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Reverse weighted l_p- norm inequalities for convolution type integrals

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Abstract

By using reverse Hölder's inequalities we obtain reverse weighted L_p norm inequalities for various iterated convolutions.

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1 Introduction

In order to study stability of some inverse problems, in a series of papers, S. Saitoh, M. Yamamoto and the last author ([10, 11, 12]) derived reverse inequalities for convolution in some weighted L_p spaces by using the following reverse Hölder's inequality:

Proposition 1.1 *([14]; see also [4, pp. 125-126]) For two positive functions f and g satisfying*

$$0 < m \leq \frac{f}{g} \leq M < \infty \tag{1.1}$$

on the set X, and for $p, q > 1,\ p^{-1} + q^{-1} = 1$,

$$\left(\int_X f d\mu\right)^{\frac{1}{p}} \left(\int_X g d\mu\right)^{\frac{1}{q}} \leq A_{p,q}\left(\frac{m}{M}\right) \int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{1.2}$$

if the right hand side integral converges. Here

$$A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} \frac{t^{-\frac{1}{pq}}(1-t)}{\left(1-t^{\frac{1}{p}}\right)^{\frac{1}{p}} \left(1-t^{\frac{1}{q}}\right)^{\frac{1}{q}}}.$$

Then, by using Proposition 1.1 the first two authors obtained the following

Proposition 1.2 *([3]) Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leq F_1(\boldsymbol{z}) \leq M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \leq F_2(\boldsymbol{z}) \leq M_2^{\frac{1}{p}} < \infty, \quad \boldsymbol{z} \in \mathbb{R}^n. \tag{1.3}$$

Then for any positive continuous functions ρ_1 and ρ_2 on $\mathbb{R}^n$, we have the following reverse weighted $L_p(p > 1)-$norm inequality for convolution

$$\begin{aligned} &\left\| ((F_1\rho_1) * (F_2\rho_2))(\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^n)} \\ &\geq A_{p,q}^{-n}\left(\frac{m_1 m_2}{M_1 M_2}\right) \|F_1\|_{L_p(\mathbb{R}^n,\rho_1)} \|F_2\|_{L_p(\mathbb{R}^n,\rho_2)}. \end{aligned} \tag{1.4}$$

In [8, 3] we gave various applications of (1.4) from the viewpoint of stability in inverse problems.

Later in [7], we introduced several iterated convolution type transformations. Using the Hölder's inequalities we established weighted $L_p, p > 1$, norm inequalities for these iterated convolutions. In this paper, by using the reverse Hölder's inequalities we will derive reverse type inequalities for the iterated convolutions.

2 Preliminaries

Throughout this paper, by $\boldsymbol{x} = (x_1, ..., x_n),\ x_j \in \mathbb{R},\ j = 1, 2, ..., n$, we denote a vector in $\mathbb{R}^n$. In particular,

$$\boldsymbol{1} = (1, 1, ..., 1),\ \boldsymbol{2} = (2, 2, ..., 2), ... \tag{2.1}$$

We shall write $\boldsymbol{x} > \boldsymbol{y}$ to denote $x_j > y_j,\ j = 1, 2, ..., n$. Anologously one has to understand $\boldsymbol{x} \geq \boldsymbol{y},\ \boldsymbol{x} < \boldsymbol{y},\ \boldsymbol{x} \leq \boldsymbol{y}$.

We always assume that ρ, ρ_j and $\rho_{j,r}(j = 1, 2, ..., m;\ r = 1, 2, ..., s)-$ the weight functions, to be nonnegative, and $1 < p < \infty$. When we write $A \leq B$, we understand that if B is finite, then A is also finite, and bounded above by B.

We shall denote some subsets of $\mathbb{R}^n$

$$\mathbb{R}_+^n(\boldsymbol{t}) = \{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{0} < \boldsymbol{x} < \boldsymbol{t}\} \tag{2.2}$$

$$\mathbb{R}_{\boldsymbol{t}}^n = \{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{t} < \boldsymbol{x} < \boldsymbol{\infty}\}. \tag{2.3}$$

Then, by using Proposition 1.1 we obtain the following lemma

Lemma 2.1 *Let Ω be a measurable set in $\mathbb{R}^n$ and ρ be a positive function belonging to $L_1(\Omega)$. For a positive function f satisfying*

$$0 < m^{1/p} \le f(\boldsymbol{x}) \le M^{1/p} < \infty \tag{2.4}$$

on the set Ω, and for $p, q > 1,\ p^{-1} + q^{-1} = 1$,

$$\left(\int_\Omega [f(\boldsymbol{x})]^p \rho(\boldsymbol{x}) d\boldsymbol{x}\right)^{\frac{1}{p}} \left(\int_\Omega \rho(\boldsymbol{x}) d\boldsymbol{x}\right)^{\frac{1}{q}} \le A^n_{p,q}\left(\frac{m}{M}\right) \int_\Omega f(\boldsymbol{x})\rho(\boldsymbol{x}) d\boldsymbol{x}. \tag{2.5}$$

Let D be a domain in $\mathbb{R}^n$, $F_i(.) : D \to \mathbb{R},\ i = 1, \cdots, q$, and $\varphi(.,.), \psi(.,.) : D \times D \to D$.

