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Fuzzification of Ideals and Filters in Γ-Semigroups

A. Iampan

Department of Mathematics,
School of Science,
University of Phayao,
Phayao 56000, Thailand
aiyared.ia@up.ac.th

Abstract

In this paper, we characterize the relationship between the fuzzy ideals (fuzzy filters) and the characteristic mappings of fuzzy ideals (fuzzy filters) in Γ-semigroups.

Key Words: fuzzy ideal, fuzzy filter, characteristic mapping, Γ-semigroup
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1 Introduction and Prerequisites

The notion of a Γ-semigroup was introduced by Sen [7] in 1981 and that of fuzzy sets by Zadeh [10] in 1965, the fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. Rosenfeld [3] was the first who studied fuzzy sets in the structure of groups.

As we know, Γ-semigroups are a generalization of semigroups. The algebraic structures of Γ-semigroups were studied by many authors, for example, Prince Williams, Latha and Chandrasekeran [2] studied the fuzzification of bi-Γ-ideals in Γ-semigroups and investigate some of their related properties. Uçkun, Öztürk and Jun [9] studied the intuitionistic fuzzification of several types of a Γ-ideal in Γ-semigroups. Chinram [1] studied rough prime ideals and rough fuzzy prime ideals in Γ-semigroups. Sardar, Davvaz and Majumder [6] studied interior ideals of Γ-semigroups and investigate some of their basic properties.

In this paper, we consider a fuzzification of the concepts of a Γ-subsemigroup, an ideal and a filter in Γ-semigroups and some properties of such Γ-subsemigroups, ideals and filters are investigated.

Definition 1. *Let M be a set. A fuzzy subset of M is an arbitrary mapping $f\colon M \to [0,1]$ where $[0,1]$ is the unit segment of the real line.*

Definition 2. *Let M be a set and $A \subseteq M$. The characteristic mapping $f_A\colon M \to [0,1]$ defined via*

$$x \mapsto f_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

By the definition of characteristic mapping, f_A is a mapping of M into $\{0,1\} \subset [0,1]$. Hence f_A is a fuzzy subset of M.

Definition 3. *[7] Let M and Γ be any two nonempty sets. Then (M,Γ) is called a Γ-semigroup if there exists a mapping $M \times \Gamma \times M \to M$, written as $(a,\gamma,b) \mapsto a\gamma b$, satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$. A nonempty subset K of M is called a Γ-subsemigroup of M if $a\gamma b \in K$ for all $a,b \in K$ and $\gamma \in \Gamma$.*

Definition 4. *Let (M,Γ) be a Γ-semigroup.*

(i) *A nonempty subset A of M is called a left ideal of M if $M\Gamma A \subseteq A$.*

(ii) *A nonempty subset A of M is called a right ideal of M if $A\Gamma M \subseteq A$.*

(iii) *A nonempty subset A of M is called an ideal of M if it is both a left and a right ideal of M. That is, $M\Gamma A \subseteq A$ and $A\Gamma M \subseteq A$.*

Definition 5. *Let (M,Γ) be a Γ-semigroup. A Γ-subsemigroup F of M is called a filter of M if for any $a,b \in M$ and $\gamma \in \Gamma, a\gamma b \in F$ implies $a,b \in F$.*

Definition 6. *Let (M,Γ) be a Γ-semigroup.*

(i) *A fuzzy subset f of M is called a fuzzy left ideal of M if $f(a\gamma b) \geq f(b)$ for all $a,b \in M$ and $\gamma \in \Gamma$.*

(ii) *A fuzzy subset f of M is called a fuzzy right ideal of M if $f(a\gamma b) \geq f(a)$ for all $a,b \in M$ and $\gamma \in \Gamma$.*

(iii) *A fuzzy subset f of M is called a fuzzy ideal of M if it is both a fuzzy left and a fuzzy right ideal of M. That is, $f(a\gamma b) \geq f(b)$ and $f(a\gamma b) \geq f(a)$ for all $a,b \in M$ and $\gamma \in \Gamma$.*

Definition 7. *Let (M,Γ) be a Γ-semigroup. A fuzzy subset f of M is called a fuzzy filter of M if $f(a\gamma b) = \min\{f(a), f(b)\}$ for all $a,b \in M$ and $\gamma \in \Gamma$.*

Definition 8. *Let (M,Γ) be a Γ-semigroup. A fuzzy subset f of M is called a fuzzy Γ-subsemigroup of M if $f(a\gamma b) \geq \min\{f(a), f(b)\}$ for all $a,b \in M$ and $\gamma \in \Gamma$.*

Definition 9. *Let (M,Γ) be a Γ-semigroup and f a fuzzy subset of M. The mapping*

$$f' \colon M \to [0,1] \text{ defined via } f'(x) = 1 - f(x)$$

is a fuzzy subset of M *called the complement of* f *in* S*.*

Definition 10. *Let* (M, Γ) *be a* Γ*-semigroup. A fuzzy subset* f *of* M *is called prime if* $f(a\gamma b) \leq \max\{f(a), f(b)\}$ *for all* $a, b \in M$ *and* $\gamma \in \Gamma$*.*

2 Main Results

In this section, we give some interesting characterizations of the fuzzy ideals (fuzzy filters) and the characteristic mappings of fuzzy ideals (fuzzy filters) in Γ-semigroups.

