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On a Convergence of the Modified Fourier-Pade Approximations

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Abstract. We consider convergence acceleration of the modified Fourier expansions by trigonometric-rational corrections which lead to the modified Fourier-Pade approximations. Exact constants of the asymptotic errors are derived for smooth functions and the comparison with the corresponding errors of the modified Fourier expansions is performed.

Key Words: Modified Fourier Expansion, Convergence Acceleration, Rational Approximation
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Introduction

The modified Fourier basis

$$\mathcal{H} = \{\cos \pi n x : n \in \mathbb{Z}_+\} \cup \{\sin \pi(n - \tfrac{1}{2})x : n \in \mathbb{N}\}$$

was originally proposed by Krein [1] and then thoroughly investigated in a series of papers [2–9]. The set $\mathcal{H}$ is an orthonormal basis of $L_2[-1,1]$ ([2]), as $\mathcal{H}$ consists of eigenfunctions of the Sturm-Liouville operator $\mathcal{L} = -d^2/dx^2$ with Neumann boundary conditions $u'(1) = u'(-1) = 0$. Both the orthogonality and the density in $L_2[-1,1]$ follow from the classical spectral theory ([10]).

Let $\mathcal{M}_N(f,x)$ be the truncated modified Fourier series

$$\mathcal{M}_N(f,x) = \frac{1}{2} f_0^c + \sum_{n=1}^{N} [f_n^c \cos \pi n x + f_n^s \sin \pi(n - \tfrac{1}{2})x], \tag{1}$$

where

$$f_n^c = \int_{-1}^{1} f(x) \cos \pi n x dx, \;\; f_n^s = \int_{-1}^{1} f(x) \sin \pi(n - \tfrac{1}{2}) x dx. \tag{2}$$

Important property of $\{f_n^c\}$ and $\{f_n^s\}$, which explains the better convergence of the modified expansions as compared to the classical truncated Fourier series, is the faster decay of the coefficients

$$f_n^c, f_n^s = O(n^{-2}),\ n \to \infty$$

for smooth but non-periodic functions. The benefit of using such expansions to approximate a smooth but non-periodic function f is a faster convergence rate. Moreover the convergence is uniform in $[-1, 1]$ and there is no Gibbs phenomenon on the boundary.

The first results on the pointwise convergence of the modified series were proved in [2]:

Theorem 1 *[2] Suppose that f is Riemann integrable in $[-1, 1]$ and that*

$$f_n^c, f_n^s = O(n^{-1}),\ \ n \gg 1.$$

If f is Lipschitz at $x \in (-1, 1)$ then $\mathcal{M}_N(f, x) \to f(x)$ as $N \to \infty$. Moreover, this progression to a limit is uniform in $[\alpha, \beta]$, where $-1 < \alpha < \beta < 1$, provided that $f \in C[\alpha, \beta]$.

Theorem 2 *[2] If f is an odd and analytic function, then $M_N(f, x)$ uniformly converges to $f(x)$ in $[-1, 1]$ and, moreover,*

$$\mathcal{M}_N(f, \pm 1) = f(\pm 1) + O(N^{-1}),\ N \to \infty.$$

Given that $\mathcal{M}_N(f, \pm 1) = f(\pm 1) + O(N^{-1})$ for any even and analytic f in $[-1, 1]$, we conclude that the modified Fourier expansions for all analytic functions converge at the endpoints by the rate $O(N^{-1})$ ([2]).

Olver in [6] proved the convergence of the modified expansions at the endpoints without requiring that f is analytic:

Theorem 3 *[6] Suppose that $f \in C^2[-1, 1]$ and f'' has bounded variation. Then*

$$\mathcal{M}_N(f, \pm 1) \to f(\pm 1).$$

Olver also proved that the convergence rate of the modified expansions is $O(N^{-2})$ away from the endpoints:

Theorem 4 *[6] Suppose that $f \in C^2[-1, 1]$, f'' has bounded variation. If $x \in (-1, 1)$, then*

$$f(x) - \mathcal{M}_N(f, x) = O(N^{-2}),\ \ |x| < 1.$$

Under some additional requirements, the convergence rate is faster:

Theorem 5 *[5, 6] Suppose that $f \in C^{2k+2}(-1,1)$, $f^{(2k+2)}$ has bounded variation and f obeys the first k derivative conditions*

$$f^{(2r+1)}(\pm 1) = 0,\ r = 0, \dots, k-1. \tag{3}$$

Then, the error

$$f(x) - \mathcal{M}_N(f,x) = O(N^{-2k-2}),$$

uniformly for x in compact subsets of $(-1,1)$.

We see that the convergence rate remains slow if the function f does not obey the first derivative conditions (3). The uniform error is $O(N^{-1})$ on $[-1,1]$ and $O(N^{-2})$ inside. This is due to function jumps in certain derivatives at the endpoints $x = \pm 1$. If these jumps are known, the convergence acceleration can be achieved by well-known polynomial subtraction approach. For the classical Fourier series this approach has a very long history (see [9, 11–17]). For modified expansions these approach is explored in [4, 8, 9]. More specifically, we write f (see [9]) in the terms of its Lanczos representation

$$f = (f - g_k) + g_k,$$

where g_k is chosen such that conditions

$$f^{(2r+1)}(\pm 1) = g_k^{(2r+1)}(\pm 1),\ r = 0, \dots, k-1.$$

Since $f - g_k$ obeys the first k derivative conditions, the new approximation

$$\mathcal{M}_N^k(f,x) = \mathcal{M}_N(f - g_k, x) + g_k$$

will converge with the same rate as if f obeyed those conditions. This is the polynomial subtraction technique known also as Krylov-Lanczos approach.

Adcock in [5, 9] explored the convergence of the modified expansions in the Sobolev spaces $H^q, q > 0$. In particular, he proved that the modified basis is dense in $H^1(-1,1)$ and $\|f - \mathcal{M}_N(f,x)\|_\infty \to 0$ as $N \to \infty$ for $\forall f \in H^1(-1,1)$.

Multivariate modified Fourier expansions were investigated in [3, 7–9].

In this paper, we consider convergence acceleration of the truncated modified Fourier series along the ideas of the Fourier-Pade approximations ([18–21]).

The paper is organized as follows. Section 1 provides exact constants of the asymptotic errors for the modified expansions. Both L_2 and pointwise (see also [9] for similar results) convergences are considered. We need those results for further comparisons. Section 2 explains the construction of the rational approximations by application of trigonometric-rational corrections to the errors of modified expansions. Rational corrections contain some unknown parameters. Their determination is carried out along the ideas of the Fourier-Pade approximations. That is why those approximations are named as modified Fourier-Pade approximations. Sections 3 and 4 investigate the pointwise and L_2 convergence, correspondingly. Section 5 gives some concluding remarks.

1 Convergence of the Modified Fourier Expansions

In this section, we derive exact estimates for the asymptotic errors of the modified Fourier expansions that we need for further comparisons. We focus our attention to $L_2(-1,1)$ and pointwise convergence on $(-1,1)$.

Let $f^{(2q+1)} \in C[-1,1]$, $q \geq 0$ and denote

$$A_{2k+1}(f) = \left(f^{(2k+1)}(1) - f^{(2k+1)}(-1)\right)(-1)^k,\ k = 0,\dots,q,$$
$$B_{2k+1}(f) = \left(f^{(2k+1)}(1) + f^{(2k+1)}(-1)\right)(-1)^k,\ k = 0,\dots,q.$$

The following lemmas (see also [2, 6, 7] for similar estimates) are the cornerstones for all asymptotic expansions provided in this paper.

