ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱՅԻ ՏԵՂԵԿԱԳԻՐ

ИЗВЕСТИЯ НАЦИОНАЛЬНОЙ АКАДЕМИИ АРМЕНИИ

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ELASTIC WAVES PROPAGATION IN AN ELASTIC LAYER WITH CUBIC ANISOTROPIC PROPERTIES

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Առաձգական ալիքների տարածումը խորանարդային սիմետրիայի հատկություններով օժտված անիզոտրոպ շերտում

Քննարկվում է առաձգական ալիքների տարածումը շերտում, որի նյութը ունի խորանարդային բյուրեղի հատկություններ: Ստացված է խնդրի դիսպերսիոն հավասարումը կապակցված երկայնական և լայնական ալիքների փուլային արագությունների նկատմամբ:

Երկար ալիքների մոտավորությամբ ստացված են անջատված հավասարումներ, որոնք որոշում են հեծանի լայնական տատանումների հաճախությունները և ծռում երկայնական ալիքի փուլային արագությունը: Կարճ ալիքների մոտավորությամբ խնդիրը հանգում է Ռելեյի տիպի մակերևութային ալիքի ուսումնասիրությանը: Հետազոտված է բյուրեղի անիզոտրոպ հատկության ազդեցությունը:

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Распространение упругих волн в слое с анизотропными свойствами кубической симметрии

Обсуждается задача распространения упругих волн в слое из материала со свойствами кубического кристалла. Получено дисперсионное уравнение задачи относительно связанных фазовых скоростей продольных и поперечных волн.

В длинноволновом приближении получены раздельные уравнения относительно частот колебаний поперечных и продольных волн балки. В коротковолновом приближении задача сводится к изучению поверхностных волн Рэлея. Исследовано влияние анизотропных свойств кристалла.

This paper discusses a dinamic problem of an elastic wave propogation along an infinite layer with anisotropic properties of a single cubic crystal. The dispersion equation are deduced relating the coupled SP and P waves patterns phase velocities. In long a wave approximation the beam flexural and extensional vibration equations are obtained. In a short waves approximation the equation of the surface Relegh type waves is deduced. By means of the Kirchoff theory of plate the two dimensional dynamic equation of anisotropic plate is deduced. A comparition between the obtained results is carried out and effect of anisotropy is studied.

STATEMENT OF THE PROBLEM

In the fixed Cartesian laboratory system (X_1, X_2, X_3) the governing equations are listed as follows:

Constitutive equations [7]

$$\sigma_{ik} = (c_{11} - c_{12})s_{ik} + c_{12}s; \quad (i = k)$$

$$(i\,;k = 1,2,3)$$

$$\sigma_{ik} = 2c_{44}s_{ik}; \qquad (i \neq k) \tag{1}$$

$$s = s_{11} + s_{22} + s_{33}$$

The strain- mechanical displacements equations

$$s_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i}) \tag{2}$$

The equation of motion

$$\sigma_{ik,k} = \rho u_{i,tt} \tag{3}$$

In the above σ_{ik}, s_{ik}, u_i are components of the stress and strain tensors and displacement vector. Material properties are the elastic modulus c_{11}, c_{12}, c_{44}; the mass density ρ

We consider the infinite elastic layer of the 2d thickness.

We take the Cartesian laboratory system (X_1, X_2, X_3) coincident with the crystallophysics system. The X_1, X_2 axis are directed along the interface. The X_3 axis is directed perpendicular to the interface of the layer.

It is supposed that interfaces $X_3 = \pm d$ are free from mechanical stresses.

From (1-3), it follows that the coupled SP, P- wave patterns are separated from SH - wave pattern.

SOLUTION OF THE PROBLEM

Later we will investigate the plane deformation case (the coupled SP, P-wave patterns).

Taking $X_1 = X$, $X_3 = Z$, $u_1 = u(X, Z)$, $u_3 = v(X, Z)$, from (1-3) we obtain the following governing equation with respect to the displacements U,V

$$c_l^2 u_{,xx} + c_t^2 u_{,zz} + \left[c_l^2 - c_t^2(2\gamma - 1)\right] v_{,xz} = u_{,tt}$$
$$c_t^2 v_{,xx} + c_l^2 v_{,zz} + \left[c_l^2 - c_t^2(2\gamma - 1)\right] u_{,xz} = v_{,tt} \tag{4}$$

At the free $X_3 = \pm d$ interfaces we have the following boundary conditions

$$\sigma_{xz} = 0; \quad \sigma_{zz} = 0$$
$$u_{,z} + v_{,x} = 0, \quad v_{,z} + (1 - 2\vartheta\gamma)u_{,x} = 0 \tag{5}$$

In the (4-5) $c_l = (c_{11} / \rho)^{1/2}$ is the velocity of the p wave. $c_t = (c_{44} / \rho)^{1/2}$ is the velocity of the SP wave. $\vartheta = c_t^2 / c_l^2 = c_{44} / c_{11}$; $\gamma = (c_{11} - c_{12}) / 2c_{44}$ is the coefficient of anisotropy.

