Armenian Journal of Mathematics
Volume 2, Number 3, 2009, 105–119

h-prime and h-semiprime ideals in Γ_N-semirings and Matrix Semiring $\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$

Sujit Kumar Sardar* and Bibhas Chandra Saha**

* *Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India.*
sksardarjumath@gmail.com

** *Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India.*
bibhas_sh@yahoo.co.in

Abstract

In this paper we show that h-prime and h-semiprime ideals in a Nobusawa Γ-semiring are preserved by the functions ${}^{*'}(\ )$, ${}^{+'}(\ )$, ${}^{*}(\ )$, ${}^{+}(\ )$, $\Gamma(\ )$ and $S(\ )$. This preservation property is then used to characterize h-prime and h-semiprime ideals in matrix semiring $\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$. Moreover, characterization theorems of h-regular and H-Noetherian $\Gamma-$semirings have been obtained.

Key Words: Γ_N-semiring, operator semirings, h-ideal, h-closure, h-prime ideal, h-semiprime ideal, H-Noetherian Γ-semring, h-regular Γ-semring.
Mathematics Subject Classification 2000: 16Y60, 20N10.

1 Introduction

Semiring is a well known universal algebra. If in a ring we do away with the requirement of having additive inverse of each element then the resulting algebraic structure becomes a semiring. Semiring has found applications in various branches of mathematics. In this connection reader may be referred to Golan's monograph[4]. Semiring arises very naturally as the nonnegative cone of a totally ordered ring. But the nonpositive cone of a totally ordered ring does not form a semiring because multiplication is no longer a binary composition. However, nonpositive cone of a totally ordered ring can be provided an algebraic home which is nothing but a $\Gamma-$semiring. $\Gamma-$semiring was introduced by Rao [9]. It generalizes not only semiring but also Γ-ring. Introducing the notion of operator semirings of a Γ-semiring, Dutta and Sardar enriched the theory of Γ-semirings [1][2][3]. They have studied the functions $(\ )^*$, $(\ )^+$, $(\ )^{*'}$, $(\ )^{+'}$ (definition follow) for ideals [2], prime ideals, prime radical [1], Jacobson radical, Levitzki radical, semiprime ideals, irreducible ideals [3] of Γ-semirings. For a Nobusawa Γ-semiring, introduced in [10], we find various other functions namely $^{*'}(\ )$, $^{+'}(\ )$, $^{*}(\ )$, $^{+}(\ )$, $\Gamma(\ )$ and $S(\ )$(definition follow).

In this paper we extend the results of Dutta and Sardar [1, 2, 3] to h-prime ideals and h-semiprime ideals in terms of the functions mentioned above. We then apply these results to obtain structure of matrix semiring $S_2 := \begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$, defined over a Γ_N-semiring.

It may be recalled here that though semiring is a generalization of ring, their ideal structures differ a lot, for example, an ideal of a semiring need be the kernel of some semiring morphism. To amend this gap different types of ideals namely k-ideals and h-ideals were introduced by Lattore [6] and Iizuka [5]. Olson et al [7][8], by using the notion of k-ideal and k-closure, introduced the notion of pre-prime and pre-semiprime ideals in semirings. We extended the same notion in the general setting of Γ-semirings and Γ_N-semirings. This also motivates us to introduce the notion of h-prime and h-semiprime ideals in semiring, Γ-semiring and Γ_N-semiring by using the notion of h-ideal and h-closure.

This paper is a continuation of the study of h-prime and h-semiprime ideals in Γ_N-semiring. Among other results we obtain the characterization theorems on H-Noetherianness, h-regularity for $S-$semiring Γ, $\Gamma-$semiring S and matrix semiring $\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$.

2 Preliminaries

We recall the following definitions and results which we shall use in the sequel.

Definition 2.1 [10] *Let S and Γ be two additive commutative monoids.*

(1) *If there exits a mapping $S\times\Gamma\times S\rightarrow S$, (with $(s_1,\gamma,\ s_2)\rightarrow\ s_1\gamma s_2\ \in\ S$) satisfying the following conditions:*
for all $s_1,\ s_2,\ s_3\in S$ and for all $\alpha,\beta\in\Gamma$
(i) $s_1\alpha(s_2+s_3)=s_1\alpha s_2+s_1\alpha s_3, (s_1+s_2)\alpha s_3=s_1\alpha s_3+s_2\alpha s_3, s_1(\alpha+\beta)s_2=s_1\alpha s_2+s_1\beta s_2$;
(ii) $s_1\alpha(s_2\beta s_3)=(s_1\alpha s_2)\beta s_3$;
(iii) $s_1\alpha 0=0\alpha s_1=0$, 0 is the zero element of S;
(iv) $s_1\theta s_2=0$, θ is the zero element of Γ, then S is called a Barnes Γ-semiring or simply a Γ-semiring [2].

(2) *If S is a Γ-semiring and Γ is a S-semiring and if*
(a) $s_1\alpha(s_2\beta s_3)=(s_1\alpha s_2)\beta s_3=s_1(\alpha s_2\beta)s_3$,

(b) $\alpha s_1(\beta s_2\gamma) = (\alpha s_1\beta)s_2\gamma = \alpha(s_1\beta s_2)\gamma$*,for all* $s_1,\ s_2,\ s_3 \in S$ *and for all* $\alpha, \beta, \gamma, \in \Gamma$*, then* *S is called a weak Nobusawa* Γ*-semiring or simply weak* Γ_N*-semiring .If in addition, we have the following condition* (3) *for all* $s_1,\ s_2 \in S, s_1\alpha s_2 = s_1\beta s_2$ *implies* $\alpha = \beta$*, then S is called a Nobusawa* Γ*-semiring or simply a* Γ_N*-semiring.*
Condition (3) *is called the Nobusawa condition* [12].

Note. As mentioned in the introduction, we note that if S and Γ are taken to be the non-positive cone of a totally ordered ring then S becomes a Γ-semiring where the ternary composition is the usual multiplication of integers.

