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Deformation of the Cross Section of the Surface of Erythrocyte in the Plane of Mirror Symmetry in the Field of Ionizing Radiation

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Abstract. In the frame of Helfrich spontaneous curvature model, a general equation of the curvature of the crosssection of the surface of erythrocyte in the plane of mirror symmetry in the field of isotropic radiationis derived in detail. It is shown that despite of the case of erythrocyte surface equation which contains three independent phenomenological parameters (osmotic pressure, tensile stress and spontaneous curvature), the shape of the contour of the cross section is defined via two of them: osmotic pressure and the linear combination of square of spontaneous curvature and tensile stress. In the case of small intensities of radiation is found that the intensity of radiation and the isoperimetric parameter of the contour are connected via a transcendental equation.

Keywords: erythrocyte, spontaneous curvature model, radiation

Introduction

As it is widely accepted now Helfrich spontaneous curvature model [1] describes in a good accuracy possible forms of the erythrocyte surface in equilibrium or out of equilibrium [2].The model contains three independent phenomenological parameters: osmotic pressure ΔP, tensile stress λ and spontaneous curvature c_0 and to any definite form of the surface of erythrocyte corresponds a definite domain in three dimensional "phase space" of these parameters. In equilibrium erythrocyte has rotary reflection symmetry and therefore its cross section with the symmetry plane is a circle. Under the influence of external conditions the membrane of the erythrocyte changes its physical and chemical properties which result to the change of the Helfrich's phenomenological parameters and eventually cause to the deformed surface. Obviously, the deformed surface contains information about deviations of Helfrich parameters from their magnitudes in equilibrium and this paper is dedicated to the problem of extraction of that information.

Preliminaries: shape equation

TheHelfrich model of erythrocyte is given viathe energy functional [1]

$$F = \frac{1}{2} k_c \oint (c_1 + c_2 - c_0)^2 \, dA + \lambda \oint dA + \Delta P \int dV. \tag{1}$$

Where *dA* and *dV* are surface area and volume elements, respectively, k_c – the bending rigidity, $c_{1,2}$– the two principal curvatures, and c_0– the spontaneous curvature [3,4]. The last parameters serves to describe theeffect of an asymmetry of the membrane.

The first term of Eq. (1) is the curvature-elastic energy of the vesicle membrane [5]. The second and third terms either take into account of the constraints of constant volume and area. Depending on the situation, osmotic pressure ΔP and the tensile stress λ serve as Lagrange multipliers or they are prescribed experimentally by volume or area reservoirs.

To obtain the shape equation for the surface corresponding to the surface when energy functional has extremal value, we claim first variation of (1) $\delta^{(1)}F$ is zero. Then we immediately come to the equation

$$\boldsymbol{\Delta}\mathrm{H} + (\mathrm{H} + \mathrm{c}_0/2)(2\mathrm{H}^2 - 2\mathrm{K} - \mathrm{c}_0\mathrm{H}) - (\lambda/k_c)\mathrm{H} + (\Delta\mathrm{P}/2k_c) = 0 \quad (2)$$

Which was first derived in [1]. Here $\boldsymbol{\Delta}$ is the Laplace-Beltrami operator on the surface of the erythrocyte (with the matrix g^{ij}) and *H, K* are mean and Gauss curvatures of the surface correspondingly [6]. As we see in (2), the bending rigidity k_c plays only a role of the scaling parameter and the shape of the erythrocyte is defined via three independent phenomenological parameters: osmotic pressure ΔP, tensile stress λ and spontaneous curvature c_0.

The equation (2) contains whole information related to connection of the shape of erythrocyte with the phenomenological parameters of the problem. That is very complicated, nonlinear equation only with a few known exact solutions [7] and besides, to follow after the shape of deformed surface is a hard experimental task.

Contour equation

Considering the external radiation, as a source of the non-equilibrium it is natural to assume homogeneity and isotropy of that in the scales of the erythrocyte. Then, obviously the mirror symmetry of the surface will be preserved and it is meaningful to speak about cross section of the surface of erythrocyte in the plane of symmetry. At equilibrium the cross section is a circle, but in non-equilibrium it could take any form. To find equation of the contour of the cross section we should consider Helfrich's equation (2) of the surface in surroundings of the symmetry plane of erythrocyte.

In practice, it is easy to follow after the deviation of the cross section from a circle when express analysis and our problem is to connect it with the intensity of the radiation. To solve the problem we begin with projecting of the Helfrich equation (2) into symmetry plane.