Taking solution of (4) in the form of the plane monochromatic waves

$$u = u_0 \exp[i(kX - \omega t)]; \quad v = v_0 \exp[i(kX - \omega t)] \tag{6}$$

(k is the wave number, ω is the frequency of the wave) we come to the set of ordinary differential equations

$$\vartheta u_{0,zz} - k^2(1-\beta\vartheta)u_0 + ik\left[1-\vartheta(2\gamma-1)\right]v_{0,z} = 0$$

$$v_{0,zz} - \vartheta k^2(1-\beta)v_0 + ik\left[1-\vartheta(2\gamma-1)\right]u_{0,z} = 0 \qquad (7)$$

where $\beta = \omega^2 / k^2 c_t^2$.

The solution of (7) can be written in the form

$$v_0 = A_1 \mathrm{sh}(kp_1Z) + A_2 \mathrm{ch}(kp_1Z) + A_3 \mathrm{sh}(kp_2Z) + A_4 \mathrm{ch}(kp_2Z) \qquad (8)$$

$$u_0 = \alpha_1 A_1 \mathrm{sh}(kp_1Z) + \alpha_1 A_2 \mathrm{ch}(kp_1Z) + \alpha_2 A_3 oh(kp_2Z) + \alpha_2 A_4 oh(kp_2Z)$$

$$\alpha_j = \frac{p_j^2 - \vartheta(1-\beta)}{p_j\left[1-\vartheta(2\gamma-1)\right]} \qquad (j = 1;2)$$

In (8) $\pm p_1$, $\pm p_2$ are the roots of the characteristic equation

$$p^4 - p^2\left[2(2\gamma-1) - \beta(1+\vartheta) + 4\vartheta\gamma(1-\gamma)\right] + (1-\beta\vartheta)(1-\beta) = 0 \qquad (9)$$

In the case, of an antisymmetry with respect to plane Z=0, when $v_0(Z)$ is the even function and $u_0(Z)$ is an odd function, according to the boundary conditions (5) we come to the following dispersion equation

$$\frac{\mathrm{th}(kp_1d)}{\mathrm{th}(kp_2d)} = \frac{f_1 g_2 p_1}{f_2 g_1 p_2} \qquad (10)$$

where

$$f_j = p_j^2 + (1-2\vartheta\gamma)(1-\beta); \quad g_j = p_j^2 + (1-2\vartheta\gamma+\vartheta\beta)$$

Let us consider the limiting cases.

If the length of wave is $l = 2\pi / K >> d$

then taking

$$\mathrm{th}(kp_jd) \cong kp_jd\left[1 - \frac{1}{3}(kp_jd)^2\right] \qquad (11)$$

and using eq.(10) , we come to

$$\beta(1+\vartheta-2\vartheta\gamma) = \frac{1}{3}(kd)^2\Big[p_1^2p_2^2 + (p_1^2+p_2^2)(1-2\vartheta\gamma+\vartheta\beta) + \qquad (12)$$

$$+(1-2\vartheta\gamma+\vartheta\beta)(1-2\vartheta\gamma)(1-\beta)\Big]$$

From eq. (9) we have

$$p_1^2p_2^2 = (1-\beta\vartheta)(1-\beta); \quad p_1^2+p_2^2 = 2(2\gamma-1) - \beta(1+\vartheta) + 4\vartheta\gamma(1-\gamma)$$

Then according to (10), the equation (12) can be written as

$$\beta - \frac{k^2d^2}{3}\left[4\gamma(1-\vartheta\gamma) - (3-4\vartheta\gamma)\beta - \beta^2\vartheta\right] = 0 \qquad (13)$$

Introducing the new variable $\zeta = \omega^2 / k^2c_l^2 \; (\beta = \zeta / \vartheta)$,we can rewrite eq (13) in the form

$$\zeta - \frac{k^2d^2}{3}\left[4\gamma\vartheta(1-\vartheta\gamma) - (3-4\vartheta\gamma)\zeta - \zeta^2\right] = 0 \qquad (14)$$

This equation determines the first mode of flexural (bending) vibration of the layer.

