ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ
ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК АРМЕНИИ

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A PROBLEM OF OPTIMAL STABILIZATION OF A QUADCOPTER UAV FLIGHT

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Keywords: Dynamical Systems, Optimal Control, Optimal Stabilization, Quadcopter UAV.

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Об одной задаче оптимальной стабилизации полёта квадрокоптера

Ключевые слова: Динамические системы, оптимальное управление, оптимальная стабилизация, квадрокоптер БЛА.

В работе рассматривается задача оптимальной стабилизации безпилотного летательного аппарата (квадрокоптера) в линейном приближении. Приведена система дифференциальных уравнений, опысивающая динамика вадрокоптера, проверена полная управляемость линейного приближения полученной управляемой системы, поставлена и решена задача оптимальной стабилизации этой системы методом Ляпунова-Беллмана. Получены оптимальная функция Ляпунова, оптимальные управляющие воздействия. Построены графики оптимальных управлений и оптимальных движений.

Շահինյան Ա.Ս.

Քառաթև անօդաչու թռչող սարքի օպտիմալ ստաբիլացման մի խնդիր

Հիմնաբառեր՝ դինամիկ համակարգեր, օպտիմալ ղեկավարում, օպտիմալ ստաբիլացում, քառաթև ԱԹՍ:

Աշխատանքում դիտարկվում է քառաթև անօդաչու թռչող սարքի (ԱԹՍ) օպտիմալ ստաբիլացման խնդիրը գծային մոտավորությամբ: Բերված է ԱԹՍ դինամիկան նկարագրող դիֆերենցիալ հավասարումների համակարգը, ստուգված է ստացված ղեկավարվող համակարգի գծային մոտավորության լրիվ ղեկավարելիությունը, ձևակերպված և լուծված է այդ համակարգի համար օպտիմալ ստաբիլացման խնդիրը Լյապունով-Բելմանի եղանակով: Ստացված են Լյապունովի օպտիմալ ֆունկցիան և օպտիմալ ղեկավարող ազդեցությունները: Կառուցված են օպտիմալ ղեկավարումների և օպտիմալ շարժումների գրաֆիկները:

This paper discusses an optimal stabilization problem of a quadcopter unmanned aerial vehicle (UAV). The dynamics of the UAV is presented and the linear time invariant (LTI) model of it is considered. The controllability of the LTI model is checked and an optimal stability problem is defined and solved for the LTI system using Lyapunov-Bellman method. The Lyapunov optimal function and optimal control inputs are gained. The graphs of optimal control inputs and optimal trajectories are constructed and presented.

1. Introduction: In this paper we are going to discuss a stabilization problem of a quadcopter (also called a quadrotor) unmanned aerial vehicle (UAV). A quadcopter is a helicopter equipped with four engines each of which have a propeller attached. Quadcopters are of high interest among researchers because of their simple structure. Moreover, quadcopters are agile and maneuverable which makes it easy to experiment complex control algorithms using them.

There are several papers which study the stabilization problem of quadcopters while approaching the problem from different view angles. Some papers use PID controllers, others do the job using just PD controllers, some researchers solved the problem using LQR regulator method. A short description of such papers is given in [1].
In [2] an Optimal Control problem is stated and solved for the linearized model. A numerical example is also given by presenting optimal control inputs calculated analytically and optimal trajectories of the motion.
Here in this paper we are going to approach the problem using so called Lyapunov-Bellman method. Using this method for Linear Time Invariant (LTI) systems we can find a Lyapunov optimal function, optimal control inputs and optimal trajectories.
What follows the introduction are three sections which are modelling of the system, problem definition and solution and discussion of results. The discussed system is a quadcopter, and the model is derived theoretically. Then the problem is defined and solved. The results are discussed by providing some simulations results and comments.

2. Model of the System: To derive the pure theoretical dynamics of a UAV let us fix a coordinate system $Oxyz$. Let O be the origin. We will also need another coordinate system $O_B x_B y_B z_B$ fixed in the center of mass O_B of the UAV (fig.1). The torques and forces generated by each of the propellers are shown in the Figure 1. The propellers are numbered 1 to 4 [1].

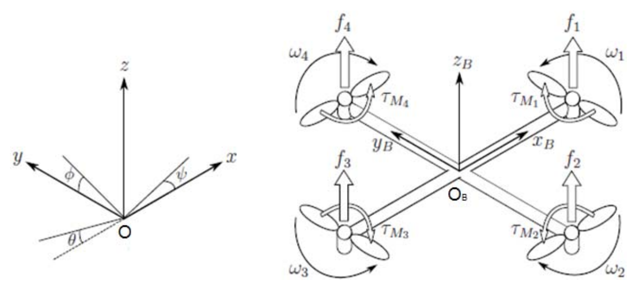

Figure 1.

