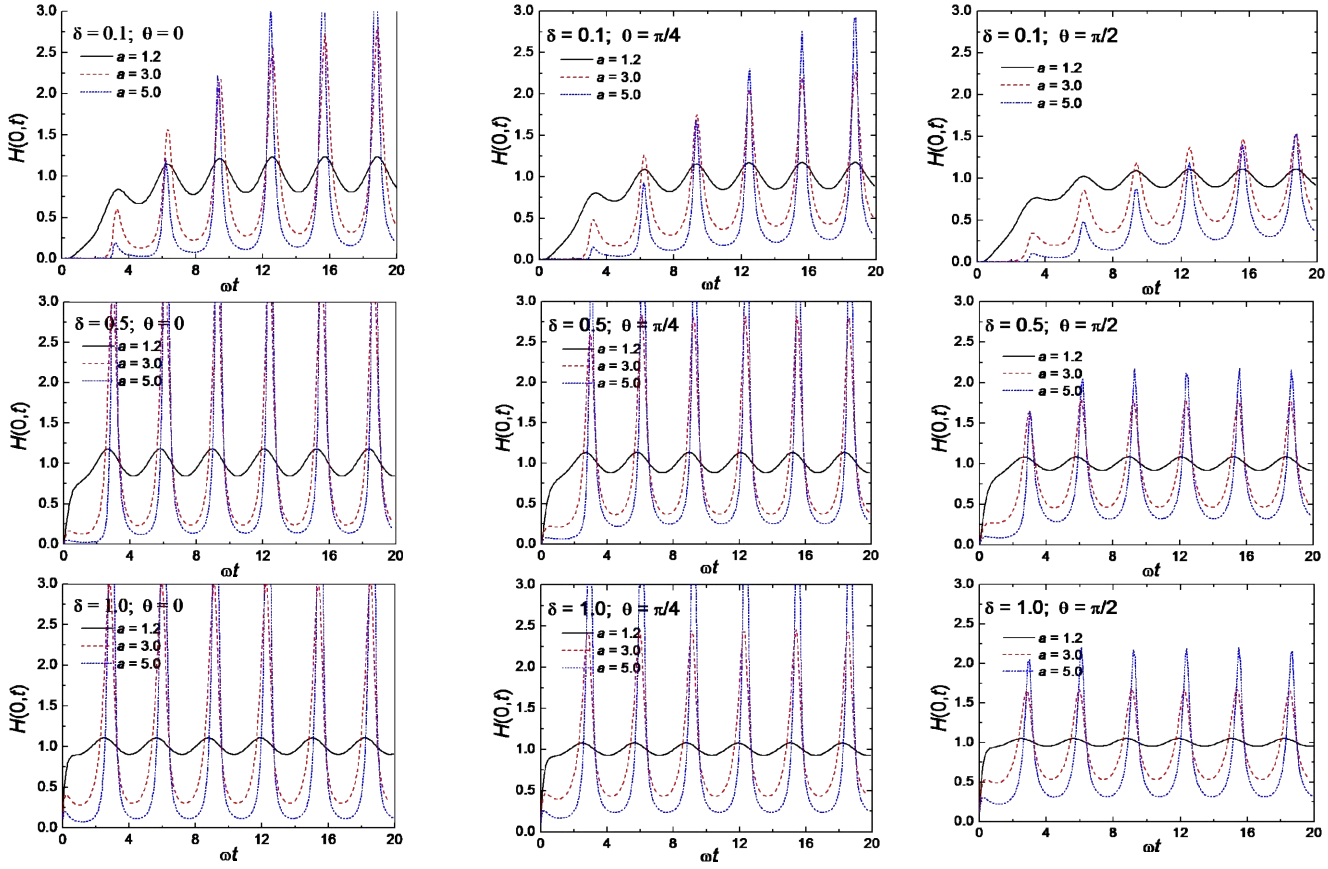

Fig. 1. The strength of the magnetic field (in units of H_0) at $r=0$ as a function of time t (in units of τ_R) for $\theta=0$, $\theta=\pi/4$, $\theta=\pi/2$, $\delta=0.1$, $\delta=0.5$, $\delta=1.0$, $a=1.2$, $a=3.0$ and $a=5.0$.

