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THE COMPLETION OF MODE DECOMPOSITION PROGRAM IN FRW SPACETIMES

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Abstract–In this paper we collect Paley–Wiener-type results for the scalar field in Friedman–Robertson–Walker spacetimes. These are then used to fulfill the functional analytical assumptions which the previously developed unified mode decomposition method is based upon.

1. Introduction

The synthesis of General Relativity and Quantum Field Theory is an open challenge in front of the mankind. From the quantum theory point of view the ultimate goal would be to have a quantum theory of gravity interacting with the remaining quantum fields. While there is no widely accepted successful theory of quantum gravity, there are several approaches, which tend to serve as approximations or germs of quantum gravity, including Quantum Field Theory in Curved Spacetimes (QFT in CST), Non-commutative Geometry (NCG), Strings, etc. The present exposition pertains to the first approach. The idea of QFT in CST is to consider the gravity as behaving classically (i.e., to neglect the possible quantum fluctuations of the gravity) and the quantum matter fields as propagating in the curved geometry created by the classical gravity according to General Relativity. The back reaction of the matter fields on the gravity is expressed by the semi-classical Einstein equation

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi\left\langle T_{\mu\nu}\right\rangle_{\omega},$$

where $G_{\mu\nu}$ and Λ are the Einstein tensor and the cosmological constant, respectively, and $\left\langle T_{\mu\nu}\right\rangle_{\omega}$ is the expectation value of the energy-momentum tensor of the quantum matter in the state ω (note that as usual in Quantum Field Theory the Heisenberg picture is adopted). Such a semi-classical approach is applicable in the theoretical cosmology, especially for the early epoch of the Universe, when the curvature of the spacetime is sufficiently strong to noticeably influence the behavior of quantum fields, but its possible quantum fluctuations can be neglected. A good account of aspects of QFT in CST can be found in [1,2].

One remarkable property of cosmology is the symmetry of the Universe at large scale. Supported by observations cosmology considers mainly spatially homogeneous and often also isotropic spacetimes. This property of the Universe is particularly convenient from the analytical point of view, as it drastically reduces the variety of possible spacetimes under consideration. Homogeneous and isotropic cosmological models are described by the

Friedman–Robertson–Walker (FRW) spacetimes where a six-dimensional Lie group of isometries acts transitively on spatial sections. There are three type of FRW models distinguished by the isometry groups G; the closed model with $G = SO(4)$, the flat model with $G = E(3)$ and the open model with $G = SO^{+}(3,1)$, ($^{+}$ here means the identity component). The spacetime metric in FRW models is given by

$$ds^2 = a^2(t)\, h_{ab} x^a x^b,$$

where $a(t)$ called the scale factor is a smooth strictly positive function which expresses the rate of the uniform expansion or contraction of the Universe, and h_{ab} is the spatial Riemannian metric canonically associated to each of three types of models. One of the advantages of such a metric form is the separation of the time variable t in the wave equation

$$\left(\Box + m^2\right)\phi = 0,$$

which describes the massive scalar field propagating in the curved spacetime. This separation in turn enables one to perform a mode decomposition of a solution ϕ in terms of independent spectral modes. This procedure is the relative of the heuristic method of expansion of the field into harmonic oscillators, and is achieved by a time-dependent Fourier transform. Roughly speaking, this amounts to examining the evolution of distribution of energy among different spectral modes of the field through the time. This decomposition is a very powerful analytical tool especially for explicit constructions. The first application of the method of mode decomposition in the cosmological context can probably be attributed to L. Parker [3], who performed it on the flat FRW spacetime and demonstrated a couple of interesting results. His derivations, however, were not carried out in complete mathematical rigor. A rigorous treatment of mode decomposition of strong solutions of the field equation in FRW spacetimes appeared in [2] and has been used by many authors afterwards [4,5]. But such problems as the mode decomposition in non-FRW spacetimes, for higher vector fields and for weak (distributional) solutions of the field equation remained open until recently. In [6] (see also [7,8]) we undertook the task of developing a unified mode decomposition method which solves all the above mentioned problems. The theory is almost complete in describing the method in its utmost generality in precise mathematical terms. However, it depends on a few conjectures which are intuitively expected but reduce to open problems in modern mathematics. The problems are mainly related to the explicit description of the image of different function spaces under the Fourier transform. Such results are usually called Paley–Wiener theorems and represent a major topic in harmonic analysis.