Since $\gamma\vartheta = (c_{11} - c_{12}) / 2c_{11}$, this equation does not depend upon the coefficient c_{44}

Using technical notations

$$c_{11} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}; \quad c_{12} = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad c_{44} = \tilde{G} \qquad (15)$$

where E is the Young modulus, ν is Poisson ratio, $\bar{G}$ is shear modulus.

($\bar{G} = E / 2(1+\nu)$ for isotropic materials)

we can rewrite eq. (14) as follows

$$\rho\omega^2 - \frac{Ek^4d^2}{3(1-\nu^2)} + \underline{\frac{\rho^2\omega^4(1+\nu)(1-2\nu)d^2}{3(1-\nu)E} + \frac{k^2d^2\rho\omega^2(1+\nu)}{3(1-\nu)}} = 0 \qquad (16)$$

Taking $-i\omega = \delta / \delta t$, $ik = \delta / \delta X$, we can rebuild the beam flexural vibration equation

$$\frac{2Ed^3}{3(1-\nu^2)} w_{,xxxx} + 2\rho d w_{,tt} - \underline{\frac{2(1+\nu)d^3\rho}{3(1-\nu)} w_{,xxtt}} - \underline{\frac{2(1+\nu)(1-2\nu)d^3\rho^2}{3(1-\nu)E} w_{,tttt}} = 0 \qquad (17)$$

In eqs.(16,17) the underlined expressions are negligible ones and in general they are not taken into account.

Thus, for the cubic crystals in a beam approximation we have the classic dynamics equation of isotropic media

In a short wave approximation, taking instead of the hyperbolic functions the value one, we can obtain the equation determining the surface wave dimensionless phase velocity

$\xi = \omega / kc_t$

$$\xi^2 = \frac{4\gamma(1-\vartheta)\sqrt{1-\xi^2}}{\sqrt{1-\xi^2} + \sqrt{1-\xi^2\vartheta}} \qquad (18)$$

The necessary and sufficient condition for the existance of the surface wave in the cubic crystal is $\vartheta < 1$.

For the isotropic materials this condition is always satisfied, since $c_t < c_l$ For the cubic crystal, this condition may be not take place.

If $\vartheta \geq 1$ the equation (18) has not real roots in the interval $0 < \xi < \vartheta^{-1/2}$

Let us consider the symmetrical case , when $v_0(Z)$ is the odd function and $u_0(Z)$ is the even function

Then, according to the boundary conditions (5) we have the following dispersion equation

$$\frac{\mathrm{th}(kp_2 d)}{\mathrm{th}(kp_1 d)} = \frac{f_1 g_2 p_1}{f_2 g_1 p_2} \qquad (19)$$

In a long wave approximation the equation (19) coincides with the equation of an isotropic beam.

In the short wave approximation we come to the eq. (18)

EQUATIONS OF A THIN PLATE

Now we will obtain the equations of a thin plate using the Kirchoff theory of plate. The investigation shall be carried out in a laboratory orthogonal system having one common axis with the crystallophysics system (X_1, X_2, X_3).

We take the rectangular Cartesian laboratory system (X, Y, Z) in such a way the coordinate plane (X, Y) coincides with the middle plane of the plate. The Z axis is directed along the normal to the plate middle plane and is coincides with the X_3 axis. The X, Y axes are twisted out on φ angle with respect to X_2, X_1 axes.

In the new system the constitutive equations have the form [6]

$$\sigma_{xx} = c_{11}s_{xx} + c_{12}(s_{yy} + s_{zz}) - \frac{c}{2}\left[(s_{xx} - s_{yy})\sin^2 2\varphi + s_{xy}\sin 4\varphi\right]$$

$$\sigma_{yy} = c_{11}s_{yy} + c_{12}(s_{xx} + s_{zz}) + \frac{c}{2}\left[(s_{xx} - s_{yy})\sin^2 2\varphi + s_{xy}\sin 4\varphi\right]$$

$$\sigma_{xy} = 2c_{44}s_{xy} - c\left[\frac{1}{4}(s_{yy} - s_{xx})\sin 4\varphi - s_{xy}\sin^2 2\varphi\right]$$