Let $\xi = (x \quad y \quad z)^T$ be the coordinates of the center of mass of the UAV with respect to the system $Oxyz$. As mentioned above, the center of the mass of the UAV coincides with the origin of the coordinate system $O_B x_B y_B z_B$. Let us describe the inclined position of the UAV about the point O_B using yaw, pitch and roll angles. Let Φ be the pitch angle, Θ be the roll angle and, finally, let Ψ be the yaw angle. Then we will have two vectors describing the position of the UAV. Those are the following:

$$\xi = (x \quad y \quad z)^T, \quad \eta = (\Phi \quad \Theta \quad \Psi)^T \tag{1}$$

In the coordinate system the linear velocities V_B and the angular velocities v are the following

$$V_B = (V_{Bx} \quad V_{By} \quad V_{Bz})^T, \qquad v = (p \quad q \quad r)^T \tag{2}$$

In this setup we will have the dynamics of the system as given below [1; 3].

$$\ddot{x} = \frac{T}{m} c_\Psi s_\Theta c_\Phi + \frac{T}{m} s_\Psi s_\Phi \ , \ \ \ddot{y} = \frac{T}{m} s_\Psi s_\Theta c_\Phi - \frac{T}{m} c_\Psi s_\Phi \ , \ \ \ddot{z} = -g + \frac{T}{m} c_\Theta c_\Phi \ ,$$

$$\dot{\Phi} = p + \frac{s_\Phi s_\Theta}{c_\Theta} q + \frac{c_\Phi s_\Theta}{c_\Theta} r, \quad \dot{\Theta} = c_\Phi q - s_\Phi r, \quad \dot{\Psi} = \frac{s_\Phi}{c_\Theta} q + \frac{c_\Phi}{c_\Theta} r,$$

$$\dot{p} = \frac{\left(I_{yy} - I_{zz}\right) qr}{I_{xx}} - I_r \frac{q}{I_{xx}} \omega_\Gamma + \frac{\tau_\Phi}{I_{xx}}, \tag{3}$$

$$\dot{q} = \frac{\left(I_{zz} - I_{xx}\right) pr}{I_{yy}} - I_r \frac{p}{I_{yy}} \omega_\Gamma + \frac{\tau_\Theta}{I_{yy}},$$

$$\dot{r} = \frac{\left(I_{xx} - I_{yy}\right) pq}{I_{zz}} - I_r \frac{q}{I_{zz}} \omega_\Gamma + \frac{\tau_\Psi}{I_{zz}}.$$

Where the following notations are used: $C_\alpha := \cos\alpha,\ S_\alpha := \sin\alpha$,

$$\tau_B = \begin{pmatrix} \tau_\Phi \\ \tau_\Theta \\ \tau_\Psi \end{pmatrix} = \begin{pmatrix} lk\left(-\omega_2^2 + \omega_4^2\right) \\ lk\left(-\omega_1^2 + \omega_3^2\right) \\ \sum_i \tau_i \end{pmatrix} \text{ and } T = \sum_i F_i = \sum_i k\omega_i^2, \quad \vec{T} = \begin{pmatrix} 0 & 0 & T \end{pmatrix}^T$$

Let us do the following notations and linearize the system around the origin.

$$\begin{aligned} &x_1 = x,\ x_2 = \dot{x},\ x_3 = y,\ x_4 = \dot{y},\ x_5 = z,\ x_6 = \dot{z}, \\ &x_7 = \Phi,\ x_8 = \Theta,\ x_9 = \Psi,\ x_{10} = p,\ x_{11} = q,\ x_{12} = r \end{aligned} \tag{4}$$

We will have

$$\begin{aligned} &\dot{x}_1 = x_2,\ \dot{x}_2 = gx_8,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -gx_7,\ \dot{x}_5 = x_6,\ \dot{x}_6 = u_1 \\ &\dot{x}_7 = x_{10},\ \dot{x}_8 = x_{11},\ \dot{x}_9 = x_{12},\ \dot{x}_{10} = \frac{u_2}{I_{xx}},\ \dot{x}_{11} = \frac{u_3}{I_{yy}},\ \dot{x}_{12} = \frac{u_4}{I_{zz}} \end{aligned} \tag{5}$$

Where $u_1 = \frac{T}{m} - g$, $u_2 = \tau_\Phi$, $u_3 = \tau_\Theta$, $u_4 = \tau_\Psi$

In [2] it is discussed and shown that the system (5) is fully controllable.