Then, we introduce a convolution type integral, called the $\varphi-$ convolution

Definition 2.2 *([7]) The $\varphi-$ convolution of F_1 and F_2, denoted by $F_1 *_\varphi F_2$, is defined by*

$$(F_1 *_\varphi F_2)(\boldsymbol{\eta}) = \int_D F_1(\boldsymbol{\xi}) F_2(\varphi(\boldsymbol{\xi}, \boldsymbol{\eta})) |\varphi_{\boldsymbol{\eta}}(\boldsymbol{\xi}, \boldsymbol{\eta})| d\boldsymbol{\xi}, \tag{2.6}$$

when this integral exists. Here,

$$|\varphi_{\boldsymbol{\eta}}(\boldsymbol{\xi}, \boldsymbol{\eta})| := \det\left(\frac{\partial}{\partial \boldsymbol{\eta}} \varphi(\boldsymbol{\xi}, \boldsymbol{\eta})\right)$$

is the Jacobian of the transformation $\boldsymbol{\eta} \to \varphi(., \boldsymbol{\eta})$.

Example 2.3 *The following convolutions are particular cases of the $\varphi-$convolution ([1], [2]):*

- *The Fourier convolution*

$$F_1 *_{\mathfrak{F}} F_2 := \int_{\mathbb{R}^n} F_1(\boldsymbol{y}) F_2(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{2.7}$$

- *The Laplace convolution*

$$F_1 *_{\mathfrak{L}} F_2 := \int_{\mathbb{R}^n_+(\boldsymbol{t})} F_1(\boldsymbol{\tau}) F_2(\boldsymbol{t} - \boldsymbol{\tau}) d\boldsymbol{\tau}, \quad \boldsymbol{t} \in \mathbb{R}^n_+. \tag{2.8}$$

- *The Mellin convolution*

$$F_1 *_{\mathfrak{M}} F_2 := \int_{\mathbb{R}^n_+} F_1(\boldsymbol{x}) F_2(\boldsymbol{t}/\boldsymbol{x}) \boldsymbol{x}^{-1} d\boldsymbol{x}, \quad \boldsymbol{t} \in \mathbb{R}^n_+. \tag{2.9}$$

Definition 2.4 *([7]) Let $G_j(.,.) : \mathbb{R}^n \times \mathbb{R}^m_+ \to \mathbb{R}$. The Fourier-Laplace convolution of $G_1(\boldsymbol{\tau}, \boldsymbol{\zeta})$ and $G_2(\boldsymbol{\tau}, \boldsymbol{\zeta})$ is defined by*

$$(G_1 *_{\mathfrak{F},\mathfrak{L}} G_2)(\boldsymbol{\xi}, \boldsymbol{\eta}) := \int_{\mathbb{R}^n_+(\boldsymbol{\xi})} d\boldsymbol{\tau} \int_{\mathbb{R}^n} G_1(\boldsymbol{\tau}, \boldsymbol{\zeta}) G_2(\boldsymbol{\xi} - \boldsymbol{\tau}, \boldsymbol{\eta} - \boldsymbol{\zeta}) d\boldsymbol{\zeta}. \tag{2.10}$$

Remark 2.5 *We observe that the Fourier-Laplace convolution is a special case of the φ - convolution.*

By Lemma 2.1, we have

Lemma 2.6 *Let ρ_1 and ρ_2 be two positive functions on D such that the convolution $\rho_1 *_\varphi \rho_2$ exists. For two positive functions F_1 and F_2 satisfying*

$$0 < m_j^{\frac{1}{p}} \le F_j \le M_j^{\frac{1}{p}} < \infty, \quad j = 1, 2, \tag{2.11}$$

on the set D, and for $p > 1$,

$$\begin{aligned} &A_{p,q}^{np}\left(\frac{m_1 m_2}{M_1 M_2}\right)[((F_1\rho_1) *_\varphi (F_2\rho_2))(\boldsymbol{\eta})]^p \\ &\ge [(\rho_1 *_\varphi \rho_2)(\boldsymbol{\eta})]^{p-1}((F_1^p\rho_1) *_\varphi (F_2^p\rho_2))(\boldsymbol{\eta}) \end{aligned} \tag{2.12}$$

for all $\boldsymbol{\eta} \in D$.

¿From the $\varphi-$ convolution and the $\psi-$ convolution, we get the following definition

Definition 2.7 *([7]) Under suitable hypotheses for the $\varphi-$ convolution $F_1 *_\varphi F_2$ and the $\psi-$ convolution $F_1 *_\psi F_2$, the $(\varphi+\psi)-$ convolution, denoted by $F_1 *_{\varphi+\psi} F_2$, is defined by*

$$(F_1 *_{\varphi+\psi} F_2)(\boldsymbol{\xi}) := \int_D F_1(\boldsymbol{\tau})\,[F_2(\varphi(\boldsymbol{\tau},\boldsymbol{\xi}))|\varphi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})| + F_2(\psi(\boldsymbol{\tau},\boldsymbol{\xi}))|\psi_{\boldsymbol{\xi}}(\boldsymbol{\tau},\boldsymbol{\xi})|]\,d\boldsymbol{\tau}. \tag{2.13}$$

Example 2.8 *The Fourier cosine convolution (see [13])*

$$F_1 *_{\mathfrak{F}_c} F_2(\boldsymbol{x}) := \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}_+^n} F_1(\boldsymbol{y})[F_2(\boldsymbol{x}+\boldsymbol{y}) + F_2(|\boldsymbol{x}-\boldsymbol{y}|)]d\boldsymbol{y} \tag{2.14}$$

is a special case of the $(\varphi+\psi)-$convolution.

