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ON ASYMPTOTIC METHOD OF STATIC AND DYNAMIC BOUNDARY PROBLEMS SOLUTION

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Լ.Ա. Աղալովյան

Ստատիկական և դինամիկական եզրային խնդիրների լուծման ասիմպտոտիկ մեթոդի մասին

Շարադրված է առաձգականության տեսության եզրային խնդիրների լուծման ասիմպտոտիկ մեթոդը բարակ մարմինների՝ հեծաններ, սալեր, թաղանթներ, լարվածա-դեֆորմացիոն վիճակները որոշելու համար: Դիտարկված են ինչպես դասական, այնպես էլ ոչ դասական եզրային խնդիրներ: Ցույց է տրված մեթոդի էֆեկտիվությունը և ստատիկական և դինամիկական խնդիրների լուծումները որոշելու հարցում: Բերված են իլյուստրացիոն բնույթի անհրաժեշտ օրինակներ:

Агаловян Л.А.

Об асимптотическом методе решения статических и динамических краевых задач

Изложена суть асимптотического метода решения краевых задач теории упругости для тонких тел — балки, пластины, оболочки. Рассмотрены как классические, так и неклассические краевые задачи. Показана эффективность асимптотического метода для определения решений и статических, и динамических задач. Приведены необходимые иллюстрационные примеры.

Abstract

The equations of elasticity theory for thin bodies (bars, beams, plates, shells) are singularly perturbed by small geometric parameter. For the solution of such systems an asymptotic method is suggested to be used. The solution of the corresponding boundary problem of elasticity theory consists of two qualitatively different types of solutions - inner problem and boundary layer. The ways of constructing these solutions and their conjunctions are described. We consider as classic boundary problems as well as nonclassic boundary problems from the point of view of the plates and shells theory on the facial surfaces the displacement vector components or mixed conditions are given. Asymptotics of the inner problem solution is established, it is proved that it sensitively reacts on the type of the boundary problems conditions of elasticity theory laid on the facial surfaces. Solutions of the boundary layers are constructed. The relation of the boundary layer with Saint-Venant principle is displaced. In case of the first boundary problem for a rectangle it is proved that Saint-Venant principle is mathematically exact. Iteration processes for the determination of the inner problem solution are built, the connection with the solutions on classical Bernoulli-Coulomb theory of beams, Kirchhoff-Love theory of plates and shells with precise theories on the base of softened hypothesis is established. The formula of calculation of the bed coefficient for a layered foundation is reduced. The asymptotic method is especially effective for the solution of nonclassical dynamic boundary problems. Free and forced vibrations of thin bodies are considered. The connections between the frequencies values of free vibrations and the velocities of propagation of elastic shear and longitudinal waves are established.

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Key words: elasticity, singularly perturbed problems, thin bodies, asymptotic method, free and forced vibrations.

Introduction

For the calculation of thin bodies of beam type, plates and shells methods of hypotheses. decompositions of sought values according to the cross coordinate or special functions were originally used. Yet, the specific character of this kind of bodies is so, that one of its sizes sharply differs from the others and if in the equations of elasticity theory we pass to dimensionless coordinates and components of the displacement vector, these equations turn to be perturbed by small geometrical parameter. That is why it will be natural to use asymptotic methods. It was found out that the perturbance by small parameter is singular. Mathematical theory of such equations and systems began to develop from the middles of the 20th century, that is why the application of the asymptotic methods has a considerably new history. The first papers where the asymptotic method for the solution of boundary value problems of elasticity theory for plates and shells are [1-3]. The first boundary value problem of elasticity theory for isotropic rectangle is solved in [4] by an asymptotic method. The asymptotic theory of isotropic shells is built in [5], and the anisotropic theory of beams. plates and shells is in [6].

The asymptotic method turned to be especially effective for the solution of nonclassical static and dynamic boundary value problems of thin bodies – on the facial surfaces the values of the displacement vector component or mixed conditions are given [6-14].

Let's stop at some key results, obtained by the asymptotic method.