3. Stabilization Problem: Let us now define the stabilization problem that we want to solve.

Problem: Given the system (1.8), the initial position of the system $x(0) = x_0$, find control inputs $u^0 = \begin{pmatrix} u_1^0 & u_2^0 & u_3^0 & u_4^0 \end{pmatrix}^T$ such that it drives the system from the given initial position to asymptotically stable state, while minimizing the given linear quadratic regulator

$$J[\bullet] = \int_0^\infty \left(\sum_{i=1}^{12} x_i^2 + \sum_{i=1}^{4} u_i^2 \right) d\tau. \tag{6}$$

Solution: We will follow Lyapunov-Bellman method to solve this problem. Notice that the system (5) can be decomposed into 4 systems which are the following.

$$\dot{x}_1 = x_2,\ \dot{x}_2 = gx_8,\ \dot{x}_8 = x_{11},\ \dot{x}_{11} = \frac{u_3}{I_{yy}} \tag{7}$$

$$\dot{x}_3 = x_4,\ \dot{x}_4 = -gx_7,\ \dot{x}_7 = x_{10},\ \dot{x}_{10} = \frac{u_2}{I_{xx}} \tag{8}$$

$$\dot{x}_5 = x_6,\ \dot{x}_6 = u_1 \tag{9}$$

$$\dot{x}_9 = x_{12},\ \dot{x}_{12} = \frac{u_4}{I_{zz}} \tag{10}$$

So, we will have 4 independent systems which are fully controllable. This means we can solve the problem for each of these systems separately. In this case (6) will be written as

$$J[\bullet] = J_1[\bullet] + J_2[\bullet] + J_3[\bullet] + J_4[\bullet]$$

where

$$J_1[\bullet] = \int_0^\infty \left(x_5^2 + x_6^2 + u_1^2\right) d\tau,\quad J_2[\bullet] = \int_0^\infty \left(x_3^2 + x_4^2 + x_7^2 + x_{10}^2 + u_2^2\right) d\tau,$$

$$J_3[\bullet] = \int_0^\infty \left(x_1^2 + x_2^2 + x_8^2 + x_{11}^2 + u_3^2\right) d\tau,\quad J_4[\bullet] = \int_0^\infty \left(x_9^2 + x_{12}^2 + u_4^2\right) d\tau.$$

As we see that system (5) can be divided into four subsystems then it is convenient to search for a Lyapunov function in the form

$$V(x_1,\dots,x_{12}) = V_1(x_5,x_6) + V_2(x_3,x_4,x_7,x_{10}) + \\ + V_3(x_1,x_2,x_8,x_{11}) + V_4(x_9,x_{12})$$

We will show the steps for one of the above subsystems (say (7)) and will present the solutions of other three systems instantly.

So, for (7) Bellman equation will be as follows.

$$\mathfrak{B}[\bullet] = \frac{\partial V_3}{\partial x_1} x_2 + \frac{\partial V_3}{\partial x_2} gx_8 + \frac{\partial V_3}{\partial x_8} x_{11} + \frac{\partial V_3}{\partial x_{11}} au_3 + x_1^2 + x_2^2 + x_8^2 + x_{11}^2 + u_3^2 \tag{11}$$

Where $a = \frac{1}{I_{xx}} = \frac{1}{I_{yy}}$. Now differentiating (11) and making it equal to 0 we get that

$$u_3^0 = -\frac{1}{2} a \frac{\partial V_3}{\partial x_{11}} \tag{12}$$

Then we will have

$$\mathfrak{B}[\bullet]\big|_{u_3 = u_3^0} = \frac{\partial V_3}{\partial x_1} x_2 + \frac{\partial V_3}{\partial x_2} gx_8 + \frac{\partial V_3}{\partial x_8} x_{11} - \frac{1}{4} a^2 \left(\frac{\partial V_3}{\partial x_{11}}\right)^2 + x_1^2 + x_2^2 + x_8^2 + x_{11}^2 = 0 \tag{13}$$

We are looking for Lyapunov function in the form shown below.

$$V_3(x_1,x_2,x_8,x_{11}) = \frac{1}{2}(c_{11}x_1^2 + c_{22}x_2^2 + c_{88}x_8^2 + c_{1111}x_{11}^2 + 2c_{12}x_1x_2 + 2c_{18}x_1x_8 + \\ + 2c_{111}x_1x_{11} + 2c_{28}x_2x_8 + 2c_{211}x_2x_{11} + 2c_{811}x_8x_{11}) \tag{14}$$