Definition 2.2 [2] *Let S be a* Γ*-semiring and F be the free additive commutative semigroup generated by* $S\times\Gamma$*. Then the relation* ρ *on F, defined by* $\sum_{i=1}^{m}(x_i, \alpha_i)\rho\sum_{j=1}^{n}(y_j, \beta_j)$ *if and only if* $\sum_{i=1}^{m} x_i\alpha_i a = \sum_{j=1}^{n} y_j\beta_j a$ *for all* $a \in S$ *(m, n*$\in Z^+$*), is a congruence on F. Congruence class containing* $\sum_{i=1}^{m}(x_i, \alpha_i)$ *is denoted by* $\sum_{i=1}^{m}[x_i, \alpha_i]$*. Then* F/ρ *is an additive commutative semigroup. Now* F/ρ *forms a semiring with the multiplication defined by* $\left(\sum_{i=1}^{m}[x_i, \alpha_i]\right)\left(\sum_{j=1}^{n}[y_j, \beta_j]\right) = \sum_{ij}[x_i\alpha_i y_j, \beta_j]$*. This semiring is denoted by L and called the left operator semiring of the* Γ*-semiring S.*

Dually the right operator semiring R of the Γ*-semiring S has been defined where*

$$R = \left\{\sum_{i=1}^{m}[\alpha_i, x_i] : \alpha_i \in \Gamma, x_i \in S, i = 1, 2, ...m; m \in Z^+\right\}$$

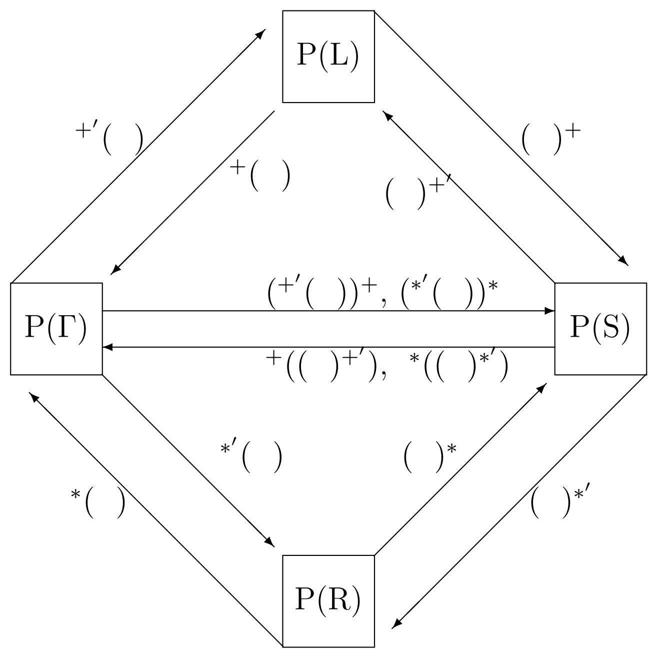
and the multiplication on R is defined as

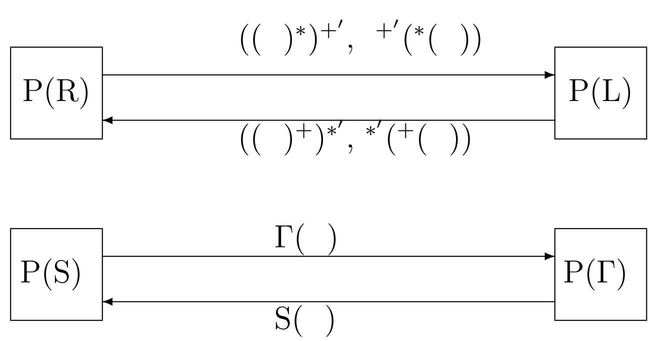
$$\left(\sum_{i=1}^{m}[\alpha_i, x_i]\right)\left(\sum_{j=1}^{n}[\beta_j, y_j]\right) = \sum_{ij}[\alpha_i, x_i\beta_j y_j].$$

Four different structures viz, S-semiring Γ, Γ-semiring S, the left operator semiring L and right operator semiring R are associated with a Γ_N-semiring S. There are various functions between power sets of any two of the above structures which are shown by the following diagrams.

The definition of the functions are as follows. Let S be a weak Γ_N-semiring and L and R be respectively the left and right operator semirings of the Γ-semring S. For P$\subseteq$L($\subseteq$R), P^+=$\{a\in S$: $[a, \Gamma]\subseteq P\}$(respectively P^*=$\{a\in S$: $[\Gamma, a]\subseteq P\}$). For Q$\subseteq$S, $Q^{+'}=\{\sum_{i=1}^{m}[x_i, \alpha_i]\in$ L: $\left(\sum_{i=1}^{m}[x_i, \alpha_i]\right)S\subseteq$Q$\}$, where $\left(\sum_{i=1}^{m}[x_i, \alpha_i]\right)$S denotes the sets of all finite sums $\sum_{i,j} x_i\alpha_i s_j,\ s_j\in$S.

Similarly, $Q^{*'}=\{\sum_{i=1}^{m}[\alpha_i, x_i]\in$ R: $S\left(\sum_{i=1}^{m}[\alpha_i, x_i]\right)\subseteq Q\}$, where $S\left(\sum_{i=1}^{m}[x_i, \alpha_i]\right)$ denotes the sets of all finite sums $\sum_{i,j} s_j\alpha_i x_i$, $s_j\in S$. For $P\subseteq R$ and $Q\subseteq L$ we define ${}^{*}P = \{\gamma \in \Gamma\colon [\gamma, S]=[\{\gamma\}, S]\subseteq P\}$, ${}^{+}Q = \{\gamma \in \Gamma\colon [S,\gamma]=[S, \{\gamma\}]\subseteq Q\}$. For $\Theta \subseteq \Gamma$ we define ${}^{*'}\Theta = \{r\in R\colon r\Gamma \subseteq \Theta\}$ and ${}^{+'}\Theta = \{l\in L\colon \Gamma l\subseteq \Theta\}$. It is known that ${}^{*}P({}^{+}P)$ is an ideal of the S-semiring Γ, where P is an ideal of R (respectively of L) and ${}^{*'}\Theta$ (${}^{+'}\Theta$) is an ideal of R (respectively of L), where Θ is an ideal of the S-semiring Γ.





$\Gamma(A)=\{\alpha \in \Gamma\colon S\alpha S\subseteq A\}$ and $S(\Phi)=\{s\in S\colon \Gamma s\Gamma \subseteq \Phi\}$, where $A\subseteq S$ and $\Phi \subseteq \Gamma$. We note that $\Gamma(A)$ is an ideal of the S-semiring Γ and $S(\Phi)$ is an ideal of the Γ-semiring S, where A is an ideal of the Γ-semiring S and Φ is an ideal of the S-semiring Γ.