Then, we have

Lemma 2.9 *Let ρ_1 and ρ_2 be two positive functions on D such that the convolution $\rho_1 *_{\varphi+\psi} \rho_2$ exists. For two positive functions F_1 and F_2 satisfying*

$$0 < m_j^{\frac{1}{p}} \le F_j \le M_j^{\frac{1}{p}} < \infty \tag{2.15}$$

on the set D, and for $p > 1$,

$$\begin{aligned} &A_{p,q}^{np}\left(\frac{m_1 m_2}{M_1 M_2}\right) A_{p,q}^{p}\left(\frac{m_2}{M_2}\right)[((F_1\rho_1) *_{\varphi+\psi} (F_2\rho_2))(\boldsymbol{\eta})]^p \\ &\ge [(\rho_1 *_{\varphi+\psi} \rho_2)(\boldsymbol{\eta})]^{p-1}((F_1^p\rho_1) *_{\varphi+\psi} (F_2^p\rho_2))(\boldsymbol{\eta}) \end{aligned} \tag{2.16}$$

for all $\boldsymbol{\eta} \in D$.

Proof. Application of the reverse Hölder's inequality gives

$$A_{p,q}\left(\frac{m_2}{M_2}\right)[F_2(\varphi(\boldsymbol{\xi},\boldsymbol{\eta}))\rho_2(\varphi(\boldsymbol{\xi},\boldsymbol{\eta}))|\varphi_{\boldsymbol{\eta}}(\boldsymbol{\xi},\boldsymbol{\eta})|+F_2(\psi(\boldsymbol{\xi},\boldsymbol{\eta}))\rho_2(\psi(\boldsymbol{\xi},\boldsymbol{\eta}))|\psi_{\boldsymbol{\eta}}(\boldsymbol{\xi},\boldsymbol{\eta})|]$$
$$\geq [F_2^p(\varphi(\boldsymbol{\xi},\boldsymbol{\eta}))\rho_2(\varphi(\boldsymbol{\xi},\boldsymbol{\eta}))|\varphi_{\boldsymbol{\eta}}(\boldsymbol{\xi},\boldsymbol{\eta})|+F_2^p(\psi(\boldsymbol{\xi},\boldsymbol{\eta}))\rho_2(\psi(\boldsymbol{\xi},\boldsymbol{\eta}))|\psi_{\boldsymbol{\eta}}(\boldsymbol{\xi},\boldsymbol{\eta})|]^{\frac{1}{p}}$$
$$\times [\rho_2(\varphi(\boldsymbol{\xi},\boldsymbol{\eta}))|\varphi_{\boldsymbol{\eta}}(\boldsymbol{\xi},\boldsymbol{\eta})|+\rho_2(\psi(\boldsymbol{\xi},\boldsymbol{\eta}))|\psi_{\boldsymbol{\eta}}(\boldsymbol{\xi},\boldsymbol{\eta})|]^{\frac{p-1}{p}}.$$

Therefore,

$$A_{p,q}^n\left(\frac{m_1m_2}{M_1M_2}\right)A_{p,q}\left(\frac{m_2}{M_2}\right)[((F_1\rho_1)*_{\varphi+\psi}(F_2\rho_2))(\boldsymbol{\eta})]$$
$$\geq A_{p,q}^n\left(\frac{m_1m_2}{M_1M_2}\right)\int_D F_1(\boldsymbol{\xi})\rho_1(\boldsymbol{\xi})$$
$$\times [F_2^p(\varphi(\boldsymbol{\xi},\boldsymbol{\eta}))\rho_2(\varphi(\boldsymbol{\xi},\boldsymbol{\eta}))|\varphi_{\boldsymbol{\eta}}(\boldsymbol{\xi},\boldsymbol{\eta})|+F_2^p(\psi(\boldsymbol{\xi},\boldsymbol{\eta}))\rho_2(\psi(\boldsymbol{\xi},\boldsymbol{\eta}))|\psi_{\boldsymbol{\eta}}(\boldsymbol{\xi},\boldsymbol{\eta})|]^{\frac{1}{p}}$$
$$\times [\rho_2(\varphi(\boldsymbol{\xi},\boldsymbol{\eta}))|\varphi_{\boldsymbol{\eta}}(\boldsymbol{\xi},\boldsymbol{\eta})|+\rho_2(\psi(\boldsymbol{\xi},\boldsymbol{\eta}))|\psi_{\boldsymbol{\eta}}(\boldsymbol{\xi},\boldsymbol{\eta})|]^{\frac{p-1}{p}}d\boldsymbol{\xi},$$

which is, by Lemma 2.1,

$$\geq [(\rho_1*_{\varphi+\psi}\rho_2)(\boldsymbol{\eta})]^{\frac{p-1}{p}}[((F_1^p\rho_1)*_{\varphi+\psi}(F_2^p\rho_2))(\boldsymbol{\eta})]^{\frac{1}{p}}.$$