1. The first boundary value problem for a rectangle. The connection of the asymptotic solution with classical theory of beams and with Saint-Venant principle

The solution of this considerably simple problem reveals the basic principles and advantages of the asymptotic method application. It is required to find the solution of the equations at a plane problem of elasticity theory in the region of $D = \{(x, y) : x \in [0, \ell], |y| \leq h, h << \ell\}$, if on the longitudinal edges $y = \pm h$ of the rectangle the values of the stresses are given

$$\sigma_{xy}(\pm h) = \pm X^{\pm}(x), \ \sigma_{yy}(\pm h) = \pm Y^{\pm}(x) \tag{1.1}$$

and when $x = 0, \ell$ are the values of stresses, displacements or mixed conditions. Passing to dimensionless coordinates $\xi = x/\ell, \ \zeta = y/h$ and displacements $U = u/\ell, V = \mathrm{v}/\ell$ the equations system of a plane problem for an isotropic rectangle is written in the following form

$$\begin{gathered}
\frac{\partial\sigma_{xx}}{\partial\xi} + \varepsilon^{-1}\frac{\partial\sigma_{xy}}{\partial\zeta} + \ell F_x(\ell\xi, h\zeta) = 0 \\
\frac{\partial\sigma_{xy}}{\partial\xi} + \varepsilon^{-1}\frac{\partial\sigma_{yy}}{\partial\zeta} + \ell F_y(\ell\xi, h\zeta) = 0 \\
\frac{\partial U}{\partial\xi} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}), \ \ \varepsilon^{-1}\frac{\partial V}{\partial\zeta} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx})
\end{gathered} \tag{1.2}$$

$$\varepsilon^{-1}\frac{\partial U}{\partial \zeta}+\frac{\partial V}{\partial \xi}=\frac{1}{G}\sigma_{xy},\ \varepsilon=h/\ell$$

where σ_{ik} are the stresses tensor components, F_x, F_y are volume forces. The solution of system (1.2), as singularly perturbed by small parameter ε system, is combined from the solution of inner problem and from the solutions of boundary layers built close to $x=0,\ell$:

$$I=Q+R_b^{(1)}+R_b^{(2)} \qquad (1.3)$$

The solution of inner problem Q, which exactly reacts the types of the stated when $y=\pm h$ conditions, is sought in the form of

$$Q=\varepsilon^{q_a+s}Q^{(s)}(\xi,\zeta), \qquad s=\overline{0,N} \qquad (1.4)$$

where q_a characterizes the asymptotic order of the given magnitude, their values must be so that substituting (1.4) into (1.2) and coefficients under the same degrees ε to get a noncontradictory system for sequential determination of values $Q^{(s)}$, i.e. the stresses tensor components and displacement vector. This is most responsible moment when applying the asymptotic method as not all the components have the same orders. In this case

$$q=-2 \text{ for } \sigma_{xx},u;\ q=-1 \text{ for } \sigma_{xy};\ q=0 \text{ for } \sigma_{yy};\ q=-3 \text{ for } v \qquad (1.5)$$

From the system for $Q^{(s)}$ all the values are expressed through functions $u^{(s)}(\xi), v^{(s)}(\xi)$, which satisfy the equations

$$E\frac{d^2u^{(s)}}{d\xi^2}=P_x^{(s)}, \quad \frac{1}{3}E\frac{d^4v^{(s)}}{d\xi^4}=P_y^{(s)} \qquad (1.6)$$

where $P_x^{(s)}$, $P_y^{(s)}$ are expressed through $X^{\pm},Y^{\pm},F_x,F_y$ consequently are known functions. The first of the equations (1.6) when $s=0$ coincides with the classic equation of the bars extension-pressure, and the second one coincides with the classical equation of beam bend. Approximations $s\geq 1$ make the results on Bernoulli-Coulomb-Euler classical theory of bars and beams precise. Derivatives of the first order from $u^{(s)}$, the third and the fourth orders from $v^{(s)}$ enter the formulae for stresses, that is why, corresponding to (1.6) the formulae of stresses will involve three arbitrary constants which should be determined from the boundary value conditions when $x=0,\ell$. Naturally, restricted only by the solution of the inner problem, it is impossible to satisfy these conditions at every point, which also indirectly proves singular perturbation of the original problem. In order to remove the arising residual it is necessary to build a qualitatively new solution.

That is the solution of the boundary layer which exponentially decreases when removing from end sections of the rectangle. In order to find the denoted solution near the end – wall $x=0$, a new change of variables is introduced $t=\xi/\varepsilon$ into system (1.2) and the solution of the transformed system is sought in the form of functions of boundary layer type:

$$R_b=\varepsilon^{\chi_b+s}R_b^{(s)}(\zeta)\exp(-\lambda t), \quad s=\overline{0,N}, \quad \mathrm{Re}\lambda>0 \qquad (1.7)$$

As inhomogeneous conditions (1.1) are satisfied by the solution of the inner problem, the boundary layer problem must satisfy the conditions

$$\sigma_{xyb} = \sigma_{yyb} = 0 \quad \text{when} \quad \zeta = \pm 1 \tag{1.8}$$