Definition 2.3 [2] *Let S be a Γ-semiring and L and R be its left operator semiring and right operator semiring respectively. If there exists an element* $\sum_{i=1}^{m}[e_i,\delta_i] \in L$ ($\sum_{j=1}^{n}[\gamma_j,x_j] \in R$) *such that* $\sum_{i=1}^{m} e_i\delta_i a = a$ ($\sum_{j=1}^{n} a\gamma_j x_j = a$)*for all $a \in S$ then S is said to have the left unity* $\sum_{i=1}^{m}[e_i,\delta_i]$ *(respectively right unity* $\sum_{j=1}^{n}[\gamma_j,x_j]$*).*

It can be noted here that $\sum_{i=1}^{m}[e_i,\delta_i]$ (respectively $\sum_{j=1}^{n}[\gamma_j,x_j]$) is the identity of L (respectively of R).

Proposition 2.4 [11] *Let S be a Γ_N-semiring with unities and A be an ideal of the Γ-semiring S and Φ be an ideal of the S-semiring Γ. Then $\Gamma(A) = \Gamma A \Gamma$ and $S(\Phi) = S\Phi S$.*

Theorem 2.5 [10] *Let S be a Γ_N-semiring with unities and A be any ideal of the Γ-semiring S and Θ be any ideal of the S-semiring Γ. Then ${}^{+}(A^{+'}) = {}^{*}(A^{*'}) = \Gamma(A)$ and $({}^{+'}\Theta)^{+} = ({}^{*'}\Theta)^{*} = S(\Theta)$. i.e., the functions $({}^{+'}(\ ))^{+}$, $({}^{*'}(\ ))^{*}$,*$\Gamma(\ )$ and S(( )) *are the same.*

Definition 2.6 [6]([2]) *An ideal I of a semiring S (respectively Γ-semring) is called an h-ideal of S if $x + y_1 + z = y_2 + z$; $y_1, y_2 \in I$; $x, z \in S$ implies $x \in I$.*

Definition 2.7 [14]([13]) *Let A be an ideal of a semiring ($\Gamma-$semiring) S. The h-closure $\overline{A}$ of A is defined as* $\overline{A} = \{x \in S : x + a_1 + z = a_2 + z, \text{ for some } a_1, a_2 \in A, z \in S\}$

For the sake of convenience, we denote $\overline{A}$ by $H(A)$. Then it can be said that H defines a mapping from the power set P(S) of S to itself where S is a semiring or a $\Gamma-$semiring.

For more on preliminaries we refer to references and their references.

3 h-prime and h-semiprime ideals in Γ_N-semirings

The following are some preliminaries on h-prime and h-semiprime ideals in $\Gamma-$semirings (semirings) which we recall from [13] for immediate application.

Definition 3.1 [13] *An ideal P of a $\Gamma-$semiring (semiring) S is said to be an h-prime ideal of S if $I\Gamma J \subseteq H(P)$ (respectively $IJ \subseteq H(P)$)implies $I \subseteq H(P)$ or $J \subseteq H(P)$, where I, J are ideals of S.*

Definition 3.2 [13] *An ideal P of a $\Gamma-$semiring (semiring) S is said to be an h-semiprime ideal of S if $I\Gamma I \subseteq H(P)$ (respectively $I^2 \subseteq H(P)$)implies $I \subseteq H(P)$, where I are ideals of S.*

Now let us recall the following theorem from [13], which state that the functions$(\ )^{+'}$ and $(\ )^{*'}$ are inclusion preserving bijections for h-prime and h-semiprime ideals.

Theorem 3.3 [13] *There exists an inclusion preserving bijection between the h-prime or h-semiprime ideals of the* Γ*-semiring* S *and that of* $L(\ R)$ *via the mapping* $I \mapsto I^{+'}$ *(respectively,* $I \mapsto I^{*'}$ *).*

By an easy application of Theorem 3.3 we can deduce the following results which show that the functions $((\ )^{*})^{+'}$, $((\ )^{+})^{*'}$ are inclusion preserving bijections for h-prime and h-semiprime ideals.

Proposition 3.4 *Let* S *be a* Γ_N*-semiring with unities and* R*,* L *be respectively the right and left operator semirings of the* Γ*-semiring* S*.*
(i) If P *is an h-prime(h-semiprime) ideal of* R *then* $(P^{*})^{+'}$ *is an h-prime (respectively h-semiprime) ideal of* L*.*
(ii) If Q *is an h-prime (h-semiprime) ideal of* L *then* $(Q^{+})^{*'}$ *is an h-prime (respectively h-semiprime) ideal of* R*.*
(iii) $P \mapsto (P^{*})^{+'}$ *gives an inclusion preserving bijection between the h-prime (h-semiprime) ideals of* R *and that of* L*.*

Now we recall some lemmas from [13], which, together with some more lemmas obtained hereinafter, play a crucial role in the development of this paper. These lemmas are, infact, on commutative property of H (cf. Definition 2.7) with the functions under consideration.

Lemma 3.5 [13] *Let* S *be a* Γ*-semiring with the identities and* I *be any ideal of the* Γ*-semiring* S*. Then* $\mathrm{H}(\mathrm{I}^{*'}) = \mathrm{H}(\ \mathrm{I}\ )^{*'}$ *and* $\mathrm{H}(\mathrm{I}^{+'}) = \mathrm{H}(\ \mathrm{I}\ )^{+'}$*. i.e., H commutes with* $(\ )^{*'}$ *and* $(\ )^{+'}$*.*

Lemma 3.6 [13] *Let* S *be a* Γ*-semiring with the identities and* Q *be any ideal of the right operator semiring* R *(left operator semiring* L*) of the* Γ*-semiring* S*. Then* $H(Q^{*}) = H(Q)^{*}$ *(respectively* $H(Q^{+}) = H(Q)^{+}$*). i.e., H commutes with* $(\ )^{*}$ *and* $(\ )^{+}$*.*

Lemma 3.7 *Let* S *be a* Γ_N*-semiring with the identities and* Φ *be an ideal of the* S*-semiring* Γ*. Then* $H({}^{*'}\Phi) = {}^{*'}H(\Phi)$ *and* $H({}^{+'}\Phi) = {}^{+'}H(\Phi)$*. i.e., H commutes with* ${}^{*'}(\ )$ *and* ${}^{+'}(\ )$*.*

Proof. Since the argument is similar for ${}^{*'}(\ )$ and ${}^{+'}(\ )$, we prove the result for ${}^{*'}(\ )$. Let $\sum_{i=1}^{m}[e_i, \delta_i]$ and $\sum_{j=1}^{n}[\gamma_j, f_j]$ be respectively the left unity and right unity of the Γ-semiring S.