Thus, the proof is complete. □

Definition 2.10 *([7]) For the $\varphi-$ convolution, define the $\varphi-$ convolution product $\prod_{j=1}^m *_\varphi F_j$ by*

$$\prod_{j=1}^m *_\varphi F_j(\boldsymbol{\xi}_m)=\left[\prod_{j=1}^{m-1}*_\varphi F_j\right]*_\varphi F_m(\boldsymbol{\xi}_m). \tag{2.17}$$

3 Reverse weighted L_p inequalities in the iterated convolutions

Our main inequality is the following:

Theorem 3.1 *Let functions ρ_j, $j=1,2,...,m$, be positive on D such that the convolution $\prod_{j=1}^m *_\varphi \rho_j$. For m positive functions F_j satisfying*

$$0<m_j^{\frac{1}{p}}\leq F_j\leq M_j^{\frac{1}{p}}<\infty \tag{3.1}$$

on the set $D(j=1,2,...,m)$, and for $p>1$,

$$\left\|\left(\prod_{j=1}^m *_\varphi(F_j\rho_j)(\boldsymbol{\xi}_m)\right)\left(\prod_{j=1}^m *_\varphi\rho_j(\boldsymbol{\xi}_m)\right)^{\frac{1}{p}-1}\right\|_{L_p(D)}$$
$$\geq \prod_{i=2}^m\left\{A_{p,q}^{-n}\left(\prod_{j=1}^i\frac{m_j}{M_j}\right)\right\}\prod_{j=1}^m\|F_j\|_{L_p(D,\rho_j)}. \tag{3.2}$$

Inequality (3.2) and others should be understood in the sense that if the left hand side is finite, then so is the right hand side, and in this case the inequality holds.

Corollary 3.2 *For positive functions $\rho_{j,r}(j=1,2,...,m;\ r=1,2,...,s)$ on D such that the convolution $\prod_{j=1}^{m} *_{\mathfrak{F}}\rho_{j,r}$ exists and for positive functions $F_{j,r}$ satisfying*

$$0 < m_{j,r}^{\frac{1}{p}} \leq F_{j,r} \leq M_{j,r}^{\frac{1}{p}} < \infty \tag{3.3}$$

on the set $D(j=1,2,...,m;\ r=1,2,...,s)$, and for $p>1$, we have

$$\begin{aligned} &\left\| \left(\sum_{r=1}^{s} \prod_{j=1}^{m} *_{\varphi}(F_{j,r}\rho_{j,r}) \right) \left(\sum_{r=1}^{s} \prod_{j=1}^{m} *_{\varphi}\rho_{j,r} \right)^{\frac{1}{p}-1} \right\|_{L_p(D)}^{p} \\ &\geq A_{p,q}^{-p} \left(\frac{m_0}{M_0} \right) \sum_{r=1}^{s} \left\{ \prod_{i=2}^{m} A_{p,q}^{-np} \left(\prod_{j=1}^{i} \frac{m_{j,r}}{M_{j,r}} \right) \prod_{j=1}^{m} \|F_{j,r}\|_{L_p(D,\rho_{j,r})}^{p} \right\}, \end{aligned} \tag{3.4}$$

where

$$m_0 = \min_{r} \left\{ \prod_{j=1}^{m} m_{j,r} \right\}, \quad M_0 = \max_{r} \left\{ \prod_{j=1}^{m} M_{j,r} \right\}.$$

Proof of Theorem 3.1 We first proof the following inequality

$$\begin{aligned} &\left(\prod_{j=1}^{m} *_{\varphi}(F_j\rho_j)(\boldsymbol{\xi}_m) \right)^{p} \left(\prod_{j=1}^{m} *_{\varphi}\rho_j(\boldsymbol{\xi}_m) \right)^{1-p} \\ &\geq \prod_{i=2}^{m} \left\{ A_{p,q}^{-np} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \prod_{j=1}^{m} *_{\varphi}(F_j^p\rho_j)(\boldsymbol{\xi}_m), \quad \boldsymbol{\xi}_m \in D. \end{aligned} \tag{3.5}$$

We use induction on m. When $m=2$, the inequality (3.5) is reduced to Lemma 2.1. Now suppose (3.5) holds for some integer $m \geq 2$. We claim that it also holds for $m+1$. For all $\boldsymbol{\xi}_{m+1}$, put

$$f_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m) = \left\{ \prod_{j=1}^{m} *_{\varphi}\rho_j(\boldsymbol{\xi}_m)\rho_{m+1}(\varphi(\boldsymbol{\xi}_m,\boldsymbol{\xi}_{m+1}))\varphi_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m,\boldsymbol{\xi}_{m+1}) \right\}^{\frac{p-1}{p}}$$

and

$$\begin{aligned} g_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m) = &\left\{ \prod_{j=1}^{m} *_{\varphi}(F_j^p\rho_j)(\boldsymbol{\xi}_m)\rho_{m+1}(\varphi(\boldsymbol{\xi}_m,\boldsymbol{\xi}_{m+1}))\varphi_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m,\boldsymbol{\xi}_{m+1}) \right\}^{\frac{1}{p}} \\ &\times F_{m+1}(\varphi(\boldsymbol{\xi}_m,\boldsymbol{\xi}_{m+1})). \end{aligned}$$