Substituting (14) into (13) leads us to following equation.

$$\left(c_{11}x_1+c_{12}x_2+c_{18}x_8+c_{111}x_{11}\right)x_2+\left(c_{12}x_1+c_{22}x_2+c_{28}x_8+c_{211}x_{11}\right)gx_8+$$
$$+\left(c_{18}x_1+c_{28}x_2+c_{88}x_8+c_{811}x_{11}\right)x_{11}-\frac{1}{4}a^2\left(c_{111}x_1+c_{211}x_2+c_{811}x_8+c_{1111}x_{11}\right)^2+ \quad (15)$$
$$+x_1^2+x_2^2+x_8^2+x_{11}^2=0$$

Because the coefficients of polynomials in opposite sides of an equity must be equal, then from (15) it simply follows that

$$-\frac{1}{4}a^2c_{111}^2+1=0,\ -\frac{1}{4}a^2c_{211}^2+c_{12}+1=0,$$
$$-\frac{1}{4}a^2c_{811}^2+gc_{28}+1=0,\ -\frac{1}{4}a^2c_{1111}^2+c_{811}+1=0,$$
$$-\frac{1}{2}a^2c_{111}c_{211}+c_{11}=0,\ -\frac{1}{2}a^2c_{111}c_{811}+gc_{12}=0, \quad (16)$$
$$-\frac{1}{2}a^2c_{111}c_{1111}+c_{18}=0,\ -\frac{1}{2}a^2c_{211}c_{811}+c_{18}+gc_{22}=0,$$
$$-\frac{1}{2}a^2c_{211}c_{1111}+c_{111}+c_{28}=0,\ -\frac{1}{2}a^2c_{811}c_{1111}+gc_{211}+c_{88}=0$$

Here the parameters have the values $a=205.93,\ g=9.81$. The solution of (16) that makes $V_3(x)$ a positive definite function is the following:

$$c_{11}=2.912,\ c_{22}=1.423,\ c_{88}=11.148,\ c_{1111}=0.00997,\ c_{12}=1.121$$
$$c_{18}=2.053,\ c_{111}=0.00971,\ c_{28}=2.979,\ c_{211}=0.0141,\ c_{811}=0.0534 \quad (17)$$

It only remains to substitute values from (17) into (14) and then substitute into (12). Thus, we will have u_3^0.

By doing the same steps we will get also $V_1,\ V_2, V_4$ and $u_1^0,\ u_2^0, u_4^0$. For $V_1,\ V_2, V_3, V_4$ we will have.

$$V_1\left(x_5,x_6\right)=1.732x_5^2+2x_5x_6+1.732x_6^2$$
$$V_2(x_3,x_4,x_7,x_{10})=1.457x_3^2+0.712x_4^2+5.574x_7^2+0.00499x_{10}^2+1.121x_3x_4-$$
$$-2.053x_3x_7-0.00971x_3x_{10}-2.979x_4x_7-0.0141x_4x_{10}+0.0534x_7x_{10} \quad (18)$$
$$V_3(x_1,x_2,x_8,x_{11})=1.457x_3^2+0.712x_4^2+5.574x_7^2+0.00499x_{10}^2+1.121x_3x_4+$$
$$+2.053x_3x_7+0.00971x_3x_{10}+2.979x_4x_7+0.0141x_4x_{10}+0.0534x_7x_{10}$$
$$V_4\left(x_9,x_{12}\right)=1.009x_9^2+0.0176x_9x_{12}+0.0089x_{12}^2$$

Hence, optimal control inputs will be:

$$u_1^0=-\frac{1}{2}\frac{\partial V_1}{\partial x_6}=-x_5-1.732x_6,u_2^0=-\frac{1}{2}a\frac{\partial V_2}{\partial x_{10}}=1.00082x_3+1.452x_4-5.498x_7-1.0276x_{10}$$
$$u_3^0=-\frac{1}{2}a\frac{\partial V_3}{\partial x_{11}}=-1.00082x_1-1.452x_2-5.498x_8-1.0276x_{11},u_4^0=-\frac{1}{2}b\frac{\partial V_4}{\partial x_{12}}=-x_9-1.0112x_{12} \quad (19)$$

Where $b=\frac{1}{I_{zz}}=113.62$. And Lyapunov function for the system (5) will be the sum of Lyapunov functions of subsystems (7) -(10). That is

$$V^0\left(x_1,...,x_{12}\right)=V_1^0\left(x_5,x_6\right)+V_2^0\left(x_3,x_4,x_7,x_{10}\right)+V_3^0\left(x_1,x_2,x_8,x_{11}\right)+V_4^0\left(x_9,x_{12}\right)$$

And the minimal value of (6) is [5]

$$J^0[\bullet] = V^0(x_{1,0},\ldots,x_{12,0})$$

Where $x_{i,0} = x_i(0)$, $i = 1,\ldots,12$.