$$\sigma_{zz} = c_{11}s_{zz} + c_{12}(s_{xx} + s_{yy}) \tag{20}$$

$$\sigma_{xz} = 2c_{44}s_{xz}; \quad \sigma_{yz} = 2c_{44}s_{yz};$$

$$c = c_{11} - c_{12} - 2c_{44}$$

Taking assumptions of the Kirchoff theory

$$s_{xz} = s_{yz} = s_{zz} = 0; \tag{21}$$

$$u_x = -Zw_{,x}; \quad v_y = -Zw_{,y}; \quad u_z = w(X, Y)$$

and supposing σ_{zz} to be small, we can obtain the following constitutive equation concerning the stresses σ_{xx}, σ_{yy}, σ_{xy} acting in the plate middle plane

$$\sigma_{xx} = -\left(B'_{11}w_{,xx} + B'_{12}w_{,yy} + 2B'_{16}w_{,xy}\right)$$

$$\sigma_{yy} = -\left(B'_{12}w_{,xx} + B'_{11}w_{,yy} + 2B'_{26}w_{,xy}\right) \tag{22}$$

$$\sigma_{xy} = -\left(B'_{16}w_{,xx} + B'_{26}w_{,yy} + 2B'_{44}w_{,xy}\right)$$

In these equations the following notations are used

$$B'_{11} = B_{11} - \frac{1}{2}\sin^2 2\varphi B_0; \quad B'_{22} = B_{11} - \frac{1}{2}\sin^2 2\varphi B_0$$

$$B'_{12} = B_{12} + \frac{1}{2}\sin^2 2\varphi B_0; \quad B'_{44} = B_{44} - \frac{1}{2}\sin^2 2\varphi B_0 \tag{23}$$

$$B'_{16} = -\frac{1}{4}\sin 4\varphi B_0; \quad B'_{26} = \frac{1}{4}\sin 4\varphi B_0$$

where

$$B_0 = B_{11} - B_{12} - 2B_{44}$$

$$B_{11} = B_{22} = \left(c_{11}^2 - c_{12}^2\right)/c_{11};$$

$$B_{12} = c_{12}\left(c_{11} - c_{12}\right)/c_{11}$$

$$B_{44} = c_{44}$$

The plate bending vibration equation has the form

$$M_{x,xx} + 2H_{,xy} + M_{y,yy} = \rho w_{,tt} \tag{24}$$

where

$$M_x = \int_{-d}^{d}\sigma_{xx} Z dZ; \quad M_y = \int_{-d}^{d}\sigma_{yy} Z dZ; \quad H = \int_{-d}^{d}\sigma_{xy} Z dZ$$

are bending and torque moments

Using technical coefficients we have

$$M_x = -D\left(w_{,xx} + \nu w_{,yy}\right) + \frac{1}{2}\bar{D}\sin^2 2\varphi\left(w_{,xx} - w_{,yy}\right) + \frac{1}{2}\bar{D}\sin 4\varphi w_{,xy} \tag{25}$$

$$M_y = -D\left(\nu w_{,xx} + w_{,yy}\right) - \frac{1}{2}\bar{D}\sin^2 2\varphi\left(w_{,xx} - w_{,yy}\right) - \frac{1}{2}\bar{D}\sin 4\varphi w_{,xy}$$

$$H = -\left(\frac{4\tilde{G}d^3}{3} + \tilde{D}\sin^2 2\varphi\right)\omega_{,xy} + \frac{1}{4}\tilde{D}\sin 4\varphi\left(w_{,xx} - w_{,yy}\right)$$

where

$$D = \frac{2d^3E}{3(1-\nu^2)}; \quad \tilde{D} = \frac{2d^3}{3}\left(\frac{E}{1+\nu} - 2\tilde{G}\right)$$

Finally, the plate vibration equation has the form

$$D\left(w_{,xxxx} + 2\left(1 - \tilde{D}/D\right)w_{,xxyy} + w_{,yyyy}\right) -$$

$$-\frac{\tilde{D}}{2}\sin^2 2\varphi\left(w_{,xxxx} - 2w_{,xxyy} + w_{,yyyy}\right) - \tag{26}$$

$$-\tilde{D}\sin 4\varphi\left(w_{,xxxy} - w_{,xyyy}\right) + 2\rho d w_{,tt} = 0$$

For isotropic material it is necessary to take $\bar{D} = 0$

The beam equation has the form

$$\left(D - \frac{\tilde{D}}{2}\sin^2 2\varphi\right)w_{,xxxx} + 2\rho d w_{,tt} = 0 \tag{27}$$

When $\varphi = 0$, we come to eq. (17).

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