For deriving noncontradictory system relative to $R_b^{(s)}(\zeta)$, it is necessary to have

$$\chi_{\sigma_s} = \chi, \quad \chi_{u_s} = \chi + 1 \tag{1.9}$$

where the full number χ will be determined during the conjunction of the inner problem solution with the boundary layer solution. After having substituting (1.7) into transformed system (1.2) all the magnitudes may be expressed through $\sigma_{yyb}^{(s)}$:

$$\sigma_{xyb}^{(s)} = \frac{1}{\lambda}\frac{d\sigma_{yyb}^{(s)}}{d\zeta}, \quad \sigma_{xxb}^{(s)} = \frac{1}{\lambda^2}\frac{d\sigma_{yyb}^{(s)}}{d\zeta}$$

$$u_p^{(s)} = -\frac{1}{\lambda^3 E}\left[\frac{d^2\sigma_{yyb}^{(s)}}{d\zeta^2} - \nu\lambda^2\sigma_{yyb}^{(s)}\right] \tag{1.10}$$

$$v_p^{(s)} = -\frac{1}{\lambda^4 E}\left[\frac{d^3\sigma_{yyb}^{(s)}}{d\zeta^3} + (2+\nu)\lambda^2\frac{d\sigma_{yyb}^{(s)}}{d\zeta}\right]$$

$\sigma_{yyb}^{(s)}$ is determined from the equations

$$\frac{d^4\sigma_{yyb}^{(s)}}{d\zeta^4} + 2\lambda^2\frac{d^2\sigma_{yyb}^{(s)}}{d\zeta^2} + \lambda^4\sigma_{yyb}^{(s)} = 0 \tag{1.11}$$

Having solved ordinary differential equation (1.11) and satisfying conditions (1.8), we find the final solution of the boundary layer:

$$\sigma_{yyb}^{(s)} = A_n^{(s)} F_n(\zeta) \tag{1.12}$$

where

$$F_n(\zeta) = \zeta \sin\lambda_n\zeta - tg\lambda_n \cos\lambda_n\zeta \quad \text{(symmetric problem-extension)}$$

$$F_n(\zeta) = \sin\lambda_n\zeta - \zeta tg\lambda_n \cos\lambda_n\zeta \quad \text{(bend)} \tag{1.13}$$

λ_n is the root of the equation $\sin 2\lambda_n + 2\lambda_n = 0$ in the symmetrical problem and the equations $\sin 2\lambda_n - 2\lambda_n = 0$ is the bend problem. In (1.12) $A_n^{(s)}$ are constant integrations by "n" summing takes place corresponding to all the roots λ_n, every λ_n is corresponded by $\bar{\lambda}_r$, totally $\sigma_{yyb}^{(s)}$ will be real.

The solution of boundary layer (1.7), (1.10), (1.12) has a number of very important properties. It is exact for every "s"; in the arbitrary cross-section $t = t_k$ the stresses $\sigma_{xxb}^{(s)}$, $\sigma_{xyb}^{(s)}$ are self-balanced:

$$\int_{-1}^{+1}\sigma_{xxb}^{(s)}d\zeta = 0, \quad \int_{-1}^{+1}\zeta\sigma_{xxb}^{(s)}d\zeta = 0, \quad \int_{-1}^{+1}\sigma_{xyb}^{(s)}d\zeta = 0 \tag{1.14}$$

It is easy to be convinced in justification of (1.14) using formulae (1.10) and conditions (1.8). This solution when removing from the end-wall into the inside the rectangle fades as $\exp(-\operatorname{Re}\lambda_1 t)$, where $\operatorname{Re}\lambda_1 \approx 2{,}106$ in symmetric, $\operatorname{Re}\lambda_1 \approx 3{,}75$ in skew-symmetric (bend) problems [6].

The denoted solution in not possible to obtain on the base of any known hypothesis, particularly, accepting the hypothesis of plane sections, this exact solution is lost.