Let $x \in H({}^{*'}\Phi)$. Then there exist $r_1, r_2 \in {}^{*'}\Phi$ and $r \in$R such that $x + r_1 + r = r_2 + r$. Then $r_1\gamma, r_2\gamma \in$I and $r\gamma \in \Gamma$, for all $\gamma \in \Gamma$. Now $x\gamma + r_1\gamma + r\gamma = r_2\gamma + r\gamma$ implies $x\gamma \in$H(I), for all $\gamma \in \Gamma$. Hence $x \in {}^{*'}H(\Phi)$. Consequently, $H({}^{*'}\Phi) \subseteq {}^{*'}H(\Phi)$.

Conversely, let $\sum_k [\alpha_k, y_k] \in^{*'} H(\Phi)$. Then $\sum_k \alpha_k y_k \gamma \in H(\Phi)$, for all $\gamma \in \Gamma$. Hence we can find $t_1, t_2 \in \Phi$ and $\beta \in \Gamma$ such that $\sum_k \alpha_k y_k \gamma + t_1 + \beta = t_2 + \beta$. Then in particular, $\sum_k \alpha_k y_k \gamma_j + t_1 + \beta = t_2 + \beta$ for all j, which gives $\left[\sum_k \alpha_k y_k \gamma_j, s\right] + [t_1, s] + [\beta, s] = [t_2, s] + [\beta, s]$, for all $s \in$S and for all j whence $\left[\sum_k \alpha_k y_k \gamma_j, f_j\right] + [t_1, f_j] + [\beta, f_j] = [t_2, f_j] + [\beta, f_j]$, for all j.

Summing over j we deduce that

$$\sum_j \left[\sum_k \alpha_k y_k \gamma_j, f_j\right] + \sum_j [t_1, f_j] + \sum_j [\beta, f_j] = \sum_j [t_2, f_j] + \sum_j [\beta, f_j]$$

which gives

$$\sum_k [\alpha_k, y_k] \sum_j [\gamma_j, f_j] + \sum_j [t_1, f_j] + \sum_j [\beta, f_j] = \sum_j [t_2, f_j] + \sum_j [\beta, f_j].$$

Hence $\sum_k [\alpha_k, y_k] + \sum_j [t_1, f_j] + \sum_j [\beta, f_j] = \sum_j [t_2, f_j] + \sum_j [\beta, f_j]$..........(A)

Since Φ is an ideal of the S-semiring Γ and $t_1, t_2 \in \Phi$, we see that $\sum_j t_1 f_j \gamma$, $\sum_j t_2 f_j \gamma \in \Phi$, for all $\gamma \in \Gamma$. Hence $\sum_j [t_1, f_j], \sum_j [t_2, f_j] \in^{*'} \Phi$. Again $\beta \in \Gamma$, so $\sum_j [\beta, f_j] \in$R. Therefore, from (A), $\sum_k [\alpha_k, y_k] \in H({}^{*'}\Phi)$. Thus ${}^{*'}H(\Phi) \subseteq H({}^{*'}\Phi)$. Consequently, $H({}^{*'}\Phi) ={}^{*'} H(\Phi)$.□

Lemma 3.8 *Let S be a Γ_N-semiring with the identities and Q be an ideal of the right operator semiring R (left operator semiring L) of the Γ-semiring S. Then H(${}^*Q$) =*H(Q) (respectively H(${}^+Q$) =${}^+$H(Q)). i.e., H commutes with *( ) and ${}^+$( ).*

Proof. Since the argument is similar for *( ) and ${}^+$( ), we prove the result for *( ). Let $\sum_{i=1}^{m} [e_i, \delta_i]$ and $\sum_{j=1}^{n} [\gamma_j, f_j]$ be respectively the left unity and right unity of the Γ-semiring S.

Let x∈H(${}^*Q$). Then there exist $y_1, y_2 \in^* Q$ and $\alpha \in \Gamma$ such that $x + y_1 + \alpha = y_2 + \alpha$. This gives$[x + y_1 + \alpha, s] = [y_2 + \alpha, s]$, for all $s \in$S whence
$[x, s] + [y_1, s] + [\alpha, s] = [y_2, s] + [\alpha, s]$, for all for all $s \in$S.
Now $[y_1, s], [y_2, s] \in$Q and $[x, s], [\alpha, s] \in$R. Hence $[x, s] \in$H(Q) for all $s \in$S whence $x \in^*$H(Q). Consequently, H(${}^*Q$) $\subseteq^* H(Q)$.

To prove the reverse containment, let $x \in^*$H(Q). Then $[x, s] \in$H(Q), for all $s \in$S. So we can find $q_1, q_2 \in$Q and $r \in$R such that $[x, s] + q_1 + r = q_2 + r$, for all $s \in$S. In particular, $[x, e_i] + q_1 + r = q_2 + r$, for all i which gives $[x, e_i]\gamma + q_1\gamma + r\gamma = q_2\gamma + r\gamma$, for all i and for all $\gamma \in$

Γ. In particular, $[x, e_i]\delta_i + q_1\delta_i + r\delta_i = q_2\delta_i + r\delta_i$, for all i whence $xe_i\delta_i + q_1\delta_i + r\delta_i = q_2\delta_i + r\delta_i$, for all i. Summing over i we easily deduce that $\sum_i xe_i\delta_i + \sum_i q_1\delta_i + \sum_i r\delta_i = \sum_i q_2\delta_i + \sum_i r\delta_i$ which gives $x + \sum_i q_1\delta_i + \sum_i r\delta_i = \sum_i q_2\delta_i + \sum_i r\delta_i$. Now $\sum_i q_1\delta_i, \sum_i q_2\delta_i \in^* Q$ and $\sum_i r\delta_i \in \Gamma$. Hence it follows that $x \in H(^*Q)$. Thus $^*H(Q) \subseteq$H($^*Q$). Consequently, H($^*Q$) =*H(Q).□

Lemma 3.9 *Let S be a Γ_N-semiring with unities and P be an ideal of the Γ-semiring S and Φ be an ideal of the S-semiring Γ. Then $H(\Gamma(P)) = \Gamma(H(P))$ and $H(S(\Phi)) = S(H(\Phi))$. i.e., H commutes with $\Gamma(\ )$ and $S(\ )$.*

