Armenian Journal of Physics, 2010, vol. 3, issue 3, pp. 178-186

A METHOD OF GENERALIZED TRANSFER MATRIX FOR A PROBLEM OF ELECTROMAGNETIC WAVE PROPAGATION THROUGH AN ARBITRARY ONE-DIMENSIONAL ABSORBING MEDIUM

A. Zh. Khachatrian

State Engineering University of Armenia, Yerevan, E-mail: ashot.khachatrian@gmail.com

Received 09 June, 2010

Abstract: A generalized transfer matrix method for a problem of electromagnetic wave propagation through an arbitrary one-dimensional absorbing medium is suggested. Connections between the scattering amplitudes of the left and right scattering problems are found. It is shown that the problem of scattering amplitude determination can be reduced to solution of a one set of differential equations with two different initial conditions.

1. Introduction

Photonic crystals represent a new kind of optical materials, which possess many interesting properties and render many novel applications as well [1–6]. The existence of photonic band gaps in photonic crystals, owing to multiple Bragg scatterings, leads to many interesting phenomena [1–9]. Most photonic crystals fabricated so far are made from two dielectric materials. Usually, photonic band gaps of dielectric photonic crystals are not large. Combinations of metallic and dielectric materials may lead to more interesting properties comparing with dielectric photonic crystals. The introduction of metal sheets into dielectric photonic crystal can increase photonic band gaps considerably [10-14]. For example, the absorption of the bulk metal can be enhanced by inserting a dielectric layer periodically to form one-dimensional metallic-dielectric photonic crystals. By a proper choice of the structural and material parameters, one can obtain a large absorption enhancement in the visible and infrared regions [15].

Despite the fact that an investigation of one-dimensional periodic structures from absorbing layers was a subject of interest for many authors and has a great interest for many years, until now an analytical solution of this problem is unknown. The problem is that the operator of a field wave is not Hermitian, so that the flux of electromagnetic wave energy does not converse. In the standard method of transfer matrix [16], the complex transmission and reflection amplitudes are derived from the elements of the transfer matrix; however, the inverse connection between complex characteristics of a scattering and transfer matrix elements is not established. We consider the mention point as a main deficiency of the standard transfer matrix method, not only for that, it does not allow exploring important details of the absorption–scattering process. It is important to note that for a multilayer system all calculations have to be performed by numerical methods only.

In this work, we find connections between elements of transfer matrix in the case when absorption present. These connections allow presenting the elements of transfer matrix with help of complex parameters of scattering problem and get some analytical results for absorbing photonic crystals.

2. The wave field as a sum of modulated plane waves

Bellow we consider the propagation of a harmonic in time electromagnetic wave in a one-dimensional absorbing media. It is known that the electric component of the field is described by the following equation:

$$\frac{d^2E}{dx^2}+\left(k^2-u(x)\right)E=0\,, \tag{1}$$

where

$$u(x)=V_1(x)+iV_2(x)\,,\ k^2=\omega^2/c^2\,,\ V_1=k^2\left(1-\varepsilon'(x)\right),\ V_2=-k^2\varepsilon''(x)$$

and $\varepsilon'(x)$, $\varepsilon''(x)$ are real and imagine parts of dielectric constant.

We introduce the functions $a(x)$ and $b(x)$ in accordance with the following formulas:

$$a(x)=\frac{1}{2}\left[E(x)-\frac{i}{k}\frac{dE(x)}{dx}\right]\exp\{-ikx\}\,, \tag{2}$$

$$b(x)=\frac{1}{2}\left[E(x)+\frac{i}{k}\frac{dE(x)}{dx}\right]\exp\{ikx\}\,. \tag{3}$$

It is easy to see that the

$$E(x)=a(x)\exp\{ikx\}+b(x)\exp\{-ikx\}\,. \tag{4}$$

To present $d\Psi(x)/dx$ with help of the functions $a(x)$ and $b(x)$, we consider the first-order derivatives of expressions (2), (3). From (2), (3) one can write down

$$\frac{da(x)}{dx}=-\frac{i}{2k}\left[k^2E(x)+\frac{d^2E(x)}{dx^2}\right]\exp\{-ikx\}\,, \tag{5}$$

$$\frac{db(x)}{dx}=\frac{i}{2k}\left[k^2E(x)+\frac{d^2E(x)}{dx^2}\right]\exp\{ikx\}\,. \tag{6}$$