Using formulae (1.3)-(1.5), (1.7), (1.9) and property (1.14), it is easy to satisfy the boundary value conditions when $x = 0, \ell$. Let the values of stresses be given when $x = 0$:

$$\sigma_{xx} = \varphi(\zeta), \quad \sigma_{xy} = \psi(\zeta) \quad \text{when} \quad x = 0 \tag{1.15}$$

When satisfying conditions with $x = 0$ it is usually ignored by the affect of the boundary layer $R_b^{(2)}$, which is equivalent to fulfillment of the conditions $1 + \exp(-\operatorname{Re}\lambda_1 \ell / h) \approx 1$, which is practically fulfilled even for a square. Then we shall have:

$$\begin{matrix} \varepsilon^{-2+s}\sigma_{xx}^{(s)} + \varepsilon^{\chi+s}\sigma_{xxb}^{(s)} = \varphi \\ \varepsilon^{-1+s}\sigma_{xy}^{(s)} + \varepsilon^{\chi+s}\sigma_{xyb}^{(s)} = \psi \end{matrix} \quad \text{when } x = 0 \ (t = 0) \tag{1.16}$$

From (1.16) noncontradictory conditions only with $\chi = -2$ follow. We have

$$\begin{gathered} \sigma_{xxb}^{(s)}(t = 0) = \varphi^{(s-2)} - \sigma_{xx}^{(s)}(\xi = 0) \\ \sigma_{xyb}^{(s)}(t = 0) = \psi^{(s-2)} - \sigma_{xy}^{(s-1)}(\xi = 0) \\ \varphi^{(0)} = \varphi, \ \varphi^{(k)} = 0 \quad \text{when} \quad k \neq 0, (\varphi, \psi) \end{gathered} \tag{1.17}$$

The right parts of (1.17) must satisfy the conditions of self-balance (1.14). From these three conditions all the three unknown constants in the solution of inner problem are determined. From this fact it follows, that the self-balanced part of the end-wall loading doesn't affect on the solution of the inner problem. This pure mathematical result expresses the validity of Saint-Venant principle. Returning again to (1.17) the right parts of which will already be known functions, constants $A_n^{(s)}$ of the boundary layer solution are determined. Let

$$\varphi(\zeta) = 2P(1 - |\zeta|), \quad \psi(\zeta) = 0 \tag{1.18}$$

the above said may be illustrated in fig. 1

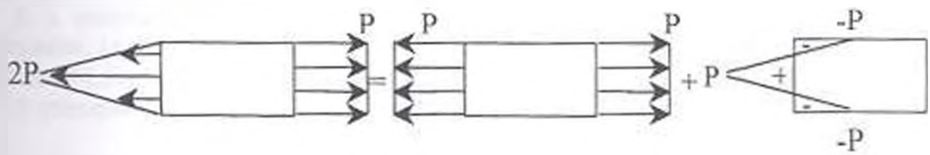

fig.1

where in the right part the first summand corresponds to the solution of the inner problem, the second one corresponds to the boundary layer.

From (1.8), (1.10) follows that the boundary layer displacements don't have the characteristics of self-balance (1.14), i.e. Saint-Venant principle for displacements is not correct and under other boundary value conditions when $x = 0, \ell$, conjunction of the inner problems and the boundary layer solutions is fulfilled by other ways – by the method of boundary collocation, by less squares and so on.

The advantage of an asymptotic method appears under the solution of more complicated problems for thin bodies. With the help of this method solutions of plane problems for an anisotropic strip-rectangle, for layered rectangle-strips are found. In all cases when loadings affecting on facial surfaces are polynomials, iteration process for inner problem terminates on certain approximation and mathematically precise solutions are obtained.

From these solutions as private cases, all the solutions obtained by Menage-Timoshenko method, follow. As an illustration the solution of inner problem for orthotropic rectangle reinforced by stringer, which stretches by the load of permanent intensively will be brought (fig. 2)

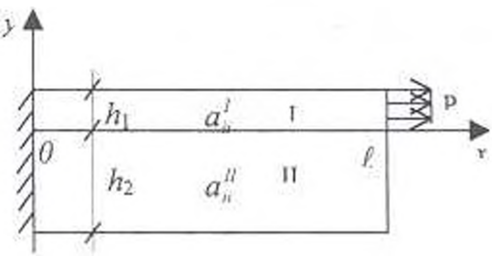

fig.2

$$\sigma'_{xx} = P\zeta_1(B_2 - B_1\zeta)\frac{1}{a'_{11}}, \quad 0 \le \zeta \le \zeta_1$$

$$\sigma'_{xy} = 0, \ \sigma'_{yy} = 0, \ u' = P\ell\zeta_1\xi(B_2 - B_1\zeta)$$

$$v' = \frac{P\ell^2}{2h_2}B_1\zeta_1\xi^2 + \frac{Ph_1}{8}\left[(1-\zeta_1)(4B_2 + (1-\zeta_1)B_1)\frac{a''_{12}}{a''_{11}} + 4\zeta(2B_2 - B_1\zeta)\frac{a'_{12}}{a'_{11}}\right]$$