By induction hypothesis, we arrive at

$$\begin{aligned}
&\prod_{i=2}^{m}\left\{A_{p,q}^{n}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}\prod_{j=1}^{m+1}*_{\varphi}(F_j\rho_j)(\boldsymbol{\xi}_{m+1})\\
&=\prod_{i=2}^{m}\left\{A_{p,q}^{n}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}\left(\prod_{j=1}^{m}*_{\varphi}(F_j\rho_j)\right)*_{\varphi}\left(F_{m+1}\rho_{m+1}\right)(\boldsymbol{\xi}_{m+1})\\
&\geq\int_D f_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m)g_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m)d\boldsymbol{\xi}_m.
\end{aligned}$$

The condition (3.1) implies

$$\prod_{j=1}^{m+1}m_j\leq\frac{g^{p}_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m)}{f^{q}_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m)}\leq\prod_{j=1}^{m+1}M_j,\quad\boldsymbol{\xi}_m\in D.$$

Hence, one can apply the reverse Hölder inequality to $f_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m)$ and $g_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m)$ to obtain

$$\begin{aligned}
&\left\{A_{p,q}^{n}\left(\prod_{j=1}^{m+1}\frac{m_j}{M_j}\right)\int_D f_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m)g_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m)d\boldsymbol{\xi}_m\right\}^{p}\\
&\geq\left\{\int_D[f_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m)]^{q}d\boldsymbol{\xi}_m\right\}^{p-1}\int_D[g_{\boldsymbol{\xi}_{m+1}}(\boldsymbol{\xi}_m)]^{p}d\boldsymbol{\xi}_m\\
&=\left\{\prod_{j=1}^{m+1}*_{\varphi}\rho_j(\boldsymbol{\xi}_{m+1})\right\}^{p-1}\prod_{j=1}^{m+1}*_{\varphi}(F_j^{p}\rho_j)(\boldsymbol{\xi}_{m+1})
\end{aligned}$$

and so the assertion follows.

Now, by taking integration of both sides of (3.5) with respect to $\boldsymbol{\xi}_m$ on D we obtain the inequality

$$\begin{aligned}
&\int_D\left(\prod_{j=1}^{m}*_{\varphi}(F_j\rho_j)(\boldsymbol{\xi}_m)\right)^{p}\left(\prod_{j=1}^{m}*_{\varphi}\rho_j(\boldsymbol{\xi}_m)\right)^{1-p}d\boldsymbol{\xi}_m\\
&\geq\prod_{i=2}^{m}\left\{A_{p,q}^{-np}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}\int_D\prod_{j=1}^{m}*_{\varphi}(F_j^{p}\rho_j)(\boldsymbol{\xi}_m)d\boldsymbol{\xi}_m.
\end{aligned}$$

Here we have, by definition,

$$\begin{aligned}
&\int_D\prod_{j=1}^{m}*_{\varphi}(F_j^{p}\rho_j)(\boldsymbol{\xi}_m)d\boldsymbol{\xi}_m\\
&=\int_D d\boldsymbol{\xi}_m\int_{D^{\varphi}_{\boldsymbol{\xi}_{m-1}}}\left[\prod_{j=1}^{m-1}*_{\varphi}(F_j^{p}\rho_j)(\boldsymbol{\xi}_{m-1})\right]\\
&\qquad\times F_m^{p}(\varphi(\boldsymbol{\xi}_{m-1},\boldsymbol{\xi}_m))\rho_m(\varphi(\boldsymbol{\xi}_{m-1},\boldsymbol{\xi}_m))|\varphi_{\boldsymbol{\xi}_m}(\boldsymbol{\xi}_{m-1},\boldsymbol{\xi}_m)|d\boldsymbol{\xi}_{m-1},
\end{aligned}$$

which is, by the Fubini's theorem and the change of variables $\boldsymbol{x}_m=\varphi(\boldsymbol{\xi}_{m-1},\boldsymbol{\xi}_m)$,

$$=\int_D\left[\prod_{j=1}^{m-1}*_{\varphi}(|F_j|^{p}\rho_j)(\boldsymbol{\xi}_{m-1})\right]d\boldsymbol{\xi}_{m-1}\int_D|F_m(\boldsymbol{x}_m)|^{p}\rho_m(\boldsymbol{x}_m)d\boldsymbol{x}_m.$$

Therefore,

$$\begin{aligned}&\int_D \left(\prod_{j=1}^{m} *_\varphi (F_j\rho_j)(\boldsymbol{\xi}_m)\right)^p \left(\prod_{j=1}^{m} *_\varphi \rho_j(\boldsymbol{\xi}_m)\right)^{1-p} d\boldsymbol{\xi}_m \\ &\geq \prod_{i=2}^{m}\left\{A_{p,q}^{-np}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}\prod_{j=1}^{m}\int_D F_j^p(\boldsymbol{x})\rho_j(\boldsymbol{x})d\boldsymbol{x}.\end{aligned} \tag{3.6}$$