By using Eq. (1) the last two equations can be written in the form

$$\frac{da(x)}{dx}=-\frac{iV(x)}{2k}E(x)\exp\{-ikx\}\,, \tag{7}$$

$$\frac{db(x)}{dx}=\frac{iV(x)}{2k}E(x)\exp\{ikx\}\,. \tag{8}$$

As it follows from (7), (8) and (4), the functions $a(x)$ and $b(x)$ satisfy the following set of equations:

$$\frac{da(x)}{dx}=-\frac{iV(x)}{2k}a(x)-\frac{iV(x)}{2k}b(x)\exp\{-i2kx\}\,, \tag{9}$$

$$\frac{db(x)}{dx}=\frac{iV(x)}{2k}b(x)+\frac{iV(x)}{2k}a(x)\exp\{i2kx\}\,. \tag{10}$$

From Eqs. (7), (8) it follows that

$$\frac{da(x)}{dx}\exp\{ikx\}+\frac{db(x)}{dx}\exp\{-ikx\}=0\,. \tag{11}$$

Taking into account Eq. (11), it is easy to get

$$\frac{dE(x)}{dx}=ik\big(a(x)\exp\{ikx\}-b(x)\exp\{-ikx\}\big). \tag{12}$$

Note that the obtained form of $dE(x)/dx$ is similar to a wave derivative getting for the case of constant dielectric constant. This similarity has a physical meaning, in particular, the functions $a(x)$ and $b(x)$ if they satisfy the set of equations (7), (8), should be considered as amplitudes of secondary waves propagating in opposite directions.

3. The transfer matrix method for an arbitrary absorbing slab

Now we discuss the problem of wave transmission through an irregularly absorbing slab when the dielectric constant has the form

$$\begin{cases} 1, & x<x_1, \\ \varepsilon'(x)+i\varepsilon''(x), & x_1<x<x_2, \\ 1, & x>x_2. \end{cases} \tag{13}$$

In accordance with the above-mentioned result, let us consider two linear independent solutions of Eq. (1) $E_1(x)$ and $E_2(x)$ as a sum of modulated plane waves propagating in opposite directions:

$$E_1(x)=a(x)\exp\{ikx\}+b(x)\exp\{-ikx\}, \tag{14}$$

$$E_2(x)=c(x)\exp\{ikx\}+d(x)\exp\{-ikx\}. \tag{15}$$

Their derivatives have the form

$$\frac{dE_1(x)}{dx}=ik\big(a(x)\exp\{ikx\}-b(x)\exp\{-ikx\}\big), \tag{16}$$

$$\frac{dE_2(x)}{dx}=ik\big(c(x)\exp\{ikx\}-d(x)\exp\{-ikx\}\big). \tag{17}$$

Note that for the case of a non-absorbing slab ($\varepsilon''(x)=0$) the second solution $E_2(x)$ can be taken as a complex conjugate of the solution $E_1(x)$, i.e.

$$d(x)=a^*(x)\,,\; c(x)=b^*(x)\,. \tag{18}$$

It is easy to check that for two arbitrary independent solutions of Eq. (1) the following conservation law takes place:

$$E_1(x)\frac{dE_2}{dx}-E_2(x)\frac{dE_1}{dx}=const\,, \tag{19}$$

which by using Eqs. (14)–(17) can be written as

$$a(x)d(x)-b(x)c(x)=const\,. \tag{20}$$

The asymptotic behaviors of the fields $E_1(x)$ and $E_2(x)$ in the region outside the slab can be presented in the form:

$$E_1(x)=\begin{cases} a_1\exp\{ikx\}+b_1\exp\{-ikx\}, & x<x_1, \\ a_2\exp\{ikx\}+b_2\exp\{-ikx\}, & x>x_2, \end{cases} \quad (21)$$