To complete the program of mode decomposition pursued in [6] one needs to prove the conjectures for all possible models. This is an extensive task, because the methods of harmonic analysis differ dramatically for various geometries. In this paper we will show how these conjectures can be satisfied in case of the scalar field in FRW spacetimes.

2. Paley–Wiener-Type Results of the Scalar Field on FRW Spacetimes

Here we will present the necessary results in very explicit terms. Although no major proof or computation is involved, but the explicit answers to these problems seems not to be available in the literature at least in the cosmological context.

Let us start with the closed FRW model. Its spatial section Σ is $SU(2)$ with the canonical $SO(4)$-invariant Riemannian metric and the associated Laplace operator Δ. The eigenfunctions of Δ are spherical harmonics ζ_α with $\alpha = (k,l,m)$, $0 \le l \le k$ and $m^2 \le l$, $l,m \in \mathbb{Z}$ $k \in \mathbb{N}_0$. The eigenvalues are

$$\lambda = -k(k+2).$$

The spectral Fourier transform is given by

$$\tilde{f}(\alpha) = \langle \overline{\zeta}_\alpha, f \rangle_\Sigma,$$

where

$$\langle f,h \rangle \Sigma = \int_M dx f(x) h(x).$$

We are interested in the images $\tilde{\mathcal{D}}(\Sigma)$ and $\tilde{\mathcal{E}}(\Sigma)$ of function spaces $\mathcal{D}(\Sigma)$ and $\mathcal{E}(\Sigma)$ under this Fourier transform, respectively. Because $SU(2)$ is compact, $\mathcal{D}(\Sigma) = \mathcal{E}(\Sigma)$, we only need to examine the smoothness property. It turns out that $\tilde{\mathcal{D}}(\Sigma)$ is the space of functions $\tilde{f}(k,l,m)$ which are of rapid decay in k. This can be seen, for instance, from the observation that on a compact manifold a function is smooth if and only if it is L^2 along with all powers of Δ acting on it.

Next we go on to the flat FRW model, of which the spatial section is $\Sigma = \mathbb{R}^3$ with the canonical $E(3)$-invariant Riemannian metric and the associated Laplace operator Δ. The eigenfunctions are the usual exponentials

$$\zeta_\alpha = (2\pi)^{3/2} e^{i\{\mathbf{k},\mathbf{x}\}}, \quad \alpha = \mathbf{k} \in \mathbb{R}^3,$$

and the eigenvalues are

$$\lambda = -|\mathbf{k}|^2$$

($\{,\}$ is the Euclidean product). By classical Paley–Wiener theorems we have that $\tilde{\mathcal{D}}(\Sigma)$ is the space of entire (in each variable) functions on $\mathbb{C}^3$ which are of rapid decay on the real line and of finite exponential type. Turning to $\tilde{\mathcal{E}}(\Sigma)$ we must say that its straightforward characterization is very difficult because of unpredictable behavior at infinity. We may instead realize $\tilde{\mathcal{E}}(\Sigma)$ as the dual space to the space of Fourier transformed distributions of compact support,

$$\tilde{\mathcal{E}}(\Sigma)=\left[\widetilde{\mathcal{E}(\Sigma)'}\right]'.$$

By another classical theorem (Theorem IX.12 in [9]) we have that the Fourier transform of a distribution of compact support is an entire function in each variable on $\mathbb{C}^3$ which is polynomially bounded on the real line and has finite exponential type. The Fourier transformed smooth functions are thus linear continuous functionals on this function space.