Raising both sides of the inequality (3.6) to power $1/p$ yields the inequality (3.1). □

Proof of Corollary 3.2 For every $\boldsymbol{\xi}_m \in D$, take

$$a_r = \left(\prod_{j=1}^{m} *_\varphi (F_{j,r}\rho_{j,r})(\boldsymbol{\xi}_m)\right)\left(\prod_{j=1}^{m} *_\varphi \rho_{j,r}(\boldsymbol{\xi}_m)\right)^{-1/q}$$

and

$$b_r = \left(\prod_{j=1}^{m} *_\varphi \rho_{j,r}(\boldsymbol{\xi}_m)\right)^{1/q}, \quad r = 1, 2, ..., s.$$

The condition (3.3) implies

$$m_0 \leq \prod_{j=1}^{m} m_{j,r} \leq \frac{a_r^p}{b_r^q} \leq \prod_{j=1}^{m} M_{j,r} \leq M_0, \quad r = 1, 2, ..., s.$$

Hence, we obtain

$$\left(\sum_{r=1}^{s} a_r^p\right)\left(\sum_{r=1}^{s} b_r^q\right)^{p-1} \leq A_{p,q}^p\left(\frac{m_0}{M_0}\right)\left(\sum_{r=1}^{s} a_r b_r\right)^p$$

or, equivalent

$$\begin{aligned}&\sum_{r=1}^{s}\left(\prod_{j=1}^{m} *_\varphi (F_{j,r}\rho_{j,r})(\boldsymbol{\xi}_m)\right)^p\left(\prod_{j=1}^{m} *_\varphi \rho_{j,r}(\boldsymbol{\xi}_m)\right)^{1-p}\left\{\sum_{r=1}^{s}\prod_{j=1}^{m} *_\varphi \rho_{j,r}(\boldsymbol{\xi}_m)\right\}^{p-1} \\ &\leq A_{p,q}^p\left(\frac{m_0}{M_0}\right)\left(\sum_{r=1}^{s}\prod_{j=1}^{m} *_\varphi (F_{j,r}\rho_{j,r})(\boldsymbol{\xi}_m)\right)^p.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(\sum_{r=1}^{s}\prod_{j=1}^{m} *_\varphi (F_{j,r}\rho_{j,r})(\boldsymbol{\xi}_m)\right)^p\left(\sum_{r=1}^{s}\prod_{j=1}^{m} *_\varphi \rho_{j,r}(\boldsymbol{\xi}_m)\right)^{1-p} \\ &\geq A_{p,q}^{-p}\left(\frac{m_0}{M_0}\right)\sum_{r=1}^{s}\left(\prod_{j=1}^{m} *_\varphi (F_{j,r}\rho_{j,r})(\boldsymbol{\xi}_m)\right)^p\left(\prod_{j=1}^{m} *_\varphi \rho_{j,r}(\boldsymbol{\xi}_m)\right)^{1-p}.\end{aligned} \tag{3.7}$$

Now, by taking integration of both sides of (3.7) with respect to $\boldsymbol{\xi}_m$ on D we obtain

$$\begin{aligned}
&\int_D \left(\sum_{r=1}^{s}\prod_{j=1}^{m} *_{\varphi}(F_{j,r}\rho_{j,r})(\boldsymbol{\xi}_m)\right)^p \left(\sum_{r=1}^{s}\prod_{j=1}^{m} *_{\varphi}\rho_{j,r}(\boldsymbol{\xi}_m)\right)^{1-p} d\boldsymbol{\xi}_m \\
&\geq A_{p,q}^{-p}\left(\frac{m_0}{M_0}\right)\int_D \sum_{r=1}^{s}\left(\prod_{j=1}^{m} *_{\varphi}(F_{j,r}\rho_{j,r})(\boldsymbol{\xi}_m)\right)^p \left(\prod_{j=1}^{m} *_{\varphi}\rho_{j,r}(\boldsymbol{\xi}_m)\right)^{1-p} d\boldsymbol{\xi}_m \\
&= A_{p,q}^{-p}\left(\frac{m_0}{M_0}\right)\sum_{r=1}^{s}\int_D \left(\prod_{j=1}^{m} *_{\varphi}(F_{j,r}\rho_{j,r})(\boldsymbol{\xi}_m)\right)^p \left(\prod_{j=1}^{m} *_{\varphi}\rho_{j,r}(\boldsymbol{\xi}_m)\right)^{1-p} d\boldsymbol{\xi}_m,
\end{aligned}$$

which is, by Theorem 3.1,

$$\geq A_{p,q}^{-p}\left(\frac{m_0}{M_0}\right)\sum_{r=1}^{s}\left\{\prod_{i=2}^{m} A_{p,q}^{-pn}\left(\prod_{j=1}^{i}\frac{m_{j,r}}{M_{j,r}}\right)\prod_{j=1}^{m}\|F_{j,r}\|^p_{L_p(D,\rho_{j,r})}\right\},$$

that completes the proof of (3.4). □

¿From Theorem 3.1, we obtain an inequality for the iterated Fourier-Laplace convolution here.