Lemma 1 *Assume that $f^{(2q+1)} \in AC[-1,1]$, $q \geq 0$. Then, the following asymptotic expansions are valid*

$$f_n^c = (-1)^n \sum_{k=0}^{q} \frac{A_{2k+1}(f)}{(\pi n)^{2k+2}} + o(n^{-2q-2}),\ n \to \infty,$$

$$f_n^s = (-1)^{n+1} \sum_{k=0}^{q} \frac{B_{2k+1}(f)}{(\pi(n-\frac{1}{2}))^{2k+2}} + o(n^{-2q-2}),\ n \to \infty.$$

Proof. The proof immediately follows from the following expansions derived by means of integration by parts

$$f_n^c = (-1)^n \sum_{k=0}^{q} \frac{A_{2k+1}(f)}{(\pi n)^{2k+2}} + \frac{(-1)^{q+1}}{(\pi n)^{2q+2}} \int_{-1}^{1} f^{(2q+2)}(x) \cos \pi n x dx, \quad (4)$$

$$f_n^s = (-1)^{n+1} \sum_{k=0}^{q} \frac{B_{2k+1}(f)}{(\pi(n-\frac{1}{2}))^{2k+2}} + \frac{(-1)^{q+1}}{(\pi(n-\frac{1}{2}))^{2q+2}} \int_{-1}^{1} f^{(2q+2)}(x) \sin \pi (n-\tfrac{1}{2}) x dx. \quad (5)$$

□

In particular, if $f^{(2q+1)} \in AC[-1,1]$, $q \geq 0$ and $f^{(2k+1)}(\pm 1) = 0$, $k = 0, \dots, q-1$, then Lemma 1 implies

$$f_n^c = (-1)^n \frac{A_{2q+1}(f)}{(\pi n)^{2q+2}} + o(n^{-2q-2}),\ n \to \infty \tag{6}$$

and

$$f_n^s = (-1)^{n+1} \frac{B_{2q+1}(f)}{(\pi(n-\frac{1}{2}))^{2q+2}} + o(n^{-2q-2}),\ n \to \infty. \tag{7}$$

Lemma 2 *Assume that $f^{(2q+2)} \in AC[-1,1]$, $q \geq 0$. Then, the following asymptotic expansions are valid*

$$f_n^c = (-1)^n \sum_{k=0}^{q} \frac{A_{2k+1}(f)}{(\pi n)^{2k+2}} + o(n^{-2q-3}),\ n \to \infty,$$

$$f_n^s = (-1)^{n+1} \sum_{k=0}^{q} \frac{B_{2k+1}(f)}{(\pi(n-\frac{1}{2}))^{2k+2}} + o(n^{-2q-3}),\ n \to \infty.$$

Proof. In view of higher smoothness of f, from (4) and (5) we have

$$f_n^c = (-1)^n \sum_{k=0}^{q} \frac{A_{2k+1}(f)}{(\pi n)^{2k+2}} + \frac{(-1)^q}{(\pi n)^{2q+3}} \int_{-1}^{1} f^{(2q+3)}(x) \sin \pi n x dx,$$

$$f_n^s = (-1)^{n+1} \sum_{k=0}^{q} \frac{B_{2k+1}(f)}{(\pi(n-\frac{1}{2}))^{2k+2}} + \frac{(-1)^{q+1}}{(\pi(n-\frac{1}{2}))^{2q+3}} \int_{-1}^{1} f^{(2q+3)}(x) \cos \pi(n-\tfrac{1}{2}) x dx,$$

which conclude the proof. □

In particular, if $f^{(2q+2)} \in AC[-1,1]$, $q \geq 0$ and $f^{(2k+1)}(\pm 1) = 0$, $k = 0, \dots, q-1$, then Lemma 2 implies

$$f_n^c = (-1)^n \frac{A_{2q+1}(f)}{(\pi n)^{2q+2}} + o(n^{-2q-3}),\ n \to \infty \tag{8}$$

and

$$f_n^s = (-1)^{n+1} \frac{B_{2q+1}(f)}{(\pi(n-\frac{1}{2}))^{2q+2}} + o(n^{-2q-3}),\ n \to \infty. \tag{9}$$

Now denote

$$R_N(f,x) = f(x) - M_N(f,x).$$

Assuming that $M_N(f,x)$ converges pointwise to $f(x)$, we can write

$$R_N(f,x) = \sum_{n=N+1}^{\infty} \left(f_n^c \cos \pi n x + f_n^s \sin \pi (n-\tfrac{1}{2})x\right) = R_N^c(f,x) + R_N^s(f,x), \tag{10}$$

where

$$R_N^c(f,x) = \sum_{n=N+1}^{\infty} f_n^c \cos \pi n x,\ R_N^s(f,x) = \sum_{n=N+1}^{\infty} f_n^s \sin \pi (n-\tfrac{1}{2})x.$$

Next theorem deals with L_2-convergence of the modified Fourier expansions.

Theorem 6 *Assume that* $f^{(2q+1)} \in AC[-1,1]$, $q \geq 0$ *and*

$$f^{(2k+1)}(\pm 1) = 0,\ k = 0,\dots,q-1.$$

Then, the following estimate holds

$$\lim_{N\to\infty} N^{2q+\frac{3}{2}} ||R_N||_{L_2} = c(q)\sqrt{A_{2q+1}^2(f) + B_{2q+1}^2(f)}, \tag{11}$$

where

$$c(q) = \frac{1}{\pi^{2q+2}\sqrt{4q+3}}.$$

Proof. Estimates (6) and (7) together with (10) imply

$$\begin{aligned} ||R_N(f,x)||_{L_2}^2 &= \sum_{n=N+1}^{\infty} (f_n^c)^2 + (f_n^s)^2 \\ &= \frac{A_{2q+1}^2(f)}{\pi^{4q+4}} \sum_{n=N+1}^{\infty} \frac{1}{n^{4q+4}} + \frac{B_{2q+1}^2(f)}{\pi^{4q+4}} \sum_{n=N+1}^{\infty} \frac{1}{(n-\frac{1}{2})^{4q+4}} \\ &+ o(N^{-4q-3}),\ N \to \infty. \end{aligned}$$

This concludes the proof. □

Now, we continue with the pointwise convergence. Denote

$$\Delta_n^0(f_n) = f_n,\ \Delta_n^k(f_n) = \Delta_n^{k-1}(f_n) + \Delta_{n-1}^{k-1}(f_n),\ k \geq 1. \tag{12}$$

Lemma 3 *Let* $l \geq 0$, *and*

$$P_{l,n} := \frac{(-1)^n}{(\pi n)^{2l+2}},\ Q_{l,n} := \frac{(-1)^n}{(\pi(n-\frac{1}{2}))^{2l+2}}.$$

Then

$$\Delta_n^p(P_{l,n}) = P_{l,n}\frac{(-1)^p p!}{n^p}\binom{p+2l+1}{2l+1} + O(n^{-2l-p-3}),$$

and

$$\Delta_n^p(Q_{l,n}) = Q_{l,n}\frac{(-1)^p p!}{n^p}\binom{p+2l+1}{2l+1} + O(n^{-2l-p-3}).$$

Proof. We have

$$\begin{aligned}
\Delta_n^p(P_{l,n}) &= \sum_{k=0}^{p}\binom{p}{k}P_{l,n-k} = \sum_{k=0}^{p}\binom{p}{k}\frac{(-1)^{n+k}}{(\pi(n-k))^{2l+2}}\\
&= \frac{(-1)^n}{(\pi n)^{2l+2}}\sum_{k=0}^{p}\binom{p}{k}\frac{(-1)^k}{(1-\frac{k}{n})^{2l+2}}\\
&= P_{l,n}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\sum_{j=2l+1}^{\infty}\binom{j}{2l+1}\left(\frac{k}{n}\right)^{j-2l-1}\\
&= P_{l,n}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\sum_{j=0}^{\infty}\binom{j+2l+1}{2l+1}\\
&= P_{l,n}\sum_{j=0}^{\infty}\binom{j+2l+1}{2l+1}\frac{1}{n^j}\sum_{k=0}^{p}(-1)^k\binom{p}{k}k^j\\
&= P_{l,n}\sum_{j=0}^{\infty}\binom{j+2l+1}{2l+1}\frac{1}{n^j}\alpha_{p,j},
\end{aligned}$$

where (see [17])

$$\alpha_{k,m} = \sum_{s=0}^{k}\binom{k}{s}(-1)^s s^m,\ m\geq 0. \tag{13}$$

This concludes the proof as $\alpha_{p,j}=0,\ j=0,\ldots,p-1$ and $\alpha_{p,p}=(-1)^p p!$.

