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The Wake Function and Beam Energy Modulation in a Helical Undulator with the Circular Waveguide

M. Ivanyan*, A. Tsakanian, T.Vardanyan

CANDLE Synchrotron Research Institute, Acharyan 31, 0040 Yerevan, Armenia

*e-mail: mikayel.ivanyan@gmail.com

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Abstract – The explicit expression for the point-charge longitudinal wake function of helical undulator radiation in circular waveguide is given. The peculiarities of the helical undulator radiation and wake functions in circular waveguides are examined. For the charge distribution and discrete spectrum of the radiated electromagnetic fields the coherent part of the radiation and the beam energy modulation are studied.

1. Introduction

The helical undulators have an essential value in advanced synchrotron light sources and free electron lasers by producing circular polarized intense radiation. The properties of the undulatror radiation in free space are well known [1,2]. However, in reality, the beam is moving in metallic vacuum chamber that modifies the radiation fields. The general approach for studying the electromagnetic fields excitation in waveguides is the modal expansion technique [3,4]. The properties of undulator radiation in waveguides are studied in Refs. [5-12].

An important feature of the undulator radiation in waveguide is the radiation discrete spectrum conditioned by the excited waveguide modes. Actually, the waveguide modifies the continious spectrum of undulator radiation into a small number of sharp peaks, each corresponding to an excited waveguide mode. By proper choice of the undulator and waveguide parameters, the selected modes can be enhanced thus improving the undulator radiation source performance.

For the charge longitudinal distribution the particles within the bunch interact with the radiated electromagnetic fields that can be evaluated in terms of the wake function [13,14]. The low frequency part of the wakefields formes the coherent part of the radiation, while the high frequency part lead to energy modulation within the bunch. The knowledge of the undulator radiation wake field effect is important in order to optimize the facility performance both from the beam dynamics and the radiation points of view.

In this paper, the pecularities of the helical undulator radiation in circular waveguide are studied. The explicit expression for the point-charge longitudinal wake function for helical undulator radiation in waveguide is obtained. The energy modulation for the beam Gaussian and rectangular longitudinal distributions are analyzed for the discrete spectrum of the radiated electromagnetic fields.

2. Electromagnetic Fields and Wake Functions

Consider the helical undulator with the perfectly conducting vacuum beam pipe of circular cross section and radius b. The driving charge Q follows the helical orbit with the radius $a = \beta_\perp c / \omega_0$ and angular frequency $\omega_0 = 2\pi\beta_{||} c / \lambda_0$, where λ_0 is the undulator period, $\beta_\perp$ and $\beta_{||}$ indicate the charge transverse and longitudinal velocities respectively, $\beta = v / c$, v is the charge velocity and c

is the velocity of light. The cylindrical coordinate system (r,φ,z) will be in used to evaluate the problem.

The solution for electromagnetic fields produced by the charge in the waveguide is defined by the expansion of the fields over the TM and TE modes [3]. Using inhomogeneous Maxwell equations, the orthogonality of the modes and the methodology of integration in the complex plane, the longitudinal electromagnetic field components for TM and TE modes in time-space domain can be presented as follows:

$$E_{z,nm} = \frac{Q}{2\pi\varepsilon_0 b^2}\frac{J_n(j_{nm}\, a/b)}{J^2_{n-1}(j_{nm})}J_n(j_{nm}\, r/b)e^{i(n\varphi-\omega^{j}_{nm}t+k^{j}_{nm}z)}$$
$$H_{z,nm} = \frac{Qa^2\omega_0}{2\pi b^3}\frac{i\,\nu^3_{nm}}{\nu^2_{nm}-n^2}\frac{J'_n(\nu_{nm}\, a/b)}{f(\nu_{nm})J^2_n(\nu_{nm})}J_n(\nu_{nm}\, r/b)e^{i(n\varphi-\omega^{\nu}_{nm}t+k^{\nu}_{nm}z)} \tag{1}$$