$$E_2(x)=\begin{cases} c_1\exp\{ikx\}+d_1\exp\{-ikx\}, & x<x_1, \\ c_2\exp\{ikx\}+d_2\exp\{-ikx\}, & x>x_2, \end{cases} \quad (22)$$

where a_1, b_1, c_1, d_1 and a_2, b_2, c_2, d_2 are the values of the functions $a(x)$, $b(x)$, $c(x)$, $d(x)$ in the points $x=x_1$ and $x=x_2$, correspondingly. By using (14)-(17) it is easy to see that the continuity of the functions $E_1(x)$, $E_2(x)$ and their derivatives at the slab boundaries takes place:

$$E_1(x_1-0)=E_1(x_1+0),\ \ E_1(x_2-0)=E_1(x_2+0), \quad (23)$$

$$E_2(x_1-0)=E_2(x_1+0),\ E_2(x_2-0)=E_2(x_2+0), \quad (24)$$

$$\frac{dE_1(x_1-0)}{dx}=\frac{dE_1(x_1+0)}{dx},\ \ \frac{dE_1(x_2-0)}{dx}=\frac{dE_1(x_2+0)}{dx}, \quad (25)$$

$$\frac{dE_2(x_1-0)}{dx}=\frac{dE_2(x_1+0)}{dx},\ \frac{dE_2(x_2-0)}{dx}=\frac{dE_2(x_2+0)}{dx}. \quad (26)$$

Any solution $E(x)$ of Eq. (1) having the asymptotic behavior of the form

$$E(x)=\begin{cases} A_1\exp\{ikx\}+B_1\exp\{-ikx\}, & x<x_1, \\ A_2\exp\{ikx\}+B_2\exp\{-ikx\}, & x>x_2, \end{cases} \quad (27)$$

can be presented as a linear combination of the functions $E_1(x)$, $E_2(x)$:

$$E(x)=v_1E_1(x)+v_2E_2(x). \quad (28)$$

Taking into account Eqs. (14)-(17) and (27), (28), from the standard conditions of wave continuity at the slab boundaries x_1, x_2 one can write down

$$A_1=v_1a_1+v_2d_1,\ B_1=v_1b_1+v_2c_1, \quad (29)$$

$$A_2=v_1a_2+v_2d_2,\ B_2=v_1b_2+v_2c_2. \quad (30)$$

From Eqs. (29), (30) one can obtain that the linearity between the coefficients A_2, B_2 and A_1, B_1 is presented with help of the following transfer matrix:

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}=\hat{T}\begin{pmatrix} A_1 \\ B_1 \end{pmatrix},\quad \hat{T}=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (31)$$

where

$$\alpha=\frac{d_1a_2-b_1c_2}{d_1a_1-b_1c_1}, \quad (32)$$

$$\beta=\frac{a_1c_2-c_1a_2}{d_1a_1-b_1c_1}, \quad (33)$$

$$\gamma = \frac{d_1 b_2 - b_1 d_2}{d_1 a_1 - b_1 c_1}, \tag{34}$$

$$\delta = \frac{a_1 d_2 - c_1 b_2}{d_1 a_1 - b_1 c_1}. \tag{35}$$

It is easy to check that

$$\alpha\delta - \gamma\beta = 1. \tag{39}$$

Note that the obtained transfer matrix (31) provides the transition between the coefficients of the solutions $E_1(x)$, $E_2(x)$ as well:

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \tag{40}$$

$$\begin{pmatrix} c_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} c_1 \\ p_1 \end{pmatrix}. \tag{41}$$

4. The transfer matrix elements and scattering amplitudes of left and right scattering problems

Now we consider the wave field for two cases, which describe the waves falled on a slab from the left and right sides, correspondingly:

$$E_{left} = \begin{cases} \exp\{ikx\} + r\exp\{-ikx\}, & x < x_1, \\ t\exp\{ikx\}, & x > x_2, \end{cases} \tag{42}$$

$$E_{right} = \begin{cases} s\exp\{-ikx\}, & x < x_1, \\ \exp\{-ikx\} + p\exp\{ikx\}, & x > x_2. \end{cases} \tag{43}$$

where t, r and s, p are transmission, reflection coefficients of left ant right scattering problems.