Similarly we prove the second estimate. □

Theorem 7 *Let* $f^{(2q+2)}\in AC[-1,1]$, $q\geq 0$ *and*

$$f^{(2k+1)}(\pm 1) = 0,\ k=0,\ldots,q-1.$$

Then, the following estimate holds for $|x|<1$

$$\begin{aligned}
R_N(f,x) &= A_{2q+1}(f)\frac{(-1)^{N+1}\cos\pi(N+\frac{1}{2})x}{2N^{2q+2}\pi^{2q+2}\cos\frac{\pi x}{2}} + B_{2q+1}(f)\frac{(-1)^N\sin\pi N x}{2N^{2q+2}\pi^{2q+2}\cos\frac{\pi x}{2}}\\
&\quad + o(N^{-2q-2})\ N\to\infty.
\end{aligned}$$

Proof. The Abel transformation implies

$$\begin{aligned}
R_N^c(f,x) &= \sum_{n=N+1}^{\infty} f_n^c\cos\pi n x = -\frac{f_N^c\cos\pi(N+\frac{1}{2})x}{2\cos\frac{\pi x}{2}}\\
&\quad + \frac{1}{2\cos\frac{\pi x}{2}}\sum_{n=N+1}^{\infty}\Delta_n^1(f_n^c)\cos\pi(n-\tfrac{1}{2})x,
\end{aligned}$$

and

$$R_N^s(f,x) = \sum_{n=N+1}^{\infty} f_n^s \sin \pi(n - \tfrac{1}{2})x = -\frac{f_N^s \sin \pi N x}{2\cos \frac{\pi x}{2}} + \frac{1}{2\cos \frac{\pi x}{2}} \sum_{n=N+1}^{\infty} \Delta_n^1(f_n^s) \sin \pi(n-1)x.$$

Reiteration of this transformation leads to the following expansions

$$R_N^c(f,x) = -\frac{f_N^c \cos \pi(N+\frac{1}{2})x}{2\cos \frac{\pi x}{2}} - \frac{\Delta_N^1(f_N^c) \cos \pi N x}{4\cos^2 \frac{\pi x}{2}} + \frac{1}{4\cos^2 \frac{\pi x}{2}} \sum_{n=N+1}^{\infty} \Delta_n^2(f_n^c) \cos \pi(n-1)x,$$

and

$$R_N^s(f,x) = -\frac{f_N^s \sin \pi N x}{2\cos \frac{\pi x}{2}} - \frac{\Delta_N^1(f_N^s) \sin \pi(N-\frac{1}{2})x}{4\cos^2 \frac{\pi x}{2}} + \frac{1}{4\cos^2 \frac{\pi x}{2}} \sum_{n=N+1}^{\infty} \Delta_n^2(f_n^s) \sin \pi(n-\tfrac{3}{2})x.$$

According to estimates (8), (9) and Lemma 3, we have

$$\Delta_N^1(f_N^c) = O(N^{-2q-3}),\ \Delta_n^2(f_n^c) = o(n^{-2q-3}), \tag{14}$$

and

$$\Delta_N^1(f_N^s) = O(N^{-2q-3}),\ \Delta_n^2(f_n^s) = o(n^{-2q-3}), \tag{15}$$

which conclude the proof. □

2 Modified Fourier-Pade Approximations

In a series of papers ([20–25]) the convergence acceleration of the truncated Fourier series and trigonometric interpolation were achieved by application of trigonometric-rational functions as corrections to the corresponding errors. Rational corrections contain unknown parameters and different approaches are known for their determination. One approach leads to Fourier-Pade approximations ([18, 20]). Here, the same idea we apply for the modified expansions.

Consider a finite sequence of real numbers $\{\theta_k\}_{k=1}^p$, $p \ge 1$ and by $\Delta_n^k(\theta, f_n)$ denote the following generalized finite differences

$$\Delta_n^0(\theta, f_n) = f_n,$$
$$\Delta_n^k(\theta, f_n) = \Delta_n^{k-1}(\theta, f_n) + \theta_k \Delta_{n-1}^{k-1}(\theta, f_n),\ k \ge 1.$$

If $\theta_k = 1$, $k = 1, \dots, p$, we get the classical finite differences

$$\Delta_n^k(\theta, f_n) = \Delta_n^k(f_n).$$

The Abel transformation implies (details see in [20])

$$\begin{aligned} R_N^c(f,x) &= \frac{1}{2}\sum_{n=N+1}^{\infty} f_n^c e^{i\pi n x} + \frac{1}{2}\sum_{n=N+1}^{\infty} f_n^c e^{-i\pi n x} \\ &= -\frac{e^{i\pi(N+1)x}}{2}\sum_{k=1}^{p}\frac{\theta_k^c \Delta_N^{k-1}(\theta^c, f_n^c)}{\prod_{r=1}^{k}(1+\theta_r^c e^{i\pi x})} - \frac{e^{-i\pi(N+1)x}}{2}\sum_{k=1}^{p}\frac{\theta_k^c \Delta_N^{k-1}(\theta^c, f_n^c)}{\prod_{r=1}^{k}(1+\theta_r^c e^{-i\pi x})} \\ &\quad + \frac{1}{2\prod_{k=1}^{p}(1+\theta_k^c e^{i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta^c, f_n^c) e^{i\pi n x} \\ &\quad + \frac{1}{2\prod_{k=1}^{p}(1+\theta_k^c e^{-i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta^c, f_n^c) e^{-i\pi n x}, \end{aligned}$$

and

$$\begin{aligned} R_N^s(f,x) &= \frac{e^{-\frac{i\pi x}{2}}}{2i}\sum_{n=N+1}^{\infty} f_n^s e^{i\pi n x} - \frac{e^{\frac{i\pi x}{2}}}{2i}\sum_{n=N+1}^{\infty} f_n^s e^{-i\pi n x} \\ &= -\frac{e^{i\pi(N+\frac{1}{2})x}}{2i}\sum_{k=1}^{p}\frac{\theta_k \Delta_N^{k-1}(\theta^s, f_n^s)}{\prod_{r=1}^{k}(1+\theta_r^s e^{i\pi x})} + \frac{e^{-i\pi(N+\frac{1}{2})x}}{2i}\sum_{k=1}^{p}\frac{\theta_k \Delta_N^{k-1}(\theta^s, f_n^s)}{\prod_{r=1}^{k}(1+\theta_r^s e^{-i\pi x})} \\ &\quad - \frac{e^{\frac{i\pi x}{2}}}{2i\prod_{k=1}^{p}(1+\theta_k^s e^{-i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta^s, f_n^s) e^{-i\pi n x} \\ &\quad + \frac{e^{-\frac{i\pi x}{2}}}{2i\prod_{k=1}^{p}(1+\theta_k^s e^{i\pi x})}\sum_{n=N+1}^{\infty}\Delta_n^p(\theta^s, f_n^s) e^{i\pi n x}. \end{aligned}$$