with

$$\omega^{\lambda}_{nm} = n\omega_0 + k^{\lambda}_{nm}\beta_{||}c\,, \quad k^{\lambda}_{nm} = \frac{\gamma^2_{||}}{a}[n\beta_{||}\beta_{\perp} + f(\lambda_{nm})sign(z-Vt)]\,,$$
$$f(\lambda_{nm}) = \sqrt{n^2\beta^2_{\perp} - \gamma^{-2}_{||}\lambda^2_{nm}a^2/b^2}\,,$$

where $V = c\beta_{||}$ is the charge longitudinal velocity, $\lambda_{nm} = j_{nm}$ are the roots of the first order Bessel function $J_n(x)$ (TM modes), $\lambda_{nm} = \nu_{nm}$ are the roots of $J'_n(x)$ (TE modes) ,$\gamma_{||}$ is the particle longitudinal Lorenz factor, ω^{λ}_{nm} is the excited mode frequency and k^{λ}_{nm} is the longitudinal wave number.

The transversal components $\vec{E}_t = \vec{E}^{TM}_t + \vec{E}^{TE}_t$, $\vec{H}_t = \vec{H}^{TM}_t + \vec{H}^{TE}_t$ for TM ($\vec{E}^{TM}_t, \vec{H}^{TM}_t$) and TE ($\vec{E}^{TE}_t, \vec{H}^{TE}_t$) modes are derived from the **inhomogeneous** Maxwell equations in the following way [4]

$$\vec{E}^{TM}_t = \frac{ik^{j}_{nm}b^2}{j^2_{mn}}\nabla_t E_z\,, \quad \vec{H}^{TM}_t = \frac{\omega^{j}_{nm}}{ck^{j}_{nm}}\vec{e}_z \times E^{TM}_t$$
$$\vec{H}^{TE}_t = \frac{ik^{\nu}_{nm}b^2}{\nu^2_{nm}}\nabla_t H_z\,, \quad \vec{E}^{TE}_t = -\frac{\omega^{\nu}_{nm}}{ck^{\nu}_{nm}}\vec{e}_z \times \vec{H}^{TM}_t \tag{2}$$

where the transverse gradient in cylindrical coordinates is given as $\nabla_t = (\partial/\partial r,\, in/r)$.

The wake function is defined as the integral effect of the Lorenz force acting on the test charge q which follows at the distance s with respect to driving charge position along the same helical orbit. The test charge position then corresponds to time $t' = t + s/v$. For $s > 0$ the test charge is ahead of the driving charge, for $s < 0$ the test charge is behind the driving charge. The Lorenz force of the radiated fields acting on the test charge is given by

$$\vec{F}(t',t) = q[\vec{E}(\vec{r}(t'),t) + \vec{v}\times\mu_0\vec{H}(\vec{r}(t'),t)] \tag{3}$$

where $\vec{r}(t') = (r = a, \varphi = \omega_0 t', z = Vt')$ defines the test charge orbit.

In the coordinate frame ($\vec{e}_s, \vec{e}_r, \vec{e}_n$) ($\vec{e}_s = \vec{v}/v$, $\vec{e}_n = \vec{e}_r \times \vec{e}_s$) associated with the driving charge, the Lorenz force components are given as :

$$
\begin{gathered}
F_s = q[\frac{V}{v}E_z + \frac{a\omega_0}{v}E_\varphi] , \\
F_r = q[E_r - V\mu_0 H_\varphi + a\omega_0\mu_0 H_z] , \\
F_n = -q[\frac{V}{v}E_\varphi - \frac{a\omega_0}{v}E_z + v\mu_0 H_r]
\end{gathered} \tag{4}
$$

The longitudinal $w_s(s)$ and transverse $\vec{w}_\perp(s)$ wake functions are then defined as

$$
w_s(s) = \lim_{T\to\infty} \frac{1}{2TqQ}\int_{-T}^{T} F_s(t',t)dt , \quad \vec{w}_\perp(s) = \lim_{T\to\infty} \frac{1}{2TqQ}\int_{-T}^{T} [F_r\vec{e}_r + F_n\vec{e}_n](t',t)dt \tag{5}
$$

for $t' = t + s/v$. The longitudinal wake function defines the energy gain (or loss) of test charge per unit length and the transverse wake function defines the transverse kick of the test charge per unit length. Note, that $w_s(0)$ defines the energy loss of the point charge $U = Q^2 w_s(0)$.