4. Discussion of Results: To visualize the results, we did some simulations. To do this, we simply substituted the optimal control inputs into the system (5) and get a system of first order ordinary differential equations. By solving that system, we will get the analytic forms of optimal trajectories. It is not convenient to show them in the paper because of their enormous sizes. The initial conditions and values of the parameters are assumed to be the following:

$$x_{1,0} = 50,\ x_{2,0} = 0,\ x_{3,0} = 30,\ x_{4,0} = 0,\ x_{5,0} = 10,\ x_{6,0} = 0,\ x_{7,0} = 0,\ x_{8,0} = 0,$$

$$x_{9,0} = \frac{\pi}{2},\ x_{10,0} = 0,\ x_{11,0} = 0,\ x_{12,0} = 0,\ a = \frac{1000}{4.856},\ b = \frac{1000}{8.801}$$

So, for these initial conditions we will have our constraint optimal value equal to $J^0[\bullet] = 8766.99$. The trajectories of states will have the form given below. The results of the simulation are presented below by presenting some of the graphs of optimal trajectories numbered as Fig.2 to Fig.5.

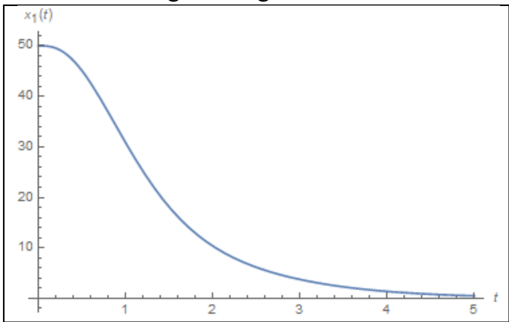

Fig. 2: Trajectory of state $x_1(t)$

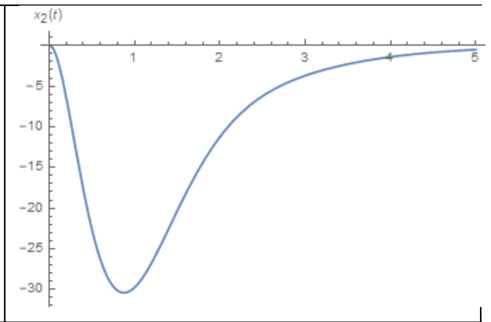

Fig. 3: Trajectory of state $x_2(t)$

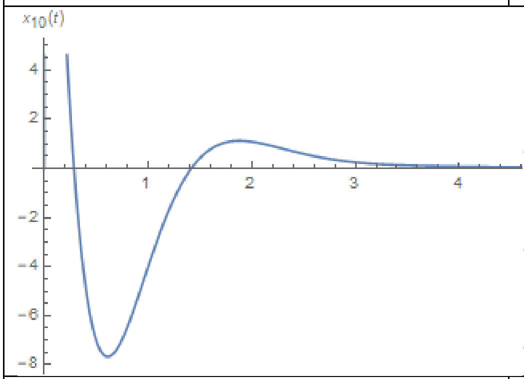

Fig. 4: Trajectory of state $x_{10}(t)$

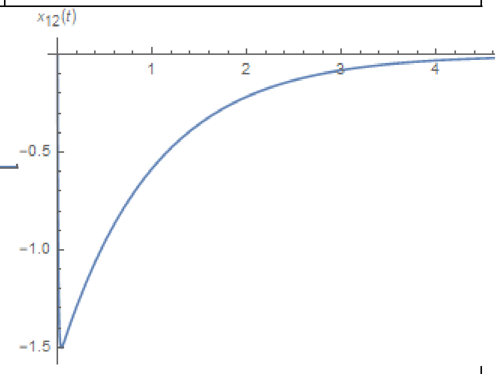

Fig. 5: Trajectory of state $x_{12}(t)$

Conclusion

The problem discussed in this paper is solved using Lyapunov-Bellman method. Lyapunov Optimal function is acquired, and the optimal control inputs are constructed. Using those results we also constructed the optimal trajectories of UAV including both geometrical coordinates and their velocities. Also, the optimal value of the energy constraint is calculated and given in discussion of results section. The results are then discussed by simulating them in MATLAB R2018a. Finally, the simulation results are shown in form of graphs.

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