After some algebraic manipulations, we derive

$$\begin{aligned} R_N^c(f,x) &= \\ &- \sum_{k=1}^{p}\frac{\theta_k^c \Delta_N^{k-1}(\theta^c, f_n^c)}{\prod_{r=1}^{k}(1+2\theta_r^c\cos\pi x + (\theta_r^c)^2)}\sum_{t=0}^{k}\gamma_t(k,\theta^c)\cos\pi(N+1-t)x \\ &+ R_{N,p}^c(f,x), \end{aligned}$$

and

$$\begin{aligned} R_N^s(f,x) &= \\ &- \sum_{k=1}^{p}\frac{\theta_k^s \Delta_N^{k-1}(\theta^s, f_n^s)}{\prod_{r=1}^{k}(1+2\theta_r^s\cos\pi x + (\theta_r^s)^2)}\sum_{t=0}^{k}\gamma_t(k,\theta^s)\sin\pi(N+\frac{1}{2}-t)x \\ &+ R_{N,p}^s(f,x), \end{aligned}$$

where

$$R^c_{N,p}(f,x) = \frac{1}{2\prod_{k=1}^p(1+\theta^c_k e^{i\pi x})}\sum_{n=N+1}^{\infty}\Delta^p_n(\theta^c, f^c_n)e^{i\pi n x} + \frac{1}{2\prod_{k=1}^p(1+\theta^c_k e^{-i\pi x})}\sum_{n=N+1}^{\infty}\Delta^p_n(\theta^c, f^c_n)e^{-i\pi n x}, \quad (16)$$

$$R^s_{N,p}(f,x) = \frac{e^{-\frac{i\pi x}{2}}}{2i\prod_{k=1}^p(1+\theta^s_k e^{i\pi x})}\sum_{n=N+1}^{\infty}\Delta^p_n(\theta^s, f^s_n)e^{i\pi n x} - \frac{e^{\frac{i\pi x}{2}}}{2i\prod_{k=1}^p(1+\theta^s_k e^{-i\pi x})}\sum_{n=N+1}^{\infty}\Delta^p_n(\theta^s, f^s_n)e^{-i\pi n x},$$

and $\gamma_t(k,\theta)$ are defined by the following identity

$$\prod_{t=1}^{k}(1+\theta_t x) = \sum_{t=0}^{k}\gamma_t(k,\theta)x^t. \quad (17)$$

These expansions lead to modified Fourier-Pade (MFP) approximations

$$\begin{aligned} M_{N,p}(f,x) &= M_N(f,x) \\ &- \sum_{k=1}^{p}\frac{\theta^c_k \Delta^{k-1}_N(\theta^c, f^c_n)}{\prod_{r=1}^k(1+2\theta^c_r\cos\pi x + (\theta^c_r)^2)}\sum_{t=0}^{k}\gamma_t(k,\theta^c)\cos\pi(N+1-t)x \\ &- \sum_{k=1}^{p}\frac{\theta^s_k \Delta^{k-1}_N(\theta, f^s_n)}{\prod_{r=1}^k(1+2\theta^s_r\cos\pi x + (\theta^s_r)^2)}\sum_{t=0}^{k}\gamma_t(k,\theta^s)\sin\pi(N+\tfrac{1}{2}-t)x \end{aligned} \quad (18)$$

with the error

$$R_{N,p}(f,x) = f(x) - M_{N,p}(f,x) = R^c_{N,p}(f,x) + R^s_{N,p}(f,x),$$

where unknown parameters $\{\theta^c_k\}$ and $\{\theta^s_k\}$, $k = 1, ..., p$ are determined from the following systems of equations

$$\Delta^p_n(\theta^c, f^c_n) = 0,\ n = N, N-1, \ldots, N-p+1, \quad (19)$$

and

$$\Delta^p_n(\theta^s, f^s_n) = 0,\ n = N, N-1, \ldots, N-p+1. \quad (20)$$

Systems (19) and (20) can be reformulated as linear systems of equations with unknowns $\gamma_k(p,\theta^c)$ and $\gamma_k(p,\theta^s)$

$$\Delta^p_n(\theta^c, f^c_n) = f^c_n + \sum_{k=1}^{p}\gamma_k(p,\theta^c)f^c_{n-k} = 0,\ \ n = N, N-1, \ldots, N-p+1, \quad (21)$$

and

$$\Delta_n^p(\theta^s, f_n^s) = f_n^s + \sum_{k=1}^{p} \gamma_k(p, \theta^s) f_{n-k}^s = 0, \ \ n = N, N-1, \dots, N-p+1. \quad (22)$$

Then, $\{\theta_k^c\}$ and $\{\theta_k^s\}$, $k = 1, \dots, p$, can be determined from (17), with $k = p$, as the roots of the corresponding polynomials.

According to systems (21) and (22), coefficients $\gamma_k(p, \theta)$ would have the following asymptotic expansions (if f is enough smooth, see below)

$$\gamma_j(p, \theta^c) = \sum_{t=0}^{\infty} \frac{\gamma_{j,t}^c}{N^t}, \ \gamma_j(p, \theta^s) = \sum_{t=0}^{\infty} \frac{\gamma_{j,t}^s}{N^t} \quad (23)$$

with some constants $\gamma_{j,t}^c$ and $\gamma_{j,t}^s$. In particular,

$$\gamma_j(p, \theta^c) = O(1), \ \gamma_j(p, \theta^s) = O(1), \ N \to \infty. \quad (24)$$

More precisely,

$$\gamma_{j,0}^s = \gamma_{j,0}^c = \binom{p}{j}. \quad (25)$$

In the further sections, we investigate the convergence of the MFP–approximations in the frameworks of the pointwise and $L_2(-1, 1)$ convergences.

3 Pointwise Convergence

First, we prove some lemmas.

Lemma 4 *Let* $f^{(2q+2p+2)} \in AC[-1, 1]$, $q \geq 0$, $p \geq 1$, *and let the systems* (21), (22) *have unique solutions. If*

$$f^{(2k+1)}(\pm 1) = 0, \ k = 0, \dots, q-1,$$

then, the following estimates are valid

$$\Delta_n^w(\Delta_n^p(\theta^c, f_n^c)) = O(n^{-w-2q-2}) + o(n^{-2q-2p-3}), \ n \geq N+1, \ N \to \infty,$$

and

$$\Delta_n^w(\Delta_n^p(\theta^s, f_n^s)) = O(n^{-w-2q-2}) + o(n^{-2q-2p-3}), \ n \geq N+1, \ N \to \infty.$$

Proof. We provide the proof for the coefficients f_n^c only. According to the estimate (8)

$$f_n^c = (-1)^n \sum_{l=q}^{q+p} \frac{A_{2l+1}(f)}{(\pi n)^{2l+2}} + o(n^{-2q-2p-3}). \quad (26)$$

Then,

$$\Delta_n^w(\Delta_n^p(\theta, f_n^c)) = \sum_{s=0}^{p} \gamma_s(p,\theta^c)\Delta_n^w(f_{n-s}^c)$$
$$= \sum_{s=0}^{p} \gamma_s(p,\theta^c) \sum_{l=q}^{q+p} A_{2l+1}(f)\Delta_n^w(P_{l,n}) + o(n^{-2q-2p-3}).$$