Finally let us turn to the open FRW model having spatial section $\Sigma = Bi(V)$ (the Bianchi V group) with the canonical $SO^+(3,1)$-invariant Riemannian metric and the associated Laplace operator Δ. Each such section is a 3-dimensional real hyperbolic (or otherwise called real Lobachevsky) space. In [6] we have suggested to slightly modify the eigenfunctions given in [10] to comply with the definition of a conventional Fourier transform. This is not only technically convenient for the aims of [6], but we will shortly see that this is precisely the form which comes from the general theory of symmetric spaces. We will again adhere to this convention. Take $\alpha=(k,\eta)\in\mathbb{R}\times S^2$, and let

$$\zeta_\alpha(\mathbf{x}) = 1/4\pi^{-3/2}\{\mathbf{x},\boldsymbol{\eta}\}^{ik-1},$$

where $\eta\in S^2$ is considered as embedded in $\mathbb{R}^3$. The eigenvalues are

$$\lambda=-(k^2+1).$$

Paley–Wiener theorems for the hyperbolic space are relatively new matters, and we will use the results suggested by the geometric theory of symmetric spaces. In terminology of [11], $G=SO^+(3,1)$, $K=SO(3)$ is a maximal compact subgroup, and $\Sigma=G/K$ (Helgason uses X instead of Σ). The Iwasawa decomposition of G can be inferred from [12]. Namely, $G=KAN$, where the Abelian subgroup $A=SO(1,1)$ and N is a normal subgroup. The stabilizer of A in K is $M=SO(2)$, and the normalizer of A in K is $M'=SO(2)\cup g_\omega SO(2)$, where g_ω is a special element. Thus the Weyl group $W=M'/M$ is isomorphic to the group $\{\pm\mathbf{1}\}$. The Lie algebra of the Abelian subgroup A is $\mathfrak{a}=\mathbb{R}$, and the adjoint action $Ad(g_\omega)\colon \mathfrak{a}\mapsto\mathfrak{a}$ of the nontrivial element g_ω of the Weyl group on $\mathfrak{a}$ can be easily found: it is the inversion $a\mapsto -a$ on $\mathbb{R}$. Thus the left action of the Weyl group W on the dual $\mathfrak{a}^*$ in the sense of [11] is again the inversion. Now look at Theorem III.5.1 in [11]. Fourier transform in this theorem is precisely our Fourier transform with substitutions $\rho=-1$ and $A(\mathbf{x},\boldsymbol{\eta})=\ln\{\mathbf{x},\boldsymbol{\eta}\}$ (in fact, historically the integral geometry of classical spaces like Lobachevsky space have served as the starting point for the analysis in general symmetric spaces). The integral symmetry condition in the definition of the space $\mathcal{H} = \left(\mathfrak{a}^* \times B\right)_W$ is precisely the integral condition we had in [6] but now in more abstract terms. At last we find from the theorem that $\tilde{\mathcal{D}}(\Sigma)$ is the space of smooth functions $\tilde{f}(k, \mathbf{\eta})$ entire in k, of rapid decay on the real line and of finite exponential type, satisfying the integral symmetry condition. We turn now to the Fourier transform of smooth functions which we again identify with the dual space of the distributions of compact support,

$$\tilde{\mathcal{E}}(\Sigma) = \widetilde{\left[\mathcal{E}(\Sigma)'\right]}'.$$

Use Corollary III.5.9 in [11] to establish that $\widetilde{\left[\mathcal{E}(\Sigma)'\right]}$ is the space of smooth functions $\tilde{\psi}(k, \mathbf{\eta})$ entire in k, of polynomial growth on the real line and of finite exponential type, satisfying the integral condition. This space has a locally convex distributional topology, and the desired space $\tilde{\mathcal{E}}(\Sigma)$ is its dual space.