Comparing expressions (42), (43) with (21), (22) we have

$$a_1 = 1,\ b_1 = r,\ a_2 = t,\ b_2 = 0, \tag{39}$$

$$c_1 = 0,\ d_1 = s,\ c_2 = p,\ d_2 = 1. \tag{40}$$

Using (39), (40) from (32)-(35) we get the elements of the transfer matrix presented with help of the scattering amplitudes of left and right scattering problems:

$$\alpha = \frac{st - pr}{s},\ \beta = \frac{p}{s},\ \gamma = -\frac{r}{s},\ \delta = \frac{1}{s}. \tag{41}$$

Substituting (41) into (34) we find

$$s = t, \tag{42}$$

which means that the transmission amplitudes of left and right scattering problems are equal.

To derive a relation between reflection amplitudes of left and right scattering problems it is necessary to note that if the solution of the wave equation has the form

$$E_1(x) = a_k(x)\exp\{ikx\} + b_k(x)\exp\{-ikx\}, \tag{43}$$

when another independent solution can be taken as

$$E_2(x) = b_{-k}(x)\exp\{ikx\} + a_{-k}(x)\exp\{-ikx\}. \tag{44}$$

Indeed, substitution of k for $-k$ into the set (7), (8) reduces the both equations to each other. Note, that in the case of nonabsorbing media the following relations exist

$$a_k^*(x) = a_{-k}(x),\ b_k^*(x) = b_{-k}(x). \tag{45}$$

In accordance with the above-mentioned in Eqs. (21), (22) we take

$$a_1 = a_k(x_1),\ a_2 = a_k(x_2),\ b_1 = b_k(x_1),\ b_2 = b_k(x_2), \tag{46}$$

$$c_1 = b_{-k}(x_1),\ c_2 = b_{-k}(x_2),\ d_1 = a_{-k}(x_1),\ d_2 = a_{-k}(x_2). \tag{47}$$

From (46), (47) and (32)-(35) for the transfer matrix $\hat{T}$ (31) one can write down

$$\hat{T} = \begin{pmatrix} \dfrac{a_k(x_2)a_{-k}(x_1) - b_k(x_1)b_{-k}(x_2)}{a_k(x_1)a_{-k}(x_1) - b_k(x_1)b_{-k}(x_1)} & \dfrac{a_k(x_1)b_{-k}(x_2) - b_{-k}(x_1)a_k(x_2)}{a_k(x_1)a_{-k}(x_1) - b_k(x_1)b_{-k}(x_1)} \\ \dfrac{a_{-k}(x_1)b_k(x_2) - b_k(x_1)a_{-k}(x_2)}{a_k(x_1)a_{-k}(x_1) - b_k(x_1)b_{-k}(x_1)} & \dfrac{a_k(x_1)a_{-k}(x_2) - b_{-k}(x_1)b_k(x_2)}{a_k(x_1)a_{-k}(x_1) - b_k(x_1)b_{-k}(x_1)} \end{pmatrix} \tag{48}$$

and

$$a_k(x)a_{-k}(x) - b_k(x)b_{-k}(x) = const. \tag{49}$$

From (48) and (31) it is easy to see that

$$\alpha(k) = \delta(-k),\ \beta(k) = \gamma(-k), \tag{50}$$

and for a nonabsorbing media (see Eq. (45))

$$\alpha^*(k) = \delta(k),\ \beta^*(k) = \gamma(k). \tag{51}$$

If we choose $a_k(x_1) = 1$ and $b_k(x_2) = 0$, when in accordance with (39) $b_k(x_1) = r(k)$, $a_k(x_2) = t(k)$, the transfer matrix elements can be presented with help of scattering amplitudes of the left scattering problem:

$$\hat{T} = \begin{pmatrix} \dfrac{1}{t(-k)} & \dfrac{-r(-k)}{t(-k)} \\ \dfrac{-r(k)}{t(k)} & \dfrac{1}{t(k)} \end{pmatrix}, \tag{52}$$

$$1 - r(k)r(-k) = t(k)t(-k). \tag{53}$$

In the case of the choice $b_k(x_1) = 0$, $a_{-k}(x_2) = 1$ in accordance with (40) the scattering amplitudes of right scattering problem will be $a_{-k}(x_1) = s(k)$, $b_{-k}(x_2) = p(k)$ and for transfer matrix elements we can write

$$\hat{T} = \begin{pmatrix} \dfrac{1}{s(-k)} & \dfrac{p(k)}{s(k)} \\ \dfrac{p(-k)}{s(-k)} & \dfrac{1}{s(k)} \end{pmatrix}, \tag{54}$$

$$1 - p(k)p(-k) = s(k)s(-k). \tag{55}$$

Comparing (52) and (54) we get

$$\frac{p(k)}{s(k)} = -\frac{r(-k)}{t(-k)}. \tag{56}$$

As it follows from (41), (42) and (52)-(55), the following connections also take place:

$$s(-k) = \frac{s(k)}{s^2(k) - r(k)p(k)}, \tag{57}$$

$$p(-k) = -\frac{r(k)}{s^2(k) - r(k)p(k)}, \tag{58}$$

$$r(-k) = -\frac{p(k)}{s^2(k) - r(k)p(k)}. \tag{59}$$

From the last two equations we obtain

$$p(k)p(-k) = r(k)r(-k), \tag{60}$$

which shows that connections between the reflection amplitudes of left and right scattering problems have more complicated form than connection existed between transmission coefficients (see (42)).

5. The scattering amplitudes as functions of a slab border point

Let us consider the transfer matrix elements as functions of the slab border points x_1, x_2, i.e. we introduce the functions $\alpha = \alpha(x_1, x_2, k)$, $\beta = \beta(x_1, x_2, k)$, $\gamma = \gamma(x_1, x_2, k)$ and $\delta = \delta(x_1, x_2, k)$ (see (31)) It is easy to see from (48), that when $x_1 = x_2$ then

$$\alpha(x_1, x_1, k) = \delta(x_1, x_1, k) = 1, \tag{61}$$

$$\beta(x_1, x_1, k) = \gamma(x_1, x_1, k) = 0. \tag{62}$$

From (9), (10) it follows that the functions $a_k(x)$, $b_k(x)$ and $a_{-k}(x)$, $b_{-k}(x)$ satisfy the same set of equations:

$$\frac{da_k(x)}{dx} = -\frac{iV(x)}{2k} a_k(x) - \frac{iV(x)}{2k} b_k(x) \exp\{-i2kx\}, \tag{63}$$

$$\frac{db_k(x)}{dx} = \frac{iV(x)}{2k} b_k(x) + \frac{iV(x)}{2k} a_k(x) \exp\{i2kx\}, \tag{64}$$

and

$$\frac{db_{-k}(x)}{dx} = -\frac{iV(x)}{2k} b_{-k}(x) - \frac{iV(x)}{2k} a_{-k}(x)\exp\{-i2kx\}, \tag{65}$$

$$\frac{da_{-k}(x)}{dx} = \frac{iV(x)}{2k} a_{-k}(x) + \frac{iV(x)}{2k} b_{-k}(x)\exp\{i2kx\}. \tag{66}$$