While obtaining the longitudinal wake function, we note that for the test charge moving along the orbit $\vec{r}'(t') = (r' = a, \varphi' = \omega_0 t', z' = Vt')$ the argument in (1) can be represented as

$$
n\varphi' - \omega_{nm}t + k_{nm}z' = g(\lambda_{nm})s \tag{6}
$$

with

$$
g(\lambda_{nm}) = \gamma_{||}^2 (a\beta)^{-1}[n\beta_\perp + \beta_{||} f(\lambda_{nm}) sign(s)]
$$

Thus, the integrand in (5) does not depend on time and depends only on the test charge relative position s with respect to drive charge along the helical orbit. The point wake function is then expressed as follows

$$
w_s(s) = \frac{1}{\pi\varepsilon_0 b^2} Re\sum_{n,m}\left(C_{nm}e^{ig(j_{nm})s} + D_{nm}e^{ig(\nu_{nm})s}\right) \tag{7}
$$

where

$$
\begin{gathered}
C_{nm}(s) = -\frac{\gamma_{||}^2 f^2(j_{nm})}{\beta}\left\{\beta_{||} + Sign(s)\frac{n\beta_\perp}{f(j_{nm})}\right\}\frac{J_n^2(j_{nm}\,a/b)}{j_{nm}^2\,a^2/b^2\,J_n'^2(j_{nm})} \\
D_{nm}(s) = -\frac{\gamma_{||}^2\beta_\perp^2}{\beta}\left\{\beta_{||}Sign(s) + \frac{n\beta_\perp}{f(\nu_{nm})}\right\}\frac{\nu_{nm}^2}{n^2 - \nu_{nm}^2}\frac{J_n'^2(\nu_{nm}\,a/b)}{J_n^2(\nu_{nm})}
\end{gathered}
$$

The loss factor k_0 is given by the longitudinal wake function at the position $s = 0$ ($k_0 = w_z(0)$), and it is the sum of the energy stored in each excited mode. As it follows from (7), contributions of TM and TE modes for fixed n have the same order with respect to a/b. For $a/b << 1$ the main contribution to the radiated fields is given by the modes with the first index $n = 1$.

3. Longitudinal Wake Potential and Beam Energy Modulation

For the bunch longitudinal distribution $\rho(s)$ the particles of the bunch interact with the radiation fields inducing particles energy spread. The modulation of the particles energy in wake fields is determined by the longitudinal wake potential $W_s(s)$ given by the convolution of the point wake function $w_s(s)$ and the charge distribution $\rho(s)$ as follows

$$W_s(s) = \int_{-\infty}^{\infty} \rho(s') w_s(s+s') ds' \tag{8}$$

Note, that the bunch longitudinal distribution $\rho(s)$ is normalized to unity.

For the perfectly conducting waveguide**,** the modes are separated to propagating (radiation) and quickly damped ones. The modes that contribute to radiation fields are given by condition $f^2(\lambda_{nm}) \geq 0$ from (1) that corresponds to the following inequality

$$\beta_\perp^2 \geq \frac{\gamma^{-2}\lambda_{nm}^2 (a/b)^2}{n^2 - \lambda_{nm}^2 (a/b)^2} = \theta_{nm}^\lambda \tag{9}$$