3. The Fulfillment of Conjectures

We start with the first conjecture of [6,7] (which is assumed rather than explicitly stated) that the time-dependent Fourier transform in all cosmological models is conventional. For the scalar field the dimension of the bundle $\dim V = n = 1$, and we indeed see that the Fourier space $\tilde{\Sigma}$ is either discrete (closed FRW model) or a connected analytical manifold (flat and open FRW models). This verifies point (i). Next, in all three models λ is an analytic function of k. On the closed model, where $\tilde{\Sigma}$ is discrete, analyticity is vacuous, and we formally call it analytic. Otherwise k is a coordinate in the analytic global chart on the manifold $\tilde{\Sigma}$. Thus point (ii) is also satisfied. Point (iii) is a direct consequence of the Paley–Wiener results given above. Finally point (iv) follows immediately from the definition of the eigenfunctions ζ_α.

Next we observe that in FRW models the Schrodinger operator D_{Σ_t} has a strictly uniform spectrum [6,7]. Indeed, because the spatial metric at any time t is given by

$$h_{ab}(t) = a^2(t) h_{ab},$$

it follows

$$\Delta(t) = \left(a^2(0)/a^2(t)\right)\Delta(0),$$

whence

$$\lambda_\alpha(t) = \left[a^2(0)/a^2(t)\right]\left(\lambda_\alpha(0) - m^2\right) + m^2.$$

We turn to the important property (1.19) in [6] (or (2.19) in [7]). Namely, we need to choose the mode solutions $T_\alpha(t)$ so that

$$\tilde{f}(\alpha)T_\alpha(t)\in\tilde{\mathcal{D}}(\Sigma), \quad \forall t\in\mathbb{R}, \quad \forall\tilde{f}(\alpha)\in\tilde{\mathcal{D}}(\Sigma).$$

For the closed FRW model, $\tilde{f}(\alpha)$ is merely a function on $\tilde{\Sigma}$ which is of rapid decay in k. Thus it suffices to choose modes $T_\alpha(t)$ such that they are polynomially bounded in k uniformly on any time interval $R\subset\mathbb{R}$. By Proposition 1.4 in [6] (Proposition 2.4 in [7])) it suffices to choose the initial data $T_\alpha(0)$ and $\dot{T}_\alpha(0)$ to be polynomially bounded in k. For flat and open FRW models $T_\alpha(t)$ also need to be entire in k for all t (actually, for the open model one needs also the solutions $T_\alpha(t)$ to satisfy the integral symmetry condition). If the scale factor $a(t)$ is not a real analytic function, then the mode equation (1.16) in [6] ((2.16) in [7]) is a smooth ordinary differential equations depending holomorphically on the parameter k. If we choose also the initial data $T_\alpha(0)$ and $\dot{T}_\alpha(0)$ as entire functions of k, then it will be reasonable to expect that the solutions will also depend holomorphically on k for all t. While this has been proven in the case of an analytic scale factor (Proposition 1.5 in [6] or Proposition 2.5 in [7]), for the case of a smooth scale factor $a(t)$ this remains unproven so far.

A short notice should be made concerning the choice of the global time function. In the section "Spatially homogeneous cosmological models" of [6,7] we assumed that the time function is chosen so that the orbits of the isometries are precisely the equal time hypersurfaces. This is obviously true for FRW models.

Finally we come to the Hadamard property. By Remark 4.3 in [6] the problem of checking the Hadamard property of a state ω reduces to the smoothness of the bi-distribution ω_2^s. This in turn reduces to determining the Fourier transforms of smooth distributions. As we have done this above for the scalar field on FRW spacetimes, the question can be answered exhaustively for this case.

We conclude by noting that the fulfillment of these properties for non-FRW models is a way more complicated a task mainly because of the absence of corresponding Paley–Wiener-type theorems. For Bianchi II-VII models this is a work in progress and we hope to be able to announce new results in the near future.

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