Corollary 3.3 *Let $F_j(\boldsymbol{x},t),\ j=1,2,...,m$, be positive functions satisfying*

$$0 < m_j^{\frac{1}{p}} \leq F_j \leq M_j^{\frac{1}{p}} < \infty \tag{3.8}$$

on the set $\mathbb{R}^n\times\mathbb{R}_+$. Then for any positive functions ρ_j on $\mathbb{R}^n\times\mathbb{R}_+$ such that there exists $\prod_{j=1}^m *_{\mathfrak{F},\mathfrak{L}}$, and for the iterated Fourier Laplace convolution $\prod_{j=1}^m *_{\mathfrak{F},\mathfrak{L}}$, we have the inequality

$$\begin{aligned}
&\left\|\left(\prod_{j=1}^{m} *_{\mathfrak{F},\mathfrak{L}}(F_j\rho_j)\right)\left(\prod_{j=1}^{m} *_{\mathfrak{F},\mathfrak{L}}\rho_j\right)^{\frac{1}{p}-1}\right\|_{L_p(\mathbb{R}^n\times\mathbb{R}_+)} \\
&\geq \prod_{i=2}^{m}\left\{A_{p,q}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}^{-(n+1)}\prod_{j=1}^{m}\|F_j\|_{L_p(\mathbb{R}^n\times\mathbb{R}_+,\rho_j)}.
\end{aligned}\tag{3.9}$$

Remark 3.4 *In formula 3.9, when $m=2$ replacing ρ_2 by 1, and $F_2(\boldsymbol{x}-\boldsymbol{\xi},t-\tau)$ by $G(\boldsymbol{x}-\boldsymbol{\xi},t-\tau)$, and integrating with respect to $\boldsymbol{x}$ from $\boldsymbol{a}$ to $\boldsymbol{b}$ and respect to t from 0 to $T(>0)$, we arrive at the following inequality*

$$\begin{aligned}
&\int_0^T dt\int_{\boldsymbol{a}}^{\boldsymbol{b}}\frac{\left(\int_0^t\int_{\mathbb{R}^n}F(\boldsymbol{\xi},\tau)\rho(\boldsymbol{\xi},\tau)G(\boldsymbol{x}-\boldsymbol{\xi},t-\tau)d\boldsymbol{\xi}d\tau\right)^p}{\left(\int_0^t\int_{\mathbb{R}^n}\rho(\boldsymbol{\xi},\tau)d\boldsymbol{\xi}d\tau\right)^{p-1}}d\boldsymbol{x} \\
&\geq \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-(n+1)p}\int_0^T d\tau\int_{\mathbb{R}^n}F^p(\boldsymbol{\xi},\tau)\rho(\boldsymbol{\xi},\tau)d\boldsymbol{\xi}\int_0^{T-\tau}dt\int_{\boldsymbol{a}-\boldsymbol{\xi}}^{\boldsymbol{b}-\boldsymbol{\xi}}G^p(\boldsymbol{x},t)d\boldsymbol{x},
\end{aligned}\tag{3.10}$$

if positive functions ρ, F and G satisfy

$$0 < m^{\frac{1}{p}} \leq F(\boldsymbol{\xi},t)G(\boldsymbol{x}-\boldsymbol{\xi},t-\tau) < M^{\frac{1}{p}} < \infty \tag{3.11}$$

for all $(\boldsymbol{x},t)\in[\boldsymbol{a},\boldsymbol{b}]\times[0,T],\ (\boldsymbol{\xi},\tau)\in\mathbb{R}^n\times[0,T]$.

Theorem 3.5 *Let* $p \geq 1,\ \mathbf{0} < \boldsymbol{\alpha} < \boldsymbol{T}_1,\ \mathbf{0} < \boldsymbol{T}_i (i = 2, ..., m),$ *and* $F_j \in L_\infty(\mathbf{0}, \boldsymbol{T}_1 + \cdots + \boldsymbol{T}_m)(j = 1, 2, ..., m),$ *satisfy*

$$0 \leq F_j \leq M < \infty, \quad \mathbf{0} < \boldsymbol{t} < \sum_{j=1}^{m} \boldsymbol{T}_j. \tag{3.12}$$

Then

$$\begin{aligned} &M^{\frac{m(1-p)}{p}} \|F_1\|_{L_p(\boldsymbol{\alpha}, \boldsymbol{T}_1)} \prod_{j=2}^{m} \|F_j\|_{L_p(\mathbf{0}, \boldsymbol{T}_j)} \\ &\leq \left(\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1 + \cdots + \boldsymbol{T}_m} d\boldsymbol{t}_m \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_m} d\boldsymbol{t}_{m-1} \cdots \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_2} F_1(\boldsymbol{t}_1) \prod_{j=2}^{m} F_j(\boldsymbol{t}_j - \boldsymbol{t}_{j-1}) d\boldsymbol{t}_1 \right)^{1/p}. \end{aligned} \tag{3.13}$$

In particular; for $\boldsymbol{\alpha} = \mathbf{0}$, *we have*

$$M^{\frac{m(1-p)}{p}} \prod_{j=1}^{m} \|F_j\|_{L_p(\mathbf{0}, \boldsymbol{T}_j)} \leq \left(\int_{\mathbf{0}}^{\boldsymbol{T}_1 + \cdots + \boldsymbol{T}_m} \prod_{j=1}^{m} *_{\mathfrak{L}} F_j(\boldsymbol{t}) d\boldsymbol{t} \right)^{1/p}. \tag{3.14}$$