As it is seen, the modes with the zero first indexes $(n=0)$ does not excite in the waveguide. For a fixed index $n \geq 1$ the number of modes contributing to the radiation pattern depends on the charge transverse velocity $\beta_\perp$ and the parameter θ_{nm}^λ. Note, that $\lambda_{nm} = j_{nm}$ for TM modes and $\lambda_{nm} = \nu_{nm}$ for the TE modes in (10). The eigenvalues λ_{nm} and parameters θ_{nm}^λ increase with the index m. The parameter θ_{nm}^λ is less than unity. As it follows from (9) both TE and TM modes contribute to wake function and the sequence of TE and TM eigenvalues with index m is characterized as $\nu_{n1} < j_{n1} < \nu_{n2} < j_{n2} < \nu_{n3} < j_{n3} \ldots$. The corresponding parameters θ_{nm}^λ are $\theta_{nm+1}^\nu < \theta_{nm+1}^j < \theta_{nm+2}^\nu < \theta_{nm+2}^j < \theta_{nm+3}^\nu < \theta_{nm+3}^j \ldots$. Thus, the first m TE and $m-1$ TM waveguide modes are excited if $\theta_{nm}^\nu < \beta_\perp^2 < \theta_{nm}^j$. In particular, for $n=1$ and $\theta_{n1}^\nu < \beta_\perp^2 < \theta_{n1}^j$ the fundamental TE_{11} mode satisfies only the radiation condition (9).

Figures 1-3 show the longitudinal wake function of point charge and longitudinal wake potentials of Gaussian bunch distributions with rms length of $\sigma = 5\mu m, 40\mu m, 1mm$ for the single (Fig.1), double (Fig.2) and three (Fig.3) excited modes in the waveguide. Calculations are carried out for the electron energy of 20 MeV ($\gamma \approx 40$) corresponding to the AREAL electron linear accelerator design parameters [15]. The plots are normalized to the value of $P = Z_0 c / \pi b^2$. The ratio a/b is taken equal to 10^{-3}. Dashed lines in the figures represent the charge longitudinal distribution. Note, that the single mode regime corresponds to $\theta_{n1}^\nu < \beta_\perp^2 < \theta_{n1}^j, \beta_\perp = 0.7 \times 10^{-4}$, double mode regime corresponds to $\theta_{n,1}^j < \beta_\perp^2 < \theta_{n,2}^\nu, \beta_\perp = 1.1 \times 10^{-4}$ and three-mode regime corresponds to $\theta_{n,2}^j < \beta_\perp^2 < \theta_{n,3}^\nu$, $\beta_\perp \approx 2 \times 10^{-4}$.

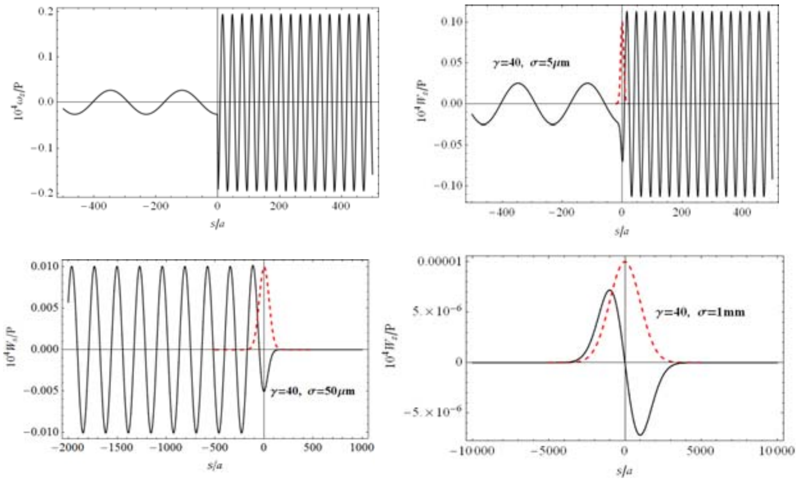

Figure 1. The single mode (TE_{11}) excitation regime. Longitudinal wake function for point charge and wake potentials for Gaussian bunches ($\sigma = 5\mu m, 40\mu m, 1mm$).