$$\sigma''_{xx} = P\zeta_1(B_2 - B_1\zeta)\frac{1}{a''_{11}}, \ \sigma''_{xy} = 0, \ \sigma''_{yy} = 0, \ -1 \le \zeta \le 0 \qquad (1.19)$$

$$u'' = P\ell\zeta_1\xi(B_2 - B_1\zeta), \ \ \xi = x/\ell, \ \zeta = y/h_2, \ \zeta_1 = h_1/h_2$$

$$v'' = \frac{P\ell^2}{2h_2}B_1\zeta_1\xi^2 + \frac{Ph_1}{8}(1-\zeta_1+2\zeta)[4B_2 + (1-\zeta_1-2\zeta)B_1]\frac{a''_{12}}{a''_{11}}$$

$$B_1 = \frac{2D_2 - \zeta_1 D_1}{2(D_1D_3 - D_2^2)}, \quad B_2 = \frac{2D_3 - \zeta_1 D_2}{2(D_1D_3 - D_2^2)}$$

$$D_1 = \frac{\zeta_1}{a'_{11}} + \frac{1}{a''_{11}}, \ D_2 = \frac{1}{2}\left(\frac{\zeta_1^2}{a'_{11}} - \frac{1}{a''_{11}}\right), \ D_3 = \frac{1}{3}\left(\frac{\zeta_1^3}{a'_{11}} + \frac{1}{a''_{11}}\right)$$

Solution (1.19) is precise in the sense of Saint-Venant, all the equations of plane problem of an orthotropic body, conditions of full contact between the layers, boundary value problems of the free edge when $y = h_1, y = -h_2$ ($\sigma_{xy} = 0, \sigma_{yy} = 0$) are satisfied. The stresses of the boundary layer σ_{xxb}, σ_{xyb} in spite of layerity in any cross-section are also self-balanced, which permitted us to satisfy the conditions when $x = 0; \ell$ integrally. From solution (1.19) it follows that on the line of contact $\sigma'_{xy} = \sigma''_{xy} = 0$. Meanwhile, in some applied models, for example, in Melan's models it is admitted the stringer behaves as a bar, on the surface of which tangential stresses arise.

The obtained precise solution (1.19) disproves such an assumption and advises to be careful when using applied models. Tangential stresses arise in the zone of boundary layer, but the stress strain state there, is not uniaxial but plane.

The established here qualitative picture is preserved for three-layered rectangular packet as well, i.e. in case of the presence of thin inclusion [15].

The described scheme of the asymptotic solution determination preserves the force for the spatial problems of plates and shells as well. It was found out that the initial approximation for the inner problem is adequate to the classical theory of Kirchhoff-Love's plates and shells. By admitting the hypotheses of classical theory, the boundary layers, which can be of two types-boundary torsion (antiplane boundary layer), plane boundary layer are eliminated. The first of these boundary layers takes into account Reissner, Hambartsumyan's theory by Timoshenko type. If we build the second approximation for the inner problem it will correspond to S.A. Hambartsumyan's iteration theory. For bending of the transversal isotropic plate the following equations relative to the plate deflection are offered:

$$D\Delta\Delta w = z - \left(2\frac{G}{G'} - 0{,}75\frac{E}{E'}\nu'\right)\frac{(2h)^2\Delta z}{10(1-\nu)} \quad \text{(asymptotic theory)}$$

$$D\Delta\Delta w = z - \left(2\frac{G}{G'} - \frac{E}{E'}\nu'\right)\frac{(2h)^2\Delta z}{10(1-\nu)} \quad \text{(Hambartsumyan's theory)} \tag{1.20}$$

For anisotropic plates and shells on the magnitude of the contribution of the following approximations are essentially influenced by the relations of the constants of elasticity, and the changeability of acting loads as well.

2. The second and the mixed boundary value problems

The asymptotic method turned out to be especially effective for the second and mixed boundary value problems solution of elasticity theory for thin bodies (nonclassical boundary value problems of beams, plates and shells). In the first case it is considered on the facial surfaces of the thin body displacements values are given. The punch problem for example refers to this.

In the second case on one of the facial surfaces the displacements values are given and on the other one the values corresponding to the stresses tensor components are given. These problems are basic in the calculation of foundations and bases of constructions by the model of a compressible layer, and during the calculation of seismic actions on the constructions as well. Mixed conditions on each of the facial surfaces may be given.

It is established that the asymptotics (1.4), (1.5) for this class of problems is not admittable, i.e. it is not possible to solve these problems on the base of plane sections and Kirchhoff-Love hypotheses.