Taking into account the estimates (24) and Lemma 3 we get the desired estimate.□

Lemma 5 *Let* $f^{(2q+2p+2)} \in AC[-1,1]$, $q \geq 0$, $p \geq 1$, *and let the systems* (21), (22) *have unique solutions. If*

$$f^{(2k+1)}(\pm 1) = 0,\ k = 0, \ldots, q-1,$$

and

$$A_{2q+1}(f)B_{2q+1}(f) \neq 0,$$

then, the following estimates are valid

$$\Delta_N^w(\Delta_n^p(\theta^c, f_n^c)) = A_{2q+1}(f)\frac{(-1)^{N+w}(2q+w+1)!}{\pi^{2q+2}N^{2q+p+w+2}(2q+1)!}\sum_{t=0}^{p}\beta_t^c(p-t)\binom{2q+w+t+1}{2q+w+1} + O(N^{-2q-w-p-3}) + o(N^{-2q-2p-3}), \quad (27)$$

$$\Delta_N^w(\Delta_n^p(\theta^s, f_n^s)) = B_{2q+1}(f)\frac{(-1)^{N+1+w}(2q+w+1)!}{\pi^{2q+2}N^{2q+p+w+2}(2q+1)!}\sum_{t=0}^{p}\beta_t^s(p-t)\binom{2q+w+t+1}{2q+w+1} + O(N^{-2q-w-p-3}) + o(N^{-2q-2p-3}), \quad (28)$$

where

$$\beta_u^c(t) = \sum_{j=0}^{p}(-1)^j\gamma_{j,t}^c j^u,\ \beta_u^s(t) = \sum_{j=0}^{p}(-1)^j\gamma_{j,t}^s j^u \quad (29)$$

and $\gamma_{j,t}^c$, $\gamma_{j,t}^s$ *are the coefficients of the asymptotic expansions*

$$\gamma_j(p,\theta^c) = \sum_{t=0}^{2p+1}\frac{\gamma_{j,t}^c}{N^t} + o(N^{-2p-1}),\ \gamma_j(p,\theta^s) = \sum_{t=0}^{2p+1}\frac{\gamma_{j,t}^s}{N^t} + o(N^{-2p-1}). \quad (30)$$

Proof. We will prove only the estimate (27). The estimate (28) can be handled similarly. The existence of the asymptotic expansions (30) follows from the smoothness of f and the solutions of the systems (21), (22) specified applying the Crammer rule. Then, we have

$$\begin{aligned}\Delta_N^w(\Delta_n^p(\theta^c, f_n^c)) &= \sum_{s=0}^{p} \gamma_s(p, \theta^c)\Delta_{N-s}^w(f_n^c) \\ &= \sum_{k=0}^{w} \binom{w}{k} \sum_{s=0}^{p} \gamma_s(p, \theta^c) f_{N-k-s}^c,\end{aligned} \tag{31}$$

where $\gamma_s(p, \theta^c)$ are the solutions of the system (21). From (26), we derive

$$\begin{aligned}f_{N-s-k}^c &= (-1)^{N-s-k} \sum_{l=q}^{q+p} \frac{A_{2l+1}(f)}{(\pi(N-k-s))^{2l+2}} + o(N^{-2q-2p-3}) \\ &=(-1)^{N-s-k} \sum_{l=q}^{q+p} \frac{A_{2l+1}(f)}{\pi^{2l+2}} \sum_{j=2l+1}^{\infty} \binom{j}{2l+1} \frac{(k+s)^{j-2l-1}}{N^{j+1}} \\ &\quad + o(N^{-2q-2p-3}) \\ &=\frac{(-1)^{N-s-k}}{(\pi N)^{2q+2}} \sum_{j=0}^{2p+1} \frac{1}{N^j} \sum_{l=0}^{[\frac{j}{2}]} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+j+1}{2q+2l+1} (k+s)^{j-2l} \\ &\quad + o(N^{-2q-2p-3}).\end{aligned}$$

Substituting this and the first equation of (30) into (31), we obtain

$$\begin{aligned}\Delta_N^w(\Delta_n^p(\theta, f_n^c)) &= \sum_{k=0}^{w} \binom{w}{k} \sum_{s=0}^{p} \gamma_s(p, \theta) f_{N-k-s}^c \\ &= \frac{(-1)^N}{(\pi N)^{2q+2}} \sum_{j=0}^{2p+1} \frac{1}{N^j} \sum_{t=0}^{j} \sum_{l=0}^{[\frac{j-t}{2}]} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+j-t+1}{2q+2l+1} \times \\ &\quad \times \sum_{u=0}^{j-t-2l} \binom{j-t-2l}{u} \alpha_{w,u} \beta_{j-t-2l-u}^c(t) \\ &\quad + o(N^{-2q-2p-3}),\end{aligned}$$

where $\alpha_{w,u}$ is defined by (13).

Taking into account that (see[17])

$$\alpha_{w,u} = 0,\ u = 0, \ldots, w-1,$$

we get

$$\Delta_N^w(\Delta_n^p(\theta, f_n^c)) = \frac{(-1)^N}{(\pi N)^{2q+2}} \sum_{j=w}^{2p+1} \frac{1}{N^j} \sum_{t=0}^{j-w} \sum_{l=0}^{[\frac{j-t-w}{2}]} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+j-t+1}{2q+2l+1}$$
$$\times \sum_{u=w}^{j-t-2l} \binom{j-t-2l}{u} \alpha_{w,u} \beta^c_{j-t-2l-u}(t) + o(N^{-2q-2p-3})$$
$$= \frac{(-1)^N}{\pi^{2q+2} N^{w+2q+2}} \sum_{j=0}^{2p-w+1} \frac{1}{N^j} \sum_{t=0}^{j} \sum_{l=0}^{[\frac{j-t}{2}]} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+j+w-t+1}{2q+2l+1}$$
$$\times \sum_{u=0}^{j-t-2l} \binom{j+w-t-2l}{u+w} \alpha_{w,u+w} \beta^c_{j-t-2l-u}(t) + o(N^{-2q-2p-3})$$
$$= \frac{(-1)^N}{\pi^{2q+2} N^{w+2q+2}} \sum_{j=0}^{2p-w+1} \frac{1}{N^j} \sum_{t=0}^{j} \sum_{l=0}^{[\frac{t}{2}]} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+t+w+1}{2q+2l+1}$$
$$\times \sum_{u=0}^{t-2l} \binom{t+w-2l}{u+w} \alpha_{w,u+w} \beta^c_{t-2l-u}(j-t) + o(N^{-2q-2p-3}).$$

Finally, we derive

$$\Delta_N^w(\Delta_n^p(\theta, f_n^c)) =$$
$$\frac{(-1)^N}{\pi^{2q+2} N^{w+2q+2}} \sum_{j=0}^{2p-w+1} \frac{1}{N^j} \sum_{t=0}^{j} \sum_{l=0}^{[\frac{t}{2}]} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+t+w+1}{2q+2l+1}$$
$$\times \sum_{u=0}^{t-2\ell} \binom{t+w-2l}{u} \alpha_{w,t-2\ell-u+w} \beta^c_u(j-t) + o(N^{-2q-2p-3}). \quad (32)$$

By similar arguments as in [20], it is possible to show that

$$\beta^c_u(j-t) = 0,\ j = 0, \dots, p-1;\ 0 \le t \le j;\ 0 \le u \le t. \quad (33)$$

Thus, from (32), we obtain

$$\Delta_N^w(\Delta_n^p(\theta, f_n^c)) =$$
$$\frac{(-1)^N}{\pi^{2q+2} N^{p+w+2q+2}} \sum_{t=0}^{p} \sum_{l=0}^{[\frac{t}{2}]} \frac{A_{2q+2l+1}(f)}{\pi^{2l}} \binom{2q+t+w+1}{2q+2l+1}$$
$$\times \sum_{u=0}^{t-2\ell} \binom{t+w-2l}{u} \alpha_{w,t-2\ell-u+w} \beta^c_u(p-t)$$
$$+ O(N^{-p-w-2q-3}) + o(N^{-2q-2p-3}).$$