Proof. Since the argument is similar for Γ(H( )) and S(H( )), we prove the result for Γ(H( )). Let $\alpha \in H(\Gamma(P))$. Then $\alpha + \beta + \gamma = \delta + \gamma$, for some $\beta, \delta \in \Gamma(P)$ and $\gamma \in \Gamma$. Hence $s_1(\alpha+\beta+\gamma)s_2 = s_1(\delta+\gamma)s_2$, for all $s_1, s_2 \in$S which gives $s_1\alpha s_2 + s_1\beta s_2 + s_1\gamma s_2 = s_1\delta s_2 + s_1\gamma s_2$, for all $s_1, s_2 \in$S. Now since $s_1\beta s_2, s_1\delta s_2 \in$P, for all $s_1, s_2 \in$S (as $\beta, \delta \in \Gamma(P)$) and $s_1\gamma s_2 \in$S, from the above relation we see that $s_1\alpha s_2 \in \overline{P}$, for all $s_1, s_2 \in$S. Hence $\alpha \in \Gamma(H(P))$. Consequently, $H(\Gamma(P)) \subseteq \Gamma(H(P))$.

Conversely, let $\alpha \in \Gamma(H(P)) = \Gamma H(P)\Gamma$ (cf. Proposition 2.4). Then there exist $\alpha_i,\ \beta_i \in \Gamma$ and $p_i \in H(P)$ $(i = 1, 2....n)$ such that $\alpha = \sum_{i=1}^{m} \alpha_i p_i \beta_i$. Again p_i being in $H(P)$, $i = 1, 2....n$, there exist $t_i, k_i \in P$ and $u_i \in$S such that $p_i + t_i + u_i = k_i + u_i$ for all $i = 1, 2....n$. This gives that $\sum_{i=1}^{m} \alpha_i(p_i + t_i + u_i)\beta_i = \sum_{i=1}^{m} \alpha_i(k_i + u_i)\beta_i$ whence we deduce that $\sum_{i=1}^{m} \alpha_i p_i\beta_i + \sum_{i=1}^{m} \alpha_i t_i\beta_i + \sum_{i=1}^{m} \alpha_i u_i\beta_i = \sum_{i=1}^{m} \alpha_i k_i\beta_i + \sum_{i=1}^{m} \alpha_i u_i\beta_i$, where $\sum_{i=1}^{m} \alpha_i t_i\beta_i$, $\sum_{i=1}^{m} \alpha_i k_i\beta_i \in \Gamma P\Gamma$, and $\sum_{i=1}^{m} \alpha_i u_i\beta_i \in \Gamma$. Hence $\sum_{i=1}^{m} \alpha_i p_i\beta_i \in H(\Gamma P\Gamma) = H(\Gamma(P))$ (cf. Proposition 2.4) whence $\alpha \in H(\Gamma(P))$. Thus $\Gamma(H(P)) \subseteq H(\Gamma(P))$. Consequently, $H(\Gamma(P)) = \Gamma(H(P))$.□

Now we apply the above lemmas to investigate the behaviour, of the functions under consideration, regarding the preservation of h-prime and h-semiprime ideals.

The following proposition shows that h-prime(h-semiprime) ideals are preserved by the functions $^*(\ )$ and $^+(\ )$.

Proposition 3.10 *Let S be a Γ_N-semiring with unities and P be an h-prime or h-semiprime ideal of the right operator semiring R (the left operator semiring L) of the Γ-semiring S . Then $^*P$ (respectively ^+P) is an h-prime or h-semiprime ideal of the S-semiring Γ according as P is h-prime or h-semiprime.*

Proof. We prove the result for h-prime ideals as the proof for the h-semiprime ideal is similar. We first prove the result for the right operator semiring R. Let P be an h-prime ideal of R and A and B be two ideals of the S-semiring Γ such that $ASB \subseteq H(^*P)$. Then $ASB \subseteq^* H(P)$ (cf. Lemma 3.8). Hence $[ASB, S] \subseteq H(P)$ (cf. Definition of $^*P$) which gives $[A, S][B, S] \subseteq H(P)$. Now since $[A, S], [B, S]$ are ideals of R and P is an h-prime ideal of R, we see that either $[A, S] \subseteq H(P)$ or $[B, S] \subseteq H(P)$ whence by definition of $^*(\ )$, $A \subseteq^* H(P) = H(^*P)$. From this by using Lemma 3.8, we easily obtain $B \subseteq^* H(P) = H(^*P)$. Consequently, $^*P$ is an h-prime ideal of the S-semiring Γ. Using Lemma 3.8, an analogous argument proves the result for the left operator semiring L.□

Now by using Lemma 3.7 we can deduce the following result on preservation of h-prime and h-semiprime ideals by the functions $^{*'}(\ )$ and $^{+'}(\ )$.

Proposition 3.11 *Let S be a Γ_N-semiring with unities and Φ be an h-prime or h-semiprime ideal of the S-semiring Γ. Then $^{*'}\Phi$ ($^{+'}\Phi$) is an h-prime or h-semiprime ideal of the right operator semiring R (respectively the left operator semiring L)of the Γ-semiring S according as Φ is h-prime or h-semiprime.*

Now, for an immediate use, we recall from [10] the following lattice isomorphism theorem between the set of ideals of the S-semiring Γ and that of L and R.

Theorem 3.12 [10] *Let S be a Γ_N-semiring with unities and R(L) be the right operator semiring (respectively the left operator semiring) of the Γ-semiring S. Then the lattices of all ideals of the S-semiring Γ and R(L) are isomorphic via the mapping $\Theta \to^{*'} \Theta$ (respectively $\Theta \to^{+'} \Theta$).*

By using the above theorem and Propositions 3.10, 3.11, we deduce below that $^{*'}(\ )$, $^{+'}(\ )$ are inclusion preserving bijections for h-prime and h-semiprime ideals.

Theorem 3.13 *Let S be a Γ_N-semiring with unities and R (L) be the right operator semiring(respectively the left operator semiring) of the Γ-semiring S. Then there exists an inclusion preserving bijection between the h-prime or h-semiprime ideals of the S-semiring Γ and that of R (respectively L) via the mapping $I \mapsto^{*'} I$ (respectively $I \mapsto^{+'} I$).*

Proof. Since the argument is similar for h-prime and h-semiprime ideals we prove the result for h-semiprime ideals. Let P be an h-prime ideal of the $S-$semiring Γ. Then by the successive use of Propositions 3.11 and 3.10 we see that $^*(^{*'}P)$ is an h-prime ideal of Γ. Again by Theorem 3.12, $^*(^{*'}P) = P$. Consequently, P is an h-prime ideal of Γ.