A principally new asymptotics is found [7]:

$$q = -1 \text{ for } \sigma_{xx}, \sigma_{xy}, \sigma_{yy};\ q = 0 \text{ for } u, \mathrm{v} \tag{2.1}$$

from where it follows that unlike the classical theory of beams and plates, here in general case all the stresses are equivalent, and the displacements are equivalent too.

In this case what was said above takes place in the case of general anisotropy. Another characteristics has appeared too – it was found out that the inner problem solution is fully determined after having satisfied the boundary value conditions on the facial surfaces, i.e. the boundary layers only correspond to the conditions under torsion sections $x = 0, \ell$. If exterior actions are polynomial closed solutions in the inner problem are obtained. We bring these solutions for two cases, corresponding to when the lower bound of the orthotropic rectangle-strip is rigidly fastened, and the upper bound is informed constant displacements or it is loaded by a load of constant intensivity. The conditions

$$u(-h) = \mathrm{v}(-h) = 0,\ u(h) = u^+,\ \mathrm{v}(h) = \mathrm{v}^+;\ u^+, \mathrm{v}^+ = \mathrm{const} \tag{2.2}$$

correspond to the solution

$$\sigma_{xx} = -\frac{a_{12}}{\Delta}\frac{v^*}{2h},\ \sigma_{xy} = G_{12}\frac{u^*}{2h},\ \sigma_{yy} = \frac{a_{11}}{\Delta}\frac{v^*}{2h}$$

$$u = \frac{u^*}{2h}(y+h),\ v = \frac{v^*}{2h}(y+h),\ \Delta = a_{11}a_{22} - a_{12}^2 \tag{2.3}$$

and the conditions

$$u(-h) = v(-h) = 0,\ \sigma_{xy}(h) = \tau^*,\ \sigma_{yy}(h) = -\sigma_2^*;\ \ \tau^*\sigma_2^* = \text{const} \tag{2.4}$$

correspond to the solution

$$\sigma_{xx} = \frac{a_{12}}{a_{11}}\sigma_2^*,\ \ \sigma_{xy} = \tau^*,\ \sigma_{yy} = -\sigma_2^*$$

$$u = \frac{\tau^*}{G_{12}}(y+h),\ \ v = -\frac{\Delta}{a_{11}}\sigma_2^*(y+h) \tag{2.5}$$

From the solution (2.5) the value of the bed coefficient for an elastic foundation of power (width) $2h$ is directly followed. Taking into account $\tau^* = 0$, we calculate the displacement under normal loading. With $y = h$ we have

$$u(h) = 0,\ \ v(h) = \frac{2\Delta h}{a_{11}}(-\sigma_2^*) \tag{2.6}$$

from where it follows

$$\sigma_2(h) = -\sigma_2^* = Kv(h),\ \text{K} = \frac{a_{11}}{2h\Delta} = \frac{E_2}{2h(1-\nu_{12}\nu_{21})} \tag{2.7}$$

For isotropic foundations coefficient K coincides with well-known bed coefficient $K = E/(2h(1-\nu^2))$. Note that in case of foundations with general anisotropy from the asymptotic solution nonapplicability of Vinkler's model follows, i.e. the sense of bed coefficient is lost.

Asymptotics (2.1) is right for layered and for inhomogeneous beams as well [6]. If Young`s module changes by the depth of the layer of power h linearly, using the values $[E_1, E_2]$, for bed coefficient we have

$$K = K_0\frac{c-1}{\ln c},\ c = \frac{E_2}{E_1},\ \ K_0 = \frac{E_1}{h(1-\nu^2)} \tag{2.8}$$

Asymptotics (1.4), (2.1) permits generalizing on anisotropic layered plates and shells, it is possible to get closed solutions [8-10].

Using the solution of the mixed boundary value problem for n-layered packet from orthotropic plates, in particular, it is possible to get the following formula of bed coefficient calculation

$$K_n = \frac{1}{\sum_{i=1}^{n} h_i A_{33}^{(i)}}$$

$$A_{33}^{(i)} = \frac{1-\nu_{12}^{(i)}\nu_{21}^{(i)} - \nu_{13}^{(i)}\nu_{31}^{(i)} - \nu_{23}^{(i)}\nu_{32}^{(i)} - \nu_{21}^{(i)}\nu_{13}^{(i)}\nu_{32}^{(i)} - \nu_{12}^{(i)}\nu_{23}^{(i)}\nu_{31}^{(i)}}{(1-\nu_{12}^{(i)}\nu_{21}^{(i)})E_3^{(i)}} \tag{2.9}$$

where $E_3^{(i)}$ is Young's module in the direction, perpendicularly to the plate of the layers contact, $\nu_{jk}^{(i)}$ is Poisson's coefficient.