Thus, as expected, the magnetic field $H(r,t)$ at $r=0$ is independent on the angle ϕ . From Fig. 1 it is seen that simultaneously with the plasma expansion the magnetic field inside grows harmonically from zero value and saturates typically at $t_{\mathbf{sat}} \simeq R_0^2/D_0 \sim \delta^{-1}$. Clearly, the saturation time of the magnetic field decreases with δ . At $t > t_{\mathbf{sat}}$ the magnetic field is constant harmonic because it is completely diffused into the expanding plasma remaining, however, smaller than H_0 . It is also noteworthy the influence of the orientation of $\mathbf{H}_0$ on the magnetic field inside the plasma cylinder at $r=0$. At weak diffusion, $\delta \ll 1$, the quantities $\mathbf{F}_1(t)$ and $\mathbf{F}_2(t)$ are evaluated using Eq. (58) with $x=0$ and $z=\delta^{-1/2}$. In this case we obtain $\mathbf{F}_1(t) \simeq (4\pi^2/\delta)^{1/4}\exp(-1/\sqrt{\delta})$ and

$\mathbf{F}_2(t) \simeq (8\pi^2/\delta)^{1/4} \exp(-\sqrt{2/\delta})$. It is seen that $\mathbf{F}_1(t) \gg \mathbf{F}_2(t)$ and the magnetic field is essentially larger for the transverse orientation of $\mathbf{H}_0$ (Fig. 1 (Row 1)). However, in the strongly diffusive regime, $\delta \gg 1$, from Eqs. (54) and (55) one derives $\mathbf{F}_1(t) \simeq 1-1/4\delta$ and $\mathbf{F}_2(t) \simeq 1-1/2\delta$, and the magnetic field is only weakly sensitive to θ approaching the value H_0 of the unperturbed magnetic field (Fig. 1(Row 3)).

6. Conclusion

In this paper, an exact solution of the purely radial expansion of a neutral, resistive plasma cylinder in the external static uniform magnetic field has been obtained. The electromagnetic fields are derived by using the appropriate initial and boundary conditions. Two type of the initial conditions have been considered assuming poorly or perfectly conducting plasma at the initial state. In the first case, the magnetic field is completely penetrated inside the plasma while in the other extreme case the magnetic field is completely excluded from the initial plasma volume. However, as expected both solutions "forget" the imposed initial conditions at $t \geq \tau_{\mathbf{D}}$, where $\tau_{\mathbf{D}}$ is the characteristic diffusion time. Using a simple model for the electrical conductivity, we have also studied the energy balance during the plasma expansion as well as the spatio-temporal distribution of the external magnetic field with an arbitrary orientation of the initial field. In contrast to the previous treatments with highly conducting plasmas [13-18], our model calculations demonstrate some new features arising due to the finite conductivity of the expanding plasma.

Going beyond the present model, which is based on several approximations, we can envisage a number of improvements. We have assumed the purely radial expansion (with a given velocity $\dot{R}(t)$ of the plasma boundary) and hence the shape of the plasma (in the xy -plane) remains isotropic during plasma expansion. For realistic laser-generated plasmas this is valid when the kinetic energy density of the expanding plasma much exceeds the magnetic field energy density [1,2]. However, after some period of accelerated motion, the plasma gets decelerated as a result of the external Lorentz force acting inward and the above mentioned condition may be violated at the later stages ($t \gg \tau_{\mathbf{D}}$) of the expansion. In this case the Lorentz force density which is anisotropic in general, cannot be neglected in Eq. (1) . Thus one can expect some deformation of the initially isotropic plasma surface [4,5] and the plasma radius $R(t,\phi)$ should be treated as a function of ϕ . On the other hand, in this regime the plasma radius $R(t,\phi)$ cannot be treated as a given function of time and should be determined self-consistently using, for instance, the equation of balance of plasma ($\sim \rho\dot{R}^2$) and magnetic field ($\sim H^2$) energies [19]. Thus, plasma dynamics can be described (at least qualitatively) inserting Eqs.(33)-(35), (38) and (39) into the energy balance equation which yields a first-order differential equation for $R(t,\phi)$. Within the scope of this analysis, the deformation of the plasma surface as well as the initial stage of plasma acceleration, the later stage of deceleration and the process of stopping at the point of maximum expansion could be examined numerically. We intend to address these issues in our forthcoming investigations.