Conversely, let P be an h-prime ideal of R. Then $^*Q$ is an h-prime ideal of Γ (cf. Proposition 3.10). Hence $^{*'}(^*Q)$ be an h-prime ideal of R (cf. Proposition 3.11). Consequently, P is an h-prime ideal of R(cf. Theorem 3.12). Now by using Theorem 3.12 we see that the function is inclusion preserving.

If we use the functions ${}^{+}(\ )$ and ${}^{+'}(\ )$ respectively instead of the functions ${}^{*}(\ )$ and ${}^{*'}(\ )$ then we can deduce the result for the left operator semiring L.□

The following are the results on preservation of h-prime and h-semiprime ideals by the functions Γ( ) and S( ).

Proposition 3.14 *Let S be a Γ_N-semiring with unities. If A is an h-prime (h-semiprime) ideal of the Γ-semiring S and Φ is an h-prime (h-semiprime) ideal of the S-semiring Γ. Then $\Gamma(A)$ is an h-prime (respectively h-semiprime) ideal of the S-semiring Γ and $S(\Phi)$ is an h-prime (respectively h-semiprime) ideal of Γ-semiring S.*

Proof. Let A be an h-prime ideal of S. Then by Theorem 3.3, $A^{*'}$ is an h-prime ideal of the right operator semiring R. Hence by Proposition 3.10, we see that ${}^{*}(A^{*'})$ is h-prime in the S$-$semiring Γ. Now by Theorem 2.5 we have $\Gamma(A) =^{*} (A^{*'})$. Consequently, $\Gamma(A)$ is an h-prime ideal of the S-semiring Γ. Now, in an analogous way, by using Theorem 2.5, Theorem 3.3 and Proposition 3.10 we can deduce the result for h-semiprime ideal.□

Now we recall the definition of H-Noetherian Γ-semiring(semiring).

Definition 3.15 [13] *A Γ-semiring (semiring) S is said to be an H-Noetherian Γ-semiring (respectively semiring) if for any ascending chain $I_1 \subseteq I_2 \subseteq$ .... of ideals of S the corresponding ascending chain $H(I_1) \subseteq H(I_2) \subseteq$ .... of ideals terminates.*

It can be recalled here that a Γ-semiring(semiring) S is H-Noetherian if it is Noetherian and a Γ$-$semiring S is H-Noetherian if and only if L(R) is H-Noetherian [13].

Now as an application of Lemmas 3.5, 3.6, 3.7, 3.8 and Theorems 6.6[2], 3.12 we obtain the following characterization theorem of H-Noetherian Γ-semiring.

Theorem 3.16 *Let S be a weak Γ_N-semiring with unities. Then the Γ-semiring S is H-Noetherian if and only if the S-semiring Γ is H-Noetherian.*

Proof. Let the Γ-semiring S be H-Noetherian and $\Theta_1 \subseteq \Theta_2 \subseteq$ .... be a sequence of ascending chain of ideals in the S-semiring Γ. Then ${}^{*'}\Theta_1 \subseteq^{*'} \Theta_2 \subseteq$ .... is a sequence of ascending chain of ideals in R (cf. Theorem 3.12). Hence $({}^{*'}\Theta_1)^{*} \subseteq ({}^{*'}\Theta_2)^{*} \subseteq$ .... is a sequence of ascending chain of ideals in Γ-semiring S (cf. Theorem 6.6 of [2]). Now, since S is H-Noetherian there exists a positive integer m such that $H(({}^{*'}\Theta_i)^{*}) = H(({}^{*'}\Theta_m)^{*})$, for all $i \geq m$. Hence $({}^{*'}H(\Theta_i))^{*} = ({}^{*'}H(\Theta_m))^{*}$, for all $i \geq m$ (successively using Lemma 3.6 and Lemma 3.7). This implies that $(({}^{*'}H(\Theta_i))^{*})^{*'} = (({}^{*'}H(\Theta_m))^{*})^{*'}$, for all $i \geq m$. Hence ${}^{*'}H(\Theta_i) =^{*'} H(\Theta_m)$, for all $i \geq m$ (cf. Theorem 6.6 of[2]) whence ${}^{*}({}^{*'}H(\Theta_i)) =^{*} ({}^{*'}H(\Theta_m))$, for all $i \geq m$. Thus we see that $H(\Theta_i) = H(\Theta_m)$, for all $i \geq m$ (cf. Theorem 3.12). Consequently, the S-semiring Γ is H-Noetherian.

If we use Lemmas 3.8 and 3.5 instead of Lemmas 3.6 and 3.7 respectively, we can deduce the converse by reversing the above argument.□

In [13] we define a Γ-semiring(semiring) S to be h-regular if every h-ideal of S is semiprime and prove that S is h-regular if and only if every ideal of S is h-semiprime. Also we have shown that S is h-regular if and only if L(R) is h-regular.

Using the above results we obtain the following theorem which characterizes an h-regular Γ-semiring.

Theorem 3.17 *Let S be a weak Γ_N-semiring. Then the* Γ*-semiring S is h-regular if and only if S-semiring* Γ *is h-regular.*

Proof. Let the Γ-semiring S be h-regular. Then R is also h-regular. Let Φ be an ideal of S-semiring Γ. Then ${}^{*'}\Phi$ is an ideal of R. Hence ${}^{*'}\Phi$ is an h-semiprime ideal of R whence ${}^{*}({}^{*'}\Phi)$ is an h-semiprime ideal of the S-semiring Γ(cf. Proposition 3.10). Now by Theorem 3.12, ${}^{*}({}^{*'}\Phi) = \Phi$. Hence Φ is an h-semiprime ideal of the S-semiring Γ. Consequently,the S-semiring Γ is h-regular.