3. Free and forced vibrations

The investigations of seismic waves actions of thin and massive bodies and so on bring to the solution of specific problems on free and forced vibrations of thin bodies. Asymptotic method shows off here too from the best side. We illustrate the above said on the example of free vibrations of an orthotropic plate $D=\{(x,y,z): x,y\in D_0, |z|\le h\}$. It is required to determine the frequencies of free vibrations of the plate, corresponding to boundary conditions

$$\begin{aligned} &\sigma_{xz}=\sigma_{yz}=\sigma_{zz}=0 \quad \text{when} \quad z=h \\ &u=\mathrm{v}=w=0 \qquad \text{when} \quad z=-h \end{aligned} \tag{3.1}$$

or

$$u=\mathrm{v}=w=0 \qquad \text{when} \quad z=\pm h \tag{3.2}$$

Looking for the solution of the equations system of special dynamic problem of elasticity theory in the form of

$$\begin{aligned} &\sigma_{\alpha\beta}(x,y,z,t)=\sigma_{jk}(x,y,z)\exp(i\omega t) \\ &(u,\mathrm{v},w)=(u_x,u_y,u_z)\exp(i\omega t),\ \alpha,\beta=x,y,z;\ j,k=1,2,3 \end{aligned} \tag{3.3}$$

where ω is the sought frequency of free vibrations and passing to dimensionless coordinates and components of displacement vector $\xi=x/\ell,\ \eta=y/\ell,\ \zeta=z/h$, $U=u_x/\ell,\ V=u_y/\ell,\ W=u_z/\ell$, ℓ is the characteristic size of the middle surface D_0 of the plate, we have the following singularly perturbed by small parameter $\varepsilon=h/\ell$ system:

$$\begin{aligned} &\frac{\partial\sigma_{11}}{\partial\xi}+\frac{\partial\sigma_{12}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{13}}{\partial\zeta}+\varepsilon^{-2}\omega_*^2U=0 \\ &\frac{\partial\sigma_{12}}{\partial\xi}+\frac{\partial\sigma_{22}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{23}}{\partial\zeta}+\varepsilon^{-2}\omega_*^2V=0 \\ &\frac{\partial\sigma_{13}}{\partial\xi}+\frac{\partial\sigma_{23}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{33}}{\partial\zeta}+\varepsilon^{-2}\omega_*^2W=0 \\ &\frac{\partial U}{\partial\xi}=a_{11}\sigma_{11}+a_{12}\sigma_{22}+a_{13}\sigma_{33} \\ &\frac{\partial V}{\partial\eta}=a_{12}\sigma_{11}+a_{22}\sigma_{22}+a_{23}\sigma_{33} \\ &\varepsilon^{-1}\frac{\partial W}{\partial\zeta}=a_{13}\sigma_{11}+a_{23}\sigma_{22}+a_{33}\sigma_{33} \\ &\varepsilon^{-1}\frac{\partial V}{\partial\zeta}+\frac{\partial W}{\partial\eta}=a_{44}\sigma_{23},\ \ \varepsilon^{-1}\frac{\partial U}{\partial\zeta}+\frac{\partial W}{\partial\xi}=a_{55}\sigma_{13} \\ &\frac{\partial U}{\partial\eta}+\frac{\partial V}{\partial\xi}=a_{66}\sigma_{12},\quad \omega_*^2=\rho h^2\omega^2 \end{aligned} \tag{3.4}$$

The solution of the singularly perturbed system (3.4) is sought in the form of

$$\begin{aligned} &\sigma_{jk}=\varepsilon^{-1+s}\sigma_{jk}^{(s)},\quad \omega_*^2=\varepsilon^s\omega_{*s}^2,\quad s=\overline{0,N} \\ &(U,V,W)=\varepsilon^s(U^{(s)},V^{(s)},W^{(s)}) \end{aligned} \tag{3.5}$$

where notation $s=\overline{0,N}$ means that by repeated index s summing in the limits $[0,N]$ takes place. Substituting (3.5) into (3.4), using Cauchy's rule of multiplying the rows, we get a new system, from where it will be possible to express the stresses tensor components through $U^{(s)}, V^{(s)}, W^{(s)}$, which, in their turn, are determined from the equations system

$$\frac{\partial^2 U^{(s)}}{\partial\zeta^2}+a_{55}\omega_{*k}^2 U^{(s-k)}=R_u^{(s)},\quad k=\overline{0,s}$$