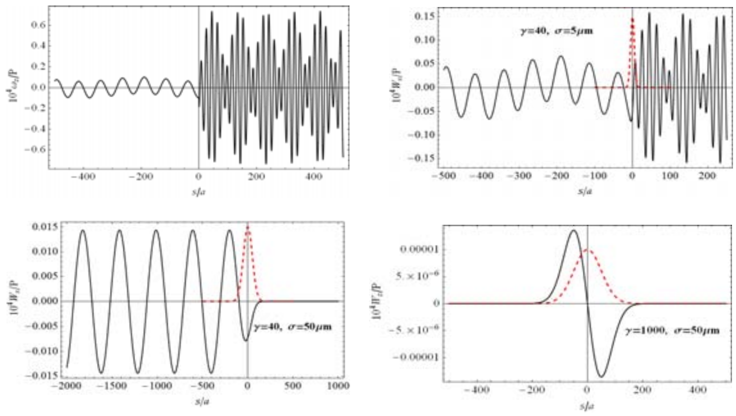

Figure 2. The double-mode (TE_{11} and TM_{11}) excitation regime. Longitudinal wake function for point charge and wake potentials for Gaussian bunches ($\sigma = 5\mu m, 40\mu m, 1mm$).

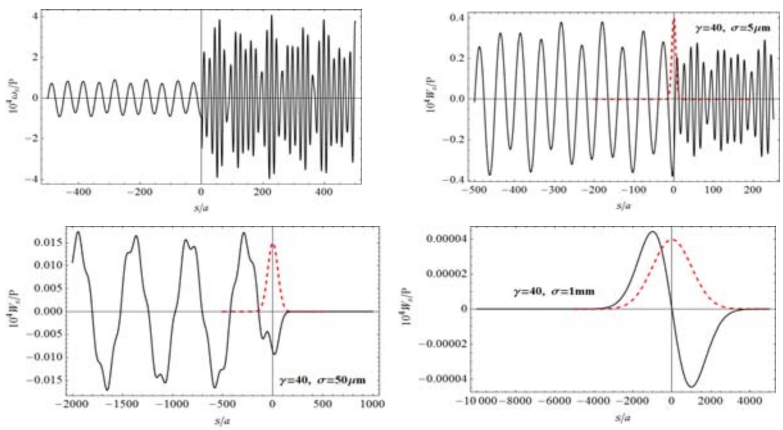

Figure 3. The three-mode (TE_{11}, TE_{12} and TM_{11}) excitation regime. Longitudinal wake function for point charge and wake potentials for Gaussian bunches ($\sigma = 5\mu m, 40\mu m, 1mm$).

As follows from the wake potential definition (8), the energy modulation within the bunch depends on the bunch longitudinal distribution. Fig.4 presents the wake potentials for the rectangular charge distribution for the single, double and three mode excitation regimes. The energy modulation within the beam takes place at the wavelength of excited modes.

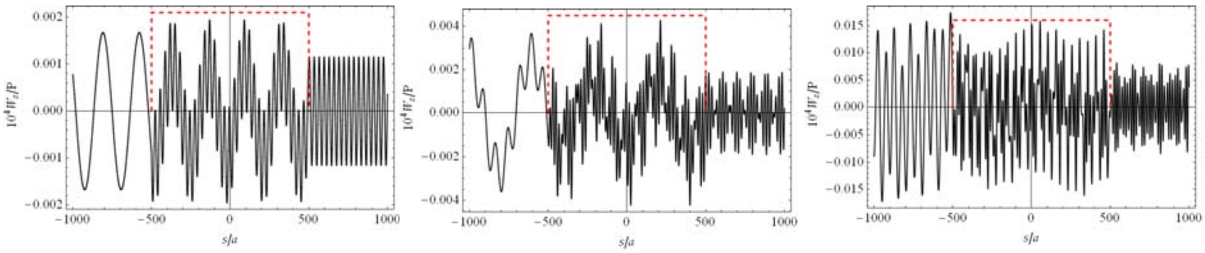

Figure 4. The longitudinal wake potential of helical coherent undulator radiation for the rectangular electron bunch distribution. The bunch rms length is $\sigma_z = 1mm$, the particle energy is 20 MeV for the single, double and three mode regimes.

4. Summary

The modal expansion technique is used to determine the explicit expression for the point wake function for helical undulator radiation in waveguide. The wake function is used to calculate the longitudinal wake potentials of the Gaussian and rectangular charge distributions for single, double and three mode excitation regimes. The results of the present paper can be used for proper evaluation of helical radiation in waveguides and the beam dynamics study in advanced electron storage rings and the free electron lasers.

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