The converse follows by slight modification of the reverse argument.□

4 h-prime and h-semiprime ideals in matrix semiring $S_2 = \begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$

It is well known that for a $\Gamma_{\mathtt{N}}$-semiring S with R and L respectively its right and left operator semirings,

$$S_2 = \left\{ \begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} : r \in R, l \in L, \gamma \in \Gamma, s \in S \right\}$$

forms a semiring with respect to the addition and multiplication defined as follows:

$$\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} + \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & \gamma_1 + \gamma_2 \\ s_1 + s_2 & l_1 + l_2 \end{pmatrix},$$

$$\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} = \begin{pmatrix} r_1 r_2 + [\gamma_1, s_2] & r_1\gamma_2 + \gamma_1 l_2 \\ s_1 r_2 + l_1 s_2 & [s_1, \gamma_2] + l_1 l_2 \end{pmatrix}$$

and $I_2 := \begin{pmatrix} \mathtt{I}^{*'} & \Gamma(\mathtt{I}) \\ \mathtt{I} & \mathtt{I}^{+'} \end{pmatrix}$ is an ideal of S_2 for an ideal I of S. Also every ideal of S_2 is of the form I_2, for some ideal I of S and an ideal I_2 is prime (semiprime) if and only if the corresponding ideal I of S is prime (respectively semiprime). Throughout this section S will denote a $\Gamma_{\mathtt{N}}$-semiring with unities and to denote the left and right operator semirings we shall use L and R respectively.

Now we establish the following lemma, on commutativity of the H-closure (cf. Definition 2.7), which we use here extensively to study the matrix semiring S_2.

Lemma 4.1 *$H(P_2) = H(P)_2$, where P is an ideal of S.*

Proof. Let $\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} \in H(P_2)$, where $s \in S, r \in R, \gamma \in \Gamma, l \in L$. Then $\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} + \begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} + \begin{pmatrix} r_3 & \gamma_3 \\ s_3 & l_3 \end{pmatrix} = \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} + \begin{pmatrix} r_3 & \gamma_3 \\ s_3 & l_3 \end{pmatrix}$, for some

$$\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix}, \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} \in P_2$$

and $\begin{pmatrix} r_3 & \gamma_3 \\ s_3 & l_3 \end{pmatrix} \in S_2$, where $s_1, s_3, s_3 \in P$, $r_1, r_3, r_3 \in P^{*'}$, $\gamma_1, \gamma_2, \gamma_3 \in \Gamma(P), l_1, l_2, l_3 \in P^{+'}$ whence we obtain

$$\begin{pmatrix} r + r_1 + r_3 & \gamma + \gamma_1 + \gamma_3 \\ s + s_1 + s_3 & l + l_1 + l_3 \end{pmatrix} = \begin{pmatrix} r_2 + r_3 & \gamma_2 + \gamma_3 \\ s_2 + s_3 & l_2 + l_3 \end{pmatrix}.$$

Hence $r + r_1 + r_3 = r_2 + r_3, \gamma + \gamma_1 + \gamma_3 = \gamma_2 + \gamma_3$, $s + s_1 + s_3 = s_2 + s_3, l + l_1 + l_3 = l_2 + l_3$. From these relations we see that $r \in H(P^{*'})$, $\gamma \in H(\Gamma(P))$, $s \in H(P), l \in H(P^{+'})$. Now by Lemma 3.5, $H(P^{*'}) = H(P)^{*'}$, by Lemma 3.9, $H(\Gamma(P)) = \Gamma(H(P))$ and by Lemma 3.5, $H(P^{+'}) = H(P)^{+'}$. Hence $r \in H(P)^{*'}$, $\gamma \in \Gamma(H(P))$, $l \in H(P)^{+'}$ whence we find that $\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} \in \begin{pmatrix} H(P)^{*'} & \Gamma(H(P)) \\ H(P) & H(P)^{+'} \end{pmatrix} = H(P)_2$. Thus $H(P_2) \subseteq H(P)_2$.
We can prove the reverse containment by reversing the above argument. Hence we conclude that $H(P_2) = H(P)_2$.□

The following theorem gives a characterization of h-prime ideals of the matrix semiring S_2.

Theorem 4.2 *Let P be an ideal of S. Then $P_2 = \begin{pmatrix} P^{*'} & \Gamma(P) \\ P & P^{+'} \end{pmatrix}$ is an h-prime(h-semiprime) ideal of S_2 if and only if P is an h-prime(respectively h-semiprime) ideal of S.*

Proof. Since the argument is similar for h-prime and h-semiprime ideal we prove the result for h-prime ideal. Let P_2 be an h-prime ideal of S_2. Then $H(P_2)$ is a prime ideal of S_2 (Olson [7]). Hence by Lemma 4.1, $H(P)_2$ is a prime ideal of S_2. Hence from the discussion made in the beginning of this section we see that $H(P)$ is a prime ideal of S. Consequently, P is an h-prime ideal of S.

The converse follows by reversing the above argument.□

Now we obtain following lemmas which we use to obtain further characterization of h-prime and h-semiprime ideals of the matrix semiring S_2.

Lemma 4.3 *Let P be an ideal of $R(L)$. Then $\Gamma(P^*) =^* P$ (respectively $\Gamma(Q^+) =^+ Q$).*

Proof. From Theorem 2.5 we see that $\Gamma(A) =^* (A^{*'})$, for any ideal A of the Γ-semiring S. Now since P is an ideal of R, P^* is an ideal of S and $P = (P^*)^{*'}$(cf.Theorem 6.6 [2]).Consequently, $\Gamma(P^*) =^* ((P^*)^{*'}) =^* P$. By using similar argument we can prove the result for L.□

The following theorem is another characterization of h-prime and h-semiprime ideal of the matrix semiring S_2.

Theorem 4.4 *Let P be an ideal of R. Then $\begin{pmatrix} P & {}^*P \\ P^* & (P^*)^{+'} \end{pmatrix}$ is an h-prime or h-semiprime ideal of S_2 if and only if P is an h-prime or h-semiprime ideal of R.*

Proof. Since P is an ideal of R, P^* is an ideal of S. Hence by definition, $(P^*)_2 = \begin{pmatrix} (P^*)^{*'} & \Gamma(P^*) \\ P^* & (P^*)^{+'} \end{pmatrix}$. Now by Theorem 6.6 of [2], $P = (P^*)^{*'}$. Again by Lemma 4.3, ${}^*P = \Gamma(P^*)$. Hence we easily deduce that $(P^*)_2 = \begin{pmatrix} P & {}^*P \\ P^* & (P^*)^{+'} \end{pmatrix}$.