Lemma 1. *Let* (M, Γ) *be a* Γ*-semigroup and* f *a fuzzy subset of* M*. Then the following statements are equivalent:*

(i) $f(x\gamma y) = \min\{f(x), f(y)\}$ *for all* $x, y \in M$ *and* $\gamma \in \Gamma$*.*

(ii) $f'(x\gamma y) = \max\{f'(x), f'(y)\}$ *for all* $x, y \in M$ *and* $\gamma \in \Gamma$*.*

Proof. Assume that $f(x\gamma y) = \min\{f(x), f(y)\}$ for $x, y \in M$ and $\gamma \in \Gamma$. Without loss of generality, we may assume that $f(x\gamma y) = f(x)$. Then $f(x) \leq f(y)$, so $f'(x\gamma y) = 1 - f(x\gamma y) = 1 - f(x) = f'(x)$ and $f'(x) = 1 - f(x) \geq 1 - f(y) = f'(y)$. Hence $f'(x\gamma y) = \max\{f'(x), f'(y)\}$.

Conversely, assume that $f'(x\gamma y) = \max\{f'(x), f'(y)\}$ for $x, y \in M$ and $\gamma \in \Gamma$. Without loss of generality, we may assume that $f'(x\gamma y) = f'(x)$. Then $f'(x) \geq f'(y)$, so $1 - f(x\gamma y) = 1 - f(x)$ and $1 - f(x) \geq 1 - f(y)$. Thus $f(x\gamma y) = f(x)$ and $f(x) \leq f(y)$. Hence $f(x\gamma y) = \min\{f(x), f(y)\}$. □

Proposition 1. *Let* (M, Γ) *be a* Γ*-semigroup and* $\emptyset \neq K \subseteq M$*. Then* K *is a* Γ*-subsemigroup of* M *if and only if the fuzzy subset* f_K *is a fuzzy* Γ*-subsemigroup of* M*.*

Proof. Obviously, f_K is a fuzzy subset of M. Let $x, y \in M$ and $\gamma \in \Gamma$. If $x \notin K$ or $y \notin K$, then $f_K(x) = 0$ or $f_K(y) = 0$ and so $f_K(x\gamma y) \geq 0 = \min\{f_K(x), f_K(y)\}$. Let $x, y \in K$. Then $f_K(x) = f_K(y) = 1$. Since $x\gamma y \in K\Gamma K \subseteq K$, we have $f_K(x\gamma y) = 1$. Thus $f_K(x\gamma y) = 1 \geq 1 = \min\{f_K(x), f_K(y)\}$. Therefore the fuzzy subset f_K is a fuzzy Γ-subsemigroup of M.

Conversely, let $x, y \in K$ and $\gamma \in \Gamma$. Then $f_K(x) = f_K(y) = 1$. Since f_K is a fuzzy Γ-subsemigroup of M, we have $f_K(x\gamma y) \geq \min\{f_K(x), f_K(y)\} = 1$. Thus $f_K(x\gamma y) = 1$ and so $x\gamma y \in K$. Therefore K is a Γ-subsemigroup of M. □

Proposition 2. *Let* (M, Γ) *be a* Γ*-semigroup and* $\emptyset \neq A \subseteq M$*. Then* A *is a prime subset of* M *if and only if the fuzzy subset* f_A *is a prime fuzzy subset of* M*.*

Proof. Obviously, f_A is a fuzzy subset of M. Let $x, y \in M$ and $\gamma \in \Gamma$. If $x\gamma y \notin A$, then $f_A(x\gamma y) = 0 \leq \max\{f_A(x), f_A(y)\}$. Let $x\gamma y \in A$. Then $f_A(x\gamma y) = 1$. Since A is a prime subset of M, we have $x \in A$ or $y \in A$. Thus $f_A(x) = 1$ or $f_A(y) = 1$ and so $f_A(x\gamma y) = 1 \leq 1 = \max\{f_A(x), f_A(y)\}$. Therefore the fuzzy subset f_A is a prime fuzzy subset of M.