It remains to notice that only the term $u = t$, $l = 0$ is not zero, and therefore,

$$\Delta_N^w(\Delta_n^p(\theta, f_n^c)) = A_{2q+1}(f)\frac{(-1)^N \alpha_{w,w}}{\pi^{2q+2}N^{p+w+2q+2}}\sum_{t=0}^{p}\binom{2q+t+w+1}{2q+1}\binom{t+w}{t}\beta_t^c(p-t) + O(N^{-p-w-2q-3}) + o(N^{-2q-2p-3}).$$

This concludes the proof since $\alpha_{w,w} = (-1)^w w!$. □

Theorem 8 *Let $f^{(2q+2p+2)} \in AC[-1,1]$, $q \geq 0$, $p \geq 1$, and let the systems* (21), (22) *have unique solutions. If*

$$f^{(2k+1)}(\pm 1) = 0, \quad k = 0, \dots, q-1,$$

and

$$A_{2q+1}(f)B_{2q+1}(f) \neq 0,$$

then, the following estimates are valid for $x \in (-1,1)$

$$R_{N,p}^c(f,x) = A_{2q+1}(f)\frac{(-1)^{N+1}(2q+p+1)!p!}{2^{2p+1}\pi^{2q+2}N^{2q+2p+2}(2q+1)!}\frac{\cos\frac{\pi x}{2}(2N-2p+1)}{\cos^{2p+1}\frac{\pi x}{2}} + o(N^{-2q-2p-2}),$$

and

$$R_{N,p}^s(f,x) = B_{2q+1}(f)\frac{(-1)^N(2q+p+1)!p!}{2^{2p+1}\pi^{2q+2}N^{2q+2p+2}(2q+1)!}\frac{\sin\frac{\pi x}{2}(2N-2p)}{\cos^{2p+1}\frac{\pi x}{2}} + o(N^{-2q-2p-2}).$$

Proof. We estimate $R_{N,p}^c(f,x)$. According to the estimate (25), we have

$$\prod_{k=1}^{p}(1+\theta_k^c e^{i\pi x}) \to (1+e^{i\pi x})^p, \; N \to \infty,$$

and it remains to estimate only the sum in the right hand side of (16)

$$\sum_{n=N+1}^{\infty}\Delta_n^p(\theta^c, f_n^c)e^{i\pi nx} = -e^{i\pi(N+1)x}\sum_{w=0}^{2p+1}\frac{\Delta_N^w(\Delta_n^p(\theta^c, f_n^c))}{(1+e^{i\pi x})^{w+1}} + \frac{1}{(1+e^{i\pi x})^{2p+2}}\sum_{n=N+1}^{\infty}\Delta_n^{2p+2}(\Delta_n^p(\theta^c, f_n^c))e^{i\pi nx}.$$

Taking into account that

$$\Delta_N^k(\Delta_n^p(\theta^c, f_n^c)) = \sum_{s=0}^{k} \binom{k}{s} \Delta_{N-s}^p(\theta^c, f_n^c)$$

we see from (19) that

$$\Delta_N^k(\Delta_n^p(\theta^c, f_n^c)) = 0,\ k = 0, \dots, p-1.$$

Therefore

$$\sum_{n=N+1}^{\infty} \Delta_n^p(\theta^c, f_n^c) e^{i\pi n x} = \\ - e^{i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta^c, f_n^c))}{(1+e^{i\pi x})^{p+1}} - e^{i\pi(N+1)x} \sum_{w=p+1}^{2p+1} \frac{\Delta_N^w(\Delta_n^p(\theta^c, f_n^c))}{(1+e^{i\pi x})^{w+1}} \\ + \frac{1}{(1+e^{i\pi x})^{2p+2}} \sum_{n=N+1}^{\infty} \Delta_n^{2p+2}(\Delta_n^p(\theta^c, f_n^c)) e^{i\pi n x}. \quad (34)$$

Lemma 4 shows that

$$\Delta_n^{2p+2}(\Delta_n^p(\theta^c, f_n^c)) = o(n^{-2q-2p-3}),\ n \to \infty.$$

Hence, the last term in the right hand side of (34) is $o(N^{-2p-2q-2})$, as $N \to \infty$. According to Lemma 5

$$\Delta_N^w(\Delta_n^p(\theta^c, f_n^c)) = O(N^{-2q-w-p-2}) + o(N^{-2q-w-p-2}),\ \ N \to \infty.$$

As in the second term of the right hand side of (34), the parameter w is ranging from $w = p+1$ to $w = 2p+1$, then this term is $O(N^{-2q-2p-3})$. Hence,

$$\sum_{n=N+1}^{\infty} \Delta_n^p(\theta^c, f_n^c) e^{i\pi n x} = -e^{i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta^c, f_n^c))}{(1+e^{i\pi x})^{2p+1}} + o(N^{-2q-2p-2}),$$

where $N \to \infty$. Similarly

$$\sum_{n=N+1}^{\infty} \Delta_n^p(\theta^c, f_n^c) e^{-i\pi n x} = -e^{-i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta^c, f_n^c))}{(1+e^{-i\pi x})^{2p+1}} + o(N^{-2q-2p-2}),\ \ (35)$$

where $N \to \infty$. Therefore,

$$R_{N,p}^c(f,x) = -e^{i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta^c, f_n^c))}{(1+e^{i\pi x})^{2p+1}} - e^{-i\pi(N+1)x} \frac{\Delta_N^p(\Delta_n^p(\theta^c, f_n^c))}{(1+e^{-i\pi x})^{2p+1}} \\ + o(N^{-2q-2p-2}),\ \ N \to \infty. \quad (36)$$

Finally, we need to estimate $\Delta_N^p(\Delta_n^p(\theta^c, f_n^c))$. Again by Lemma 5, we have

$$\begin{aligned}\Delta_N^p(\Delta_n^p(\theta^c, f_n^c)) = A_{2q+1}(f)\frac{(-1)^{N+p}}{N^{2p+2q+2}\pi^{2q+2}(2q+1)!} \\ \times \sum_{t=0}^{p} \beta_t^c(p-t)\frac{(2q+p+t+1)!}{t!} + o(N^{-2q-2p-2}).\end{aligned} \tag{37}$$

It is possible to show (details see in [20]) that the sum in the right-hand side of (37) equals to $(-1)^p p!(p+2q+1)!$.

Hence,

$$\Delta_N^p(\Delta_n^p(\theta^c, f_n^c)) = A_{2q+1}(f)\frac{(-1)^N(2q+1+p)!(p)!}{N^{2p+2q+2}\pi^{2q+2}(2q+1)!} + o(N^{-2q-2p-2}).$$

Together with (36) it implies

$$\begin{aligned}&R_{N,p}^c(f,x) = \\ &A_{2q+1}(f)\frac{(-1)^{N+1}(2q+1+p)!p!}{N^{2p+2q+2}\pi^{2q+2}(2q+1)!}Re\left[\frac{e^{i\pi(N+1)x}}{(1+e^{i\pi x})^{2p+1}}\right] + o(N^{-2q-2p-2}).\end{aligned} \tag{38}$$

Similarly, we can show that

$$\begin{aligned}&R_{N,p}^s(f,x) = \\ &B_{2q+1}(f)\frac{(-1)^N(2q+1+p)!p!}{N^{2p+2q+2}\pi^{2q+2}(2q+1)!}Re\left[\frac{e^{i\pi(N+\frac{1}{2})x}}{i(1+e^{i\pi x})^{2p+1}}\right] + \\ &o(N^{-2q-2p-2})\end{aligned}$$

which completes the proof.□

4 L_2-Convergence

In this section, we investigate L_2-convergence of the Fourier-Pade approximation. Taking into account (25), we see that $\theta_k^c, \theta_k^s \to 1$, as $N \to \infty$. Let

$$\theta_k^c = 1 - \frac{\tau_k^c}{N} + o(N^{-1}), \quad \theta_k^s = 1 - \frac{\tau_k^s}{N} + o(N^{-1}), \; k = 1, \ldots, p. \tag{39}$$

To determine $\{\tau_k^c\}$ and $\{\tau_k^s\}$ we compare two results that outline the behavior of $\Delta_n^p(\theta^c, f_n^c)$ and $\Delta_n^p(\theta^s, f_n^s)$.