$$\frac{\partial^2 V^{(s)}}{\partial\zeta^2}+a_{44}\omega_{*k}^2 V^{(s-k)}=R_v^{(s)} \tag{3.6}$$

$$A_{11}\frac{\partial^2 W^{(s)}}{\partial\zeta^2}+\omega_{*k}^2 W^{(s-k)}=R_w^{(s)}$$

where $A_{11}=1/A_{33}$, $R_u^{(s)}, R_v^{(s)}, R_w^{(s)}$ are well-known functions, in particular, $R_w^{(0)}=R_v^{(0)}=R_u^{(0)}=0$. With $s=0$ the equations of system (3.6) become independent. Having solved these equations and satisfied conditions (3.1) or (3.2), we get dispersing equations from where ω_{0n} are determined. Conditions (3.1) correspond the following three groups of main values of frequencies:

$$\omega_{0n}^{I}=\frac{\pi}{4h}\sqrt{\frac{G_{13}}{\rho}}(2n+1)=\frac{\pi}{4h}V_c^{xz}(2n+1),\ V_c^{xz}=\sqrt{\frac{G_{13}}{\rho}},\ n\in N$$

$$\omega_{0n}^{II}=\frac{\pi}{4h}V_c^{yz}(2n+1),\ V_c^{yz}=\sqrt{\frac{1}{\rho a_{44}}}=\sqrt{\frac{G_{23}}{\rho}},\ n\in N \tag{3.7}$$

$$\omega_{0n}^{III}=\frac{\pi}{4h}V_p(2n+1),\ V_p=\sqrt{\frac{A_{11}}{\rho}}=\sqrt{\frac{1}{\rho A_{33}}},\quad n\in N$$

where G_{jk} is shear module, A_{33} is calculated by formula (2.9) without ascribing index i, V_c^{xz}, V_c^{yz} are well-known in seismology and physics velocities of propagation of shear waves. Formulae (3.7) show that in the orthotropic plate free vibrations of three types – two shear and a longitudinal may arise. Their interinfluence will be perceived taking into account the approximations $s\geq 1$. The calculation of the next approximations brings to the correction of the frequency value of the order $O(\varepsilon^2)$, that is why in practical applications it is possible to be restricted by the values (3.7), which we call main values of frequencies.

Conditions (3.2) correspond to the following main values of frequencies:

$$\omega_{0n}^{I}=\frac{\pi n}{2h}V_c^{xz},\ \omega_{0n}^{II}=\frac{\pi n}{2h}V_c^{yz},\ \omega_{0n}^{III}=\frac{\pi n}{4h}V_p,\ n\in N \tag{3.8}$$

The forced vibrations are considered in the analogues way. For example, if harmonically changing in time displacements are informed to the facial surface $z=-h$ of the plate

$$u(-h) = u^{-}(\xi,\eta)\exp(i\Omega t),\ v(-h) = v^{-}(\xi,\eta)\exp(i\Omega t)$$
$$w(-h) = w^{-}(\xi,\eta)\exp(i\Omega t) \quad (3.9)$$

which take place under seismic actions on plate-like bases of constructions, the solution of the problems is sought in the form of (3.3), then (3.5) with substitution ω into Ω, ω_*^2 into $\Omega_*^2 = \rho h^2\Omega^2$. As a result the solution is expressed through the functions $U^{(s)}, V^{(s)}, W^{(s)}$, each of which is determined from the ordinary differential equations of the second order. Subjecting the solutions of these equations to the conditions (3.1), (3.9) or conditions (3.9) and $u(h) = v(h) = w(h) = 0$ the amplitudes of forced vibrations are uniquely determined. It the value of frequency Ω of the exterior action coincides with any value from (3.7) or (3.8) a resonance takes place. Note, that it is always possible to choose physical-mechanical and geometrical parameters of the plate so, that in the presence of the given interval of possible values Ω the resonance didn't arise.

Note, that the described scheme of the frequencies determination of free vibrations and amplitudes of forced vibrations is applicable for layered thin bodies as well.

4. Conclusions

Effectiveness of asymptotic method of singularly perturbed differential equations solution for the solution of boundary and dynamic problems of elasticity theory for thin bodies (beams, bars, plates, shells). Connection of asymptotic method with Saint-Venant principle. with applied theories of beams, plates and shells is established. Nonclassical problems of thin bodies are solved. The frequencies of free vibrations and the amplitudes of forced vibrations of orthotropic beams and plates to the corresponding nonclassical boundary value problems are determined.

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