Conversely, let $x, y \in M$ and $\gamma \in \Gamma$ be such that $x\gamma y \in A$. Then $f_A(x\gamma y) = 1$. Since f_A is a prime fuzzy subset of M, we have $1 = f_A(x\gamma y) \leq \max\{f_A(x), f_A(y)\}$. Thus $f_A(x) = 1$ or $f_A(y) = 1$ and so $x \in A$ or $y \in A$. Therefore A is a prime subset of M. □

Proposition 3. *Let (M, Γ) be a Γ-semigroup and $\emptyset \neq L \subseteq M$. Then L is a left ideal of M if and only if the fuzzy subset f_L is a fuzzy left ideal of M.*

Proof. Obviously, f_L is a fuzzy subset of M. Let $x, y \in M$ and $\gamma \in \Gamma$. If $y \notin L$, then $f_L(y) = 0$ and so $f_L(x\gamma y) \geq 0 = f_L(y)$. Let $y \in L$. Then $f_L(y) = 1$. Since $x\gamma y \in M\Gamma L \subseteq L$, we have $f_L(x\gamma y) = 1$. Thus $f_L(x\gamma y) = 1 \geq 1 = f_L(y)$. Therefore the fuzzy subset f_L is a fuzzy left ideal of M.

Conversely, let $x \in M, y \in L$ and $\gamma \in \Gamma$. Since f_L is a fuzzy left ideal of M, we have $f_L(x\gamma y) \geq f_L(y)$. Since $y \in L$, we have $f_L(y) = 1$. Thus $f_L(x\gamma y) = 1$ and so $x\gamma y \in L$. Therefore L is a left ideal of M. □

Corollary 1. *Let (M, Γ) be a Γ-semigroup and $\emptyset \neq I \subseteq M$. Then I is an ideal of M if and only if the fuzzy subset f_I is a fuzzy ideal of M.*

Proposition 4. *Let (M, Γ) be a Γ-semigroup and $\emptyset \neq F \subseteq M$. Then F is a filter of M if and only if the fuzzy subset f_F is a fuzzy filter of M.*

Proof. Obviously, f_F is a fuzzy subset of M. Let $x, y \in M$ and $\gamma \in \Gamma$. If $x\gamma y \notin F$, then $f_F(x\gamma y) = 0$. Since $x\gamma y \notin F$, we have $x \notin F$ or $y \notin F$. Thus $f_F(x) = 0$ or $f_F(y) = 0$, so $\min\{f_F(x), f_F(y)\} = 0$. Hence $f_F(x\gamma y) = 0 = \min\{f_F(x), f_F(y)\}$. Let $x\gamma y \in F$. Then $f_F(x\gamma y) = 1$. Since $x\gamma y \in F$, we have $x \in F$ and $y \in F$. Thus $f_F(x) = 1$ and $f_F(y) = 1$, so $\min\{f_F(x), f_F(y)\} = 1$. Hence $f_F(x\gamma y) = 1 = \min\{f_F(x), f_F(y)\}$. Therefore the fuzzy subset f_F is a fuzzy filter of M.

Conversely, let $x, y \in F$ and $\gamma \in \Gamma$. Since f_F is a fuzzy filter of M, we have $f_F(x\gamma y) = \min\{f_F(x), f_F(y)\}$. Suppose $x\gamma y \notin F$. Then $f_F(x\gamma y) = 0$, so $\min\{f_F(x), f_F(y)\} = 0$. Thus $f_F(x) = 0$ or $f_F(y) = 0$, so $x \notin F$ or $y \notin F$. It is impossible, hence $x\gamma y \in F$. Therefore F is a Γ-subsemigroup of M. Let $x, y \in M$ and $\gamma \in \Gamma$ be such that $x\gamma y \in F$. Since f_F is a fuzzy filter of M, we have $f_F(x\gamma y) = \min\{f_F(x), f_F(y)\}$. Since $x\gamma y \in F$, we have $f_F(x\gamma y) = 1$. Thus $\min\{f_F(x), f_F(y)\} = 1$ and so $f_F(x) = f_F(y) = 1$. Hence $x, y \in F$. Therefore F is a filter of M. □

Proposition 5. *Let (M, Γ) be a Γ-semigroup and f a fuzzy subset of M. Then f is a fuzzy filter of M if and only if the complement f' of f is a prime fuzzy ideal of M.*

Proof. Let $x, y \in M$ and $\gamma \in \Gamma$. Since f is a fuzzy filter of M, we have $f(x\gamma y) = \min\{f(x), f(y)\}$. By Lemma 1, we have $f'(x\gamma y) = \max\{f'(x), f'(y)\}$. Thus $f'(x\gamma y) \geq f'(x)$ and $f'(x\gamma y) \geq f'(y)$. Hence the complement f' of f is a fuzzy ideal of M. In the above proof, we have that $f'(x\gamma y) \leq \max\{f'(x), f'(y)\}$ for all $x, y \in M$ and $\gamma \in \Gamma$. Therefore the complement f' of f is a prime fuzzy ideal of M.

Conversely, let $x, y \in M$ and $\gamma \in \Gamma$. Since f' is a prime fuzzy ideal of M, we have $f'(x\gamma y) \geq f'(x), f'(x\gamma y) \geq f'(y)$, and $f'(x\gamma y) \leq \max\{f'(x), f'(y)\}$. Hence $f'(x\gamma y) = \max\{f'(x), f'(y)\}$. By Lemma 1, we have $f(x\gamma y) = \min\{f(x), f(y)\}$. Therefore f is a fuzzy filter of M. □

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