Lemma 6 *Let* $f^{(2q+p+1)} \in AC[-1,1]$, $q \geq 0$. *Let*

$$f^{(2k+1)}(\pm 1) = 0, \; k = 0, \ldots, q-1$$

and

$$A_{2q+1}(f)B_{2q+1}(f) \neq 0.$$

If systems (21), (22) *have unique solutions, then, the following estimates are valid*

$$\begin{aligned}\Delta_n^p(\theta_k^c, f_n^c) = A_{2q+1}(f)\frac{(-1)^n(2q+p+1)!}{N^p n^{2q+2}\pi^{2q+2}(2q+1)!}\left(1-\frac{n}{N}\right)^p \\ + o(N^{-p})\frac{1}{n^{2q+2}},\ n > N, N \to \infty,\end{aligned} \tag{40}$$

$$\begin{aligned}\Delta_n^p(\theta_k^s, f_n^s) = B_{2q+1}(f)\frac{(-1)^{n+1}(2q+p+1)!}{N^p n^{2q+2}\pi^{2q+2}(2q+1)!}\left(1-\frac{n-\frac{1}{2}}{N}\right)^p \\ + o(N^{-p})\frac{1}{n^{2q+2}},\ n > N, N \to \infty.\end{aligned}$$

Proof. We will prove only the first estimate. The proof, in general, imitate the one of Lemma 5, so we omit some details. Let $\gamma_{s,t}^c$ be the coefficients of the asymptotic expansion

$$\gamma_s(p,\theta^c) = \sum_{t=0}^{p}\frac{\gamma_{s,t}^c}{N^t} + o(N^{-p}).$$

We replicate the arguments in the proof of Lemma 5, then apply Lemma 1 (when p is even) or Lemma 2 (when p is odd), and at last we obtain

$$\begin{aligned}\Delta_n^p(\theta^c, f_n^c) &= \sum_{s=0}^{p}\gamma_s(p,\theta^c)f_{n-s}^c = \sum_{s=0}^{p}\left(\sum_{t=0}^{p}\frac{\gamma_{s,t}^c}{N^t} + o(N^{-p})\right)\times \\ &\times\left(\frac{(-1)^{n-s}}{(\pi n)^{2q+2}}\sum_{j=0}^{p}\frac{1}{n^j}\sum_{l=0}^{[\frac{j}{2}]}\frac{A_{2q+2l+1}(f)}{\pi^{2l}}\binom{2q+j+1}{2q+2l+1}s^{j-2l} + o(n^{-2q-p-2})\right) \\ &= \frac{(-1)^n}{(\pi n)^{2q+2}}\sum_{j=0}^{p}\frac{1}{N^j}\sum_{t=0}^{j}\frac{1}{\left(\frac{n}{N}\right)^t}\sum_{l=0}^{[\frac{t}{2}]}\frac{A_{2q+2l+1}(f)}{\pi^{2l}}\binom{2q+2l+1}{2q+t+1}\beta_{t-2l}^c(j-t) \\ &\qquad + o(N^{-p})\frac{1}{n^{2q+2}},\end{aligned}$$

where $\beta_u^c(j-t)$ are defined by (29). From the proof of Lemma 5, we know that

$$\beta_u^c(j-t) = 0,\ j = 0,\ldots,p-1;\ 0 \le p \le j;\ 0 \le u \le t.$$

Therefore

$$\Delta_n^p(\theta^c, f_n^c) = A_{2q+1}(f)\frac{(-1)^n}{N^p(\pi n)^{2q+2}}\sum_{t=0}^{p}\frac{1}{\left(\frac{n}{N}\right)^t}\binom{2q+t+1}{2q+1}\beta_t^c(p-t)+ o(N^{-p})\frac{1}{n^{2q+2}}.$$

This concludes the proof as (see [20])

$$\sum_{t=0}^{p}\frac{1}{\left(\frac{n}{N}\right)^t}\binom{2q+t+1}{2q+1}\beta_t^c(p-t) = \sum_{t=0}^{p}\frac{(-1)^t}{\left(\frac{n}{N}\right)^t}\binom{p}{t}.$$

□

Lemma 7 *Let* $f^{(2q+p+1)} \in AC[-1,1]$, $q \geq 0$. *Let*

$$f^{(2k+1)}(\pm 1) = 0,\ k = 0, \dots, q-1,$$

$$A_{2q+1}(f)B_{2q+1}(f) \neq 0,$$

and

$$\theta_k^c = 1 - \frac{\tau_k^c}{N},\ \theta_k^s = 1 - \frac{\tau_k^s}{N},\ k = 1, \dots, p.$$

Then, the following estimates hold for $n > N$, *as* $N \to \infty$

$$\Delta_n^p(\theta_k^c, f_n^c) = A_{2q+1}(f)\frac{(-1)^{n+p}}{n^{2q+2}(2q+1)!\pi^{2q+2}}\sum_{k=0}^{p}\frac{(2q+p-k+1)!(-1)^k\gamma_k(\tau^c)}{N^k n^{p-k}} + o(N^{-p})\frac{1}{n^{2q+2}},$$

$$\Delta_n^p(\theta_k^s, f_n^s) = B_{2q+1}(f)\frac{(-1)^{n+p+1}}{n^{2q+2}(2q+1)!\pi^{2q+2}}\sum_{k=0}^{p}\frac{(2q+p-k+1)!(-1)^k\gamma_k(\tau^s)}{N^k(n-\frac{1}{2})^{p-k}} + o(N^{-p})\frac{1}{n^{2q+2}},$$

where

$$\prod_{k=1}^{p}(1+\tau_k^c x) = \sum_{k=0}^{p}\gamma_k(\tau^c)x^k, \quad \prod_{k=1}^{p}(1+\tau_k^s x) = \sum_{k=0}^{p}\gamma_k(\tau^s)x^k.$$

Proof. We will prove only the estimate for $\Delta_n^p(\theta_k^c, f_n^c)$. It is not hard to prove by induction that

$$\Delta_n^p(\theta^c, f_n^c) = \sum_{k=0}^{p} \frac{(-1)^k \gamma_k(\tau^c)}{N^k} \Delta_{n-k}^{p-k}(f_n^c). \tag{41}$$

From Lemma 1 (when p is even) or Lemma 2, (when p is odd), and from Lemma 3, we get

$$\Delta_n^k(f_n^c) = A_{2q+1}(f) \frac{(-1)^{n+k}(2q+k+1)!}{n^{2q+k+2}(2q+1)!\pi^{2q+2}} + o(n^{-2q-k-2}) + o(n^{-2q-p-2}).$$

This estimate, together with (41), completes the proof.
□

Comparing Lemmas 6 and 7 we get that

$$\gamma_k(\tau^c) = \gamma_k(\tau^s) = \frac{(p+q)!}{(q+p-k)!}\binom{p}{k}. \tag{42}$$

Now, we are ready to estimate the L_2 - error.