If we consider x_1 as a constant and x_2 as a variable, then denoting $x_2 = x$ ($\alpha(x) = \alpha(x_1, x_2, k)$, $\beta(x) = \beta(x_1, x_2, k)$, $\gamma(x) = \gamma(x_1, x_2, k)$ and $\delta(x) = \delta(x_1, x_2, k)$) one can write down a problem of determination of scattering amplitudes as a Cauchy-type problem for a set of linear differential equations:

$$\frac{d\alpha(x)}{dx} = -\frac{iV(x)}{2k}\alpha(x) + \frac{iV(x)}{2k}\frac{r(x)}{t(x)}\exp\{-i2kx\}, \tag{67}$$

$$\frac{d}{dx}\frac{r(x)}{t(x)} = \frac{iV(x)}{2k}\frac{r(x)}{t(x)} - \frac{iV(x)}{2k}\alpha(x)\exp\{i2kx\}, \tag{68}$$

with initial conditions

$$\alpha(x_1) = 1,\ r(x_1)/t(x_1) = 0, \tag{69}$$

and

$$\frac{d}{dx}\frac{1}{t(x)} = \frac{iV(x)}{2k}\frac{1}{t(x)} + \frac{iV(x)}{2k}\beta(x)\exp\{i2kx\}\ . \tag{70}$$

$$\frac{d\beta(x)}{dx} = -\frac{iV(x)}{2k}\beta(x) - \frac{iV(x)}{2k}\frac{1}{t(x)}\exp\{-i2kx\}\ , \tag{71}$$

with initial conditions

$$1/t(x_1) = 1,\ \beta(x_1) = 0\ . \tag{72}$$

As it is seen from (67)-(72), the pairs of quantities $\alpha(x)$, $r(x)/t(x)$ and $\beta(x)$, $1/t(x)$ satisfy the same set of differential equations, but with different initial conditions.

5. Conclusion

In this paper, for the problem of electromagnetic wave propagation through an arbitrary one-dimensional absorbing media a generalized transfer matrix method is suggested. The developed theory allows to present transfer matrix elements with help of the scattering amplitudes of the left and right scattering problems. It is shown that for an arbitrary absorbing slab the left and right scattering amplitudes coincide with each other. In the framework of the developed method the scattering problem was formulated as a Cauchy type problem.

REFERENCES

1 **E.Yablonovitch,** *Phys. Rev. Lett.*, **58**, 2059 (1987).

2 **S.John,** *Phys. Rev. Lett.*, **58**, 2486 (1987).

3 **C.M.Soukoulis,** Photonic Band Gaps and Localization, Plenum, NY, 1993.

4 **J.D.Joannopoulos, R.D.Meade, J.N.Winn,** Photonic Crystals, Princeton University Press, NJ, 1995.

5 **C.M.Soukoulis,** Photonic Bandgap Materials, Kluwer, Dordrecht, 1996.

6 **J.D.Joannopoulos, P.R.Villeneuve, S.Fan,** *Nature*, **386**, 143 (1997).

7 **C.M.Cornelius, J.P.Dowling,** *Phys. Rev. Lett A*, **59**, 4736 (1999).

8 **S.Y.Lin, J.G.Fleming, E.Chow, J.Bur, K.K.Choi, A.Goldberg,** *Phys. Rev. Lett.*, **62**, 2243 (2000).

9 **Y.G.Xi, X.Wang, X.H.Hu, X.H.Liu, J.Zi,** *Phys. Rev. Lett.*, **19**, 1819 (2002).

10 **D.F.Sievenpiper, M.E.Sickmiller, E.Yablonovitch,** *Phys. Rev. Lett.*, **76**, 2480 (1996).

11 **J.B.Pendry, A.J.Holden, W.J.Steward, I.Youngs,** *Phys. Rev. Lett.*, **76**, 4733 (1996).

12 **A.J.Ward, J.B.Pendry, W.J.Stewart,** *J. Phys.: Condens. Matter*, **7**, 2217 (1995).

13 **M.Scalora, M.J.Bloemer, A.S.Pethel, et al.** *Appl. Phys. Rev. Lett.*, **83**, 2377 (1998).

14 **M.J.Bloemer, M.Scalora,** *Appl. Phys. Rev. Lett.*, **72**, 1676 (1998).

15 **J.Yu, Y.Shen, X.Liu, R.Fu, J.Zi, Z.Zhu,** *J. Phys.: Condens. Matter*, **16**, 51 (2004).

16 **P.Yeh.** *Optical Waves in Layered Media*, Wiley, NY, 1988.