In infinitely small distances from symmetry plane the surface of the erythrocyte behaves as cylindrical [8] and therefore the Laplace-Beltrami operator $\boldsymbol{\Delta}$ in (2) has a form

$$\boldsymbol{\Delta} = \partial^2/\partial s^2 + \partial^2/\partial z^2 \quad (3)$$

Where *z* is perpendicular coordinate to the plane of symmetry and *s* is the arc length of the cross section contour. Now we take into account assertion that the contour of cross section is a line of curvature of the surface [9] and then one of the Gauss principal curvatures becomes zero and other is coincident with the curvature of the contour. Therefore, the Gauss curvature *K* is zero and the mean curvature *H* coincident with the half of the cross section curvature.

Insertion of (3) into (2) and taking the limit at $z \to 0$ we come to second order differential equation for the curvature *k(s)* of the cross section as a function of arc length s of the contour

$$\ddot{k} - \mu k + k^3/2 + p = 0\ . \quad (4)$$

Where$p = \Delta P/k_c$ and $\mu = \lambda/k_c + c_0^2/2$. This type of equation (4) is well known in theory of the elastic loops [10-12]. As we see from (4) the cross section of erythrocyte in the field of isotropic radiation is defined by two positive parameters p and μ. It is true also reciprocal assertion: Helfrich phenomenological parameters p and μ are defining via the form of cross section contour. Really, let $k_{1,2}$ are magnitudes of the curvature of contour at two different points $s_{1,2}$ then writing (4) at these points we get

$$\mu = (\ddot{k}_1 - \ddot{k}_2)/(k_1 - k_2) + (1/2)({k_1}^3 - {k_2}^3)/(k_1 - k_2) \quad (5.a)$$

$$p = (k_2\ddot{k}_1 - k_1\ddot{k}_2)/(k_1 - k_2) - (1/2)(k_1{k_2}^3 - k_2{k_1}^3)/(k_1 - k_2) \quad (5.b)$$

For further references we recall expressions of the curvature k and its second derivative $\ddot{k}$ when the contour in *x,y* plane is given parametrically *x=x(s), y=y(s)* [6]

$$k = \dot{x}\ddot{y} - \ddot{x}\dot{y} \tag{6.a}$$

$$\ddot{k} = \ddot{x}\dddot{y} - \dddot{x}\ddot{y} + \dot{x}y^{(4)} - x^{(4)}\dot{y} \tag{6.b}$$

In the absence of radiation erythrocyte is in equilibrium with a constant curvature$k = 1/a$, where a is the radius of cross section. Insertion of these expressions into (4) leads torelation

$$\lambda/ak_c + c_0^2/2a = p + 1/2a^3 \tag{7}$$

between the Helfrich phenomenological parameters at equilibrium.

Small intensities of radiation

The equation (4) is integrable by quadrature because of it belongs into the class of equations describing conservative systems with one degree of freedom [13]. However, its solution cannot be to get analytically and in some stage numerical calculus are needed. We will publish our numerical results related to the exact solution of the equation (4) elsewhere and in this paper we restrict our consideration solving linearized equation (4), which is correct for the small intensities of external radiation.

For small intensities I of radiation one can assume linear connections of the parameters p, μ from the intensity of radiation. Introducing corresponding susceptibilities $b_{1,2}$ we assume the forms

$$p + b_1 I, \quad \mu + b_2 I \tag{8}$$

for p, μ and then we can search for the solution of (4) as

$$k(s) = 1/a + f(s) \tag{9}$$

Insertion of (9) into equation (4) and using expressions (7) we find *f(s)* in linear approximation of intensity *I*. Then, using curvature (9), we can find isoperimetric parameter $\alpha = S/P^2$ of the cross section. Where *S, P* are an area and perimeter of the cross section. In this way we come to equation

$$j_0(x) + 2\alpha x = 1 \tag{10}$$

In equation (10) $j_0(x)$ is the spherical Bessel function of the first kind [14] and x is a dimensionless variable defined as

$$x = (P/a)[1+I(b_2 - ab_1)/(a^{-2} - ap)] \tag{11}$$

Therefore having measured perimeter and area of the cross section of erythrocyte surface in mirror symmetry plane one can find intensity of radiation solving equation (10) with variable (11). At the equilibrium, when *I=0* and $x=2\pi$, the equation (10) becomes an identity as it should be.

Summary

In the field of isotropic radiation a general equation of the curvature of the cross section of the surface of erythrocyte in the plane of mirror symmetry is derived in detail in frame of the Helfrich model. It is shown that this equation is coincident with equation in theory of the elastic loops. The coefficients of equation depend of Helfrich phenomenological parameters and knowing geometrical form of the contour of the cross section it is possible to extract an information about the dynamical changes of these parameters in the field of external radiation and therefore about structural changes of the membrane.Measuring the curvature in two different points of the contour it is possible to find some of Helfrich constants. In the case of small intensities of radiation is found that the intensity of radiation and the isoperimetric parameter of the contour are connected via a transcendental equation.

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