Let h_k^c and h_k^s be the complete homogeneous symmetric polynomials (see [26]) of degree k in variables $\theta_1^c, \theta_2^c, ..., \theta_p^c$ and $\theta_1^s, \theta_2^s, ..., \theta_p^s$

$$h_k^c(\theta_1^c, \theta_2^c, ..., \theta_p^c) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le p} \theta_{i_1}^c \cdots \theta_{i_k}^c,$$

and

$$h_k^s(\theta_1^s, \theta_2^s, ..., \theta_p^s) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le p} \theta_{i_1}^s \cdots \theta_{i_k}^s.$$

The complete homogeneous symmetric polynomials h_k^c are characterized by the following identity of formal power series

$$\frac{1}{\prod_{k=1}^p (1+\theta_k^c x)} = \sum_{k=0}^{\infty} (-1)^k h_k^c(\theta_1^c, \theta_2^c, ..., \theta_p^c) x^k,$$

where

$$h_k^c(\theta_1^c, \theta_2^c, ..., \theta_p^c) = \sum_{i=1}^{p} \frac{\theta_i^{p+k-1}}{\prod_{j=1, j\neq i}^p (\theta_i^c - \theta_j^c)}. \tag{43}$$

The same we can argue for h_k^s.

For (16), we have

$$R^c_{N,p}(f,x) = \frac{1}{2\prod_{k=1}^p(1+\theta^c_k e^{i\pi x})}\sum_{n=N+1}^{\infty}\Delta^p_n(\theta^c, f^c_n)e^{i\pi nx}$$
$$+\frac{1}{2\prod_{k=1}^p(1+\theta^c_k e^{-i\pi x})}\sum_{n=N+1}^{\infty}\Delta^p_n(\theta^c, f^c_n)e^{-i\pi nx}$$
$$=\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k h^c_k e^{i\pi kx}\sum_{n=N+1}^{\infty}\Delta^p_n(\theta^c, f^c_n)e^{i\pi nx}$$
$$+\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k h^c_k e^{-i\pi kx}\sum_{n=N+1}^{\infty}\Delta^p_n(\theta^c, f^c_n)e^{-i\pi nx}$$
$$=\frac{1}{2}\sum_{s=N+1}^{\infty}(-1)^k e^{i\pi sx}\sum_{n=N+1}^{s}(-1)^n\Delta^p_n(\theta^c, f^c_n)h^c_{s-n}$$
$$+\frac{1}{2}\sum_{s=N+1}^{\infty}(-1)^k e^{-i\pi sx}\sum_{n=N+1}^{s}(-1)^n\Delta^p_n(\theta^c, f^c_n)h^c_{s-n}$$
$$=\sum_{s=N+1}^{\infty}(-1)^k\cos\pi sx\sum_{n=N+1}^{s}(-1)^n\Delta^p_n(\theta^c, f^c_n)h^c_{s-n}.$$

Performing similar manipulations for $R^s_{N,p}(f,x)$, we get

$$R^s_{N,p}(f,x) = \sum_{s=N+1}^{\infty}(-1)^k\sin\pi(s-\tfrac{1}{2})x\sum_{n=N+1}^{s}(-1)^n\Delta^p_n(\theta^s, f^s_n)h^s_{s-n}.$$

Now from orthogonality of modified Fourier series we receive

$$||R_{N,p}(f,x)||^2_{L_2} = S_1 + S_2, \tag{44}$$

where

$$S_1 = \sum_{s=N+1}^{\infty}\left|\sum_{n=N+1}^{s}(-1)^n\Delta^p_n(\theta, f^c_n)h^c_{s-n}\right|^2,$$

and

$$S_2 = \sum_{s=N+1}^{\infty}\left|\sum_{n=N+1}^{s}(-1)^n\Delta^p_n(\theta, f^s_n)h^s_{s-n}\right|^2.$$

Theorem 9 *Let* $f^{(2q+p+1)} \in AC[-1,1]$, $q \geq 0, p \geq 1$. *Let*

$$f^{(2k+1)}(\pm 1) = 0,\ k = 0,\dots,q-1,$$

and

$$A^2_{2q+1}(f) + B^2_{2q+1}(f) \neq 0.$$

If the systems (21) *and* (22) *have unique solutions, then*

$$\lim_{N\to\infty} N^{2q+\frac{3}{2}}||R_{N,p}(f,x)||_{L_2} = c_{p,q}\sqrt{A^2_{2q+1}(f)+B^2_{2q+1}(f)},$$

where

$$c_{p,q} = \frac{1}{\pi^{2q+2}}\times$$
$$\times\frac{(p+2q+1)!}{(2q+1)!}\left(\int_1^\infty dt\left|\int_1^t \frac{(1-x)^p}{x^{2q+p+2}}\sum_{j=1}^p \frac{e^{-\tau_j(t-x)}}{\prod_{k=1,k\neq j}^p(\tau_j-\tau_k)}dx\right|^2\right)^{\frac{1}{2}}$$

and τ_k *are the roots of the following Laguerre polynomial*

$$\prod_{k=1}^p(1+\tau_k x) = \sum_{k=0}^p \gamma_k(\tau)x^k$$

with $\gamma_k(\tau)$ *defined by* (42).

Proof. We use the equation (44). In view of(43) and (39) we have as $N\to\infty$

$$h^c_{s-k} = \sum_{i=1}^p \frac{(\theta^c_i)^{p+s-n-1}}{\prod_{j=1,j\neq i}^p(\theta^c_i-\theta^c_j)} = N^{p-1}\sum_{i=1}^p \frac{(1-\frac{\tau_i}{N}+o(N^{-1}))^{p+s-n-1}}{\prod_{j=1,j\neq i}^p(\tau_j-\tau_i+o(1))}.$$

Substituting this and estimate (40) into the first term of the right-hand side of (44), and tending N to infinity, we derive the limit

$$\lim_{N\to\infty} N^{4q+3}S_1 = c^2_{p,q}A^2_{2q+1}.$$

Similarly,

$$\lim_{N\to\infty} N^{4q+3}S_2 = c^2_{p,q}B^2_{2q+1}$$

which conclude the proof. □

5 Conclusion

In this article, we have considered convergence acceleration of the modified expansions (see (1)) by application of trigonometric-rational error-correction functions. Corrections contain some unknown parameters determined according to the idea of the Fourier-Pade approximations. The resulting approximations we have named modified Fourier-Pade (MFP-) approximations (see (18)). We investigated pointwise and L_2 convergence of the MFP-approximations.

Section 3 deals with the pointwise convergence. The main result of this section is Theorem 8 shows that if f obeys first q derivative conditions, then, the convergence rate of the MFP-approximation is $O(N^{-2p-2q-2})$ if $f^{(2q+2p+2)} \in AC[-1,1]$. Comparing this with the corresponding estimate for the modified expansions (see Theorem 7), we get the additional convergence rate by factor $O(N^{-2p})$. However, for modified expansions we required less smoothness $f^{(2q+2)} \in AC[-1,1]$. As we mentioned in the Introduction, if a function f does not obey the first derivative conditions, then by polynomial subtraction method, we can get the same convergence rate for the modified and MFP-approximations.

Section 4 deals with L_2-convergence. Comparison of Theorems 6 and 9 shows the same convergence rates $O(N^{-2q-3/2})$ for smooth functions. However, the constant $c_{p,q}$ (independent of f) is much smaller than the constant c_q (see [21]) and the difference is so greater, how the larger are the values of q and p.

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