Now, if P is an h-prime ideal of R, then P^* is an h-prime ideal of S. Hence by using Theorem 4.2, $(P^*)_2$ is an h-prime ideal of S_2. Consequently, $\begin{pmatrix} P & {}^*P \\ P^* & (P^*)^{+'} \end{pmatrix}$ is an h-prime ideal of S_2.

Conversely, suppose $\begin{pmatrix} P & {}^*P \\ P^* & (P^*)^{+'} \end{pmatrix}$ is an h-prime ideal of the matrix semiring S_2. Now by Lemma 4.3 and Theorem 6.6 of [2], $\begin{pmatrix} P & {}^*P \\ P^* & (P^*)^{+'} \end{pmatrix} = \begin{pmatrix} (P^*)^{*'} & \Gamma(P^*) \\ P^* & (P^*)^{+'} \end{pmatrix}$. Hence $\begin{pmatrix} (P^*)^{*'} & \Gamma(P^*) \\ P^* & (P^*)^{+'} \end{pmatrix}$ is an h-prime ideal of S_2. Consequently, by Theorem 4.2 we see that P^* is an h-prime ideal of S. Now by using Theorem 3.3, we conclude that P is an h-prime ideal of R. By adopting analogous argument we can prove the the result for h-semiprime ideal.□

By using the Lemma 4.3 and by using analogy between L and R and following the argument of the proof of Theorem 4.4 we can obtain the following characterization theorem of h-prime and h-semiprime ideals of the matrix semiring S_2 in terms of the left operator semiring L.

Theorem 4.5 *Let Q be an ideal of L. Then $\begin{pmatrix} (Q^+)^{*'} & {}^+Q \\ Q^+ & Q \end{pmatrix}$ is an h-prime (h-semiprime) ideal of S_2 if and only if Q is an h-prime (respectively h-semiprime) ideal of L.*

The following theorem characterizes H-Noetherianness of the matrix semiring S_2.

Theorem 4.6 *The matrix semiring S_2 is H-Noetherian if and only if the* Γ*-semiring S is H-Noetherian.*

Proof. Let S be H-Noetherian and $(I^1)_2 \subseteq (I^2)_2 \subseteq (I^3)_2 \subseteq (I^4)_2 \subseteq ....$ be an ascending chain of ideals of S_2. Then $H((I^1)_2) \subseteq H((I^2)_2) \subseteq H((I^3)_2) \subseteq H((I^4)_2) \subseteq ....$ is an ascending chain of ideals of S_2. Hence by Lemma 4.1 $H(I^1)_2 \subseteq H(I^2)_2 \subseteq H(I^3))_2 \subseteq H(I^4)_2 \subseteq ....$ is an ascending chain of ideals of S_2 whence we see that $H(I^1) \subseteq H(I^2) \subseteq H(I^3) \subseteq H(I^4) \subseteq$ .... is an ascending chain of ideals of S. Since S is H-Noetherian, there is a positive integer n such that $H(I^k) = H(I^n)$, for all $k \geq n$. This gives $H(I^k)_2 = H(I^n)_2$, for all $k \geq n$. Hence by Lemma 4.1 we deduce that $H((I^k)_2) = H((I^n)_2)$, for all $k \geq n$. Consequently, S_2 is an H-Noetherian semiring.
By reversing the above argument we can deduce the converse.□

Now to conclude the paper we obtain the following characterization of h-regularity of the matrix semiring S_2.

Theorem 4.7 *The matrix semiring S_2 is h-regular if and only if* Γ*-semiring S is h-regular.*

Proof. Let S be h-regular and I_2 be an ideal of S_2. Then I is an ideal of S. Hence by h-regularity of S we see that I is an h-semiprime ideal of S. This means that I_2 is an h-semiprime ideal in S_2 (cf. Theorem 4.2). Hence S_2 is h-regular. By reversing the above argument the converse can be easily deduced.□

References

[1] T.K. Dutta and S.K. Sardar, *On prime ideals and prime radicals of a* Γ*- semirings*; ANALELE STINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI, XLVI, s.l a, Matematica, 2000, f.2 319-329.

[2] T.K. Dutta and S.K. Sardar, *On the Operator semirings of a* Γ*- semiring*, Southeast Asian Bull. Math. **26**(2002), 203-213 .

[3] T.K. Dutta and S.K. Sardar, *Semiprime ideals and irreducible ideals of* Γ*-semirings*, Novi Sad J. Math. **30**(2000), no. 1 , 97-108.

[4] J.S. Golan, The Theory of semirings with application in Mathematics and Theoretical Computer Science, Pitman Monographs and surveys in Pure and Applied Mathematics, 54, Longman Sci. Tech. Harlow, 1992.

[5] K. Iizuka, O*n the Jacobson radical of a semiring*, Tohoku Math.J. **11**(1959), no. 2, 409-421.

[6] D.R. Latorre, *On h-ideals and k-ideals in hemirings*, Publ. Math, Deberecen, **12**(1965), 219-226.

[7] D.M. Olson and A.C. Nance, *A note on radicals for hemirings*, Quaestiones Mathematicae **12**(1989), 307-314.

[8] D.M. Olson, G.A.P. Heyman and H.J. Le Roux, *Weakly special classes of hemirings*, Quaestiones Mathematicae **15**(1992), 119-126.

[9] M.K. Rao, Γ-*semiring-1*, Southeast Asian Bull. of Math. **19**(1995), 49-54.

[10] S.K. Sardar and B.C. Saha, *On Nobusawa Gamma Semirings*, Universitatee Din Bacau Studii Si Cercetari Stiinifice Seria:Mathematica, **18**(2008), 283-306.

[11] S.K. Sardar and B.C. Saha, *On Nobusawa Gamma Semirings and Matrix Semiring* $\begin{pmatrix} \boldsymbol{R} & \Gamma \\ \boldsymbol{S} & \boldsymbol{L} \end{pmatrix}$, South East Asian Journal of Mathematics and Mathematical Sciences (To appear).

[12] S.K. Sardar and B.C. Saha, *Semiring Multiplications on Additive Commutative Monoids and* Γ-*semirings*, Proc. International Conf. on Semigroups and its applications, Kerala University, India(2006), 99-110.

[13] S.K. Sardar, B.C. Saha and K.P. Shum, *h-prime and h-semiprime ideals in semirings and* Γ-*semirings* (communicated).

[14] J. Zhan and W. A. Dudek, *Fuzzy h-ideals of hemirings*, Information Sciences, **177**(2007), 876-886.