7. Some summation formulas involving the Bessel functions

Using the Fourier-Bessel expansion of an arbitrary function of a real variable [25] one can derive some summation formulas involving the Bessel functions, which are used in the main text of the paper. The first relation is obtained by considering the Fourier-Bessel series for the quadratic function x^2 in the interval $0 \leq x < 1$,

$$x^2 = \sum_{n=1}^{\infty} \left(1 - \frac{4}{\lambda_n^2}\right) \frac{2J_0(\lambda_n x)}{\lambda_n J_1(\lambda_n)}, \tag{53}$$

where λ_n (with $n = 1, 2, \ldots$) are the positive zeros of the Bessel function, $J_0(\lambda_n) = 0$, arranged in ascending order of magnitude [25]. On the other hand using the known summation formula [25] (valid at $0 \leq x < 1$)

$$\sum_{n=1}^{\infty}\frac{J_0(\lambda_n x)}{\lambda_n J_1(\lambda_n)}=\frac{1}{2} \tag{54}$$

one can represent Eq.(53) in another form:

$$\sum_{n=1}^{\infty}\frac{J_0(\lambda_n x)}{\lambda_n^3 J_1(\lambda_n)}=\frac{1-x^2}{8}. \tag{55}$$

This latter relation is valid at $0\le x\le 1$.

Next, taking the x -derivatives of Eqs. (53) and (54) one obtains

$$\sum_{n=1}^{\infty}\left(1-\frac{4}{\lambda_n^2}\right)\frac{J_1(\lambda_n x)}{J_1(\lambda_n)}=-x,\ \sum_{n=1}^{\infty}\frac{J_1(\lambda_n x)}{J_1(\lambda_n)}=0,$$

respectively. The latter formula is valid at $0\le x<1$. Therefore, using the second relation in Eq.(55) the first summation formula in (55) can be represented in the form:

$$\sum_{n=1}^{\infty}\frac{J_1(\lambda_n x)}{\lambda_n^2 J_1(\lambda_n)}=\frac{x}{4}. \tag{56}$$

Finally, substituting $x=1$ in Eq. (56) we arrive at

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}=\frac{1}{4}. \tag{57}$$

Next, we derive another kind of summation formulas involving the zeros of the Bessel function $J_0(z)$. For that purpose consider the known summation formula [25]

$$\frac{I_0(xz)}{I_0(z)}=2\sum_{n=1}^{\infty}\frac{\lambda_n J_0(\lambda_n x)}{\left(\lambda_n^2+z^2\right)J_1(\lambda_n)}, \tag{58}$$

where $I_n(z)$ (with $n=1,2,...$) is the modified Bessel function of the first kind, z and x (with $0\le x<1$) are real variables. Differentiating Eq.(58) with respect to x and using the second relation in Eq. (59) we arrive at

$$\frac{I_1(xz)}{2zI_0(z)}=\sum_{n=1}^{\infty}\frac{J_1(\lambda_n x)}{\left(\lambda_n^2+z^2\right)J_1(\lambda_n)}, \tag{59}$$

which, in particular, at $x=1$ yields

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2+z^2}=\frac{I_1(z)}{2zI_0(z)}. \tag{60}$$

This relation can be used for deriving other summation formulas. For instance, after some manipulations one obtains

$$\Xi_1(z)=\sum_{n=1}^{\infty}\frac{4}{\lambda_n^2(\lambda_n^2 z+1)}=1-\frac{2I_1(\zeta)}{\zeta I_0(\zeta)}, \tag{61}$$

$$\Xi_2(z)=\sum_{n=1}^{\infty}\frac{4}{\lambda_n^2(\lambda_n^2 z+1)^2} \tag{62}$$

$$=2-\frac{4I_1(\zeta)}{\zeta I_0(\zeta)}-\left[\frac{I_1(\zeta)}{I_0(\zeta)}\right]^2.$$

Here $\zeta=1/\sqrt{z}$. We also mention the asymptotic behavior of the functions $\Xi_1(z)$ and $\Xi_2(z)$. At small argument, $z\ll 1$, these functions behave as $\Xi_1(z)\simeq\Xi_2(z)\simeq 1$ while at $z\gg 1$ they decay as $\Xi_1(z)\simeq 1/8z$, $\Xi_2(z)\simeq 1/48z^2$.

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