Proof. Since $0 \leq F_j (j = 1, 2, ..., m) \leq M$ for $\mathbf{0} < \boldsymbol{t} < \sum_{j=1}^{m} \boldsymbol{T}_j$, we have

$$\begin{aligned} &\int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_m} d\boldsymbol{t}_{m-1} \cdots \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_2} F_1^p(\boldsymbol{t}_1) \prod_{j=2}^{m} F_j^p(\boldsymbol{t}_j - \boldsymbol{t}_{j-1}) d\boldsymbol{t}_1 \\ &= \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_m} d\boldsymbol{t}_{m-1} \cdots \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_2} F_1^{p-1}(\boldsymbol{t}_1) \prod_{j=2}^{m} F_j^{p-1}(\boldsymbol{t}_j - \boldsymbol{t}_{j-1}) F_1(\boldsymbol{t}_1) \prod_{j=2}^{m} F_j(\boldsymbol{t}_j - \boldsymbol{t}_{j-1}) d\boldsymbol{t}_1 \\ &\leq M^{m(p-1)} \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_m} d\boldsymbol{t}_{m-1} \cdots \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_2} F_1(\boldsymbol{t}_1) \prod_{j=2}^{m} F_j(\boldsymbol{t}_j - \boldsymbol{t}_{j-1}) d\boldsymbol{t}_1. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1 + \cdots + \boldsymbol{T}_m} d\boldsymbol{t}_m \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_m} d\boldsymbol{t}_{m-1} \cdots \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_2} F_1^p(\boldsymbol{t}_1) \prod_{j=2}^{m} F_j^p(\boldsymbol{t}_j - \boldsymbol{t}_{j-1}) d\boldsymbol{t}_1 \\ &\leq M^{m(p-1)} \int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1 + \cdots + \boldsymbol{T}_m} d\boldsymbol{t}_m \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_m} d\boldsymbol{t}_{m-1} \cdots \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_2} F_1(\boldsymbol{t}_1) \prod_{j=2}^{m} F_j(\boldsymbol{t}_j - \boldsymbol{t}_{j-1}) d\boldsymbol{t}_1. \end{aligned} \tag{3.15}$$

On the other hand, let

$$\theta(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0, \end{cases}$$

and let

$$G(\boldsymbol{t}_{m-1}) = \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_{m-1}} d\boldsymbol{t}_{m-2} \cdots \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_2} F_1^p(\boldsymbol{t}_1) \prod_{j=2}^{m-1} F_j^p(\boldsymbol{t}_j - \boldsymbol{t}_{j-1}) d\boldsymbol{t}_1,$$

we have

$$\begin{aligned}
&\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_m} d\boldsymbol{t}_m \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_m} d\boldsymbol{t}_{m-1}\cdots\int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_2} F_1^p(\boldsymbol{t}_1)\prod_{j=2}^m F_j^p(\boldsymbol{t}_j-\boldsymbol{t}_{j-1})d\boldsymbol{t}_1\\
&=\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_m}\int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_m} G(\boldsymbol{t}_{m-1})F_m^p(\boldsymbol{t}_m-\boldsymbol{t}_{m-1})d\boldsymbol{t}_{m-1}d\boldsymbol{t}_m\\
&=\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_m}\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_m} G(\boldsymbol{t}_{m-1})F_m^p(\boldsymbol{t}_m-\boldsymbol{t}_{m-1})\theta(\boldsymbol{t}_m-\boldsymbol{t}_{m-1})d\boldsymbol{t}_{m-1}d\boldsymbol{t}_m,
\end{aligned}$$

which is, by Fubini's theorem and the change of variables $\boldsymbol{x}=\boldsymbol{t}_m-\boldsymbol{t}_{m-1}$,

$$\begin{aligned}
&=\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_m}\left(\int_{\boldsymbol{0}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_m-\boldsymbol{t}_{m-1}} F_m^p(\boldsymbol{x})d\boldsymbol{x}\right)G(\boldsymbol{t}_{m-1})d\boldsymbol{t}_{m-1}\\
&\geq\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_{m-1}}\left(\int_{\boldsymbol{0}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_m-\boldsymbol{t}_{m-1}} F_m^p(\boldsymbol{x})d\boldsymbol{x}\right)G(\boldsymbol{t}_{m-1})d\boldsymbol{t}_{m-1}\\
&\geq\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_{m-1}}\left(\int_{\boldsymbol{0}}^{\boldsymbol{T}_m} F_m^p(\boldsymbol{x})d\boldsymbol{x}\right)G(\boldsymbol{t}_{m-1})d\boldsymbol{t}_{m-1}\\
&=\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_{m-1}} G(\boldsymbol{t}_{m-1})d\boldsymbol{t}_{m-1}\int_{\boldsymbol{0}}^{\boldsymbol{T}_m} F_m^p(\boldsymbol{x})d\boldsymbol{x},
\end{aligned}$$

and so,

$$\begin{aligned}
&\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1+\cdots+\boldsymbol{T}_m} d\boldsymbol{t}_m \int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_m} d\boldsymbol{t}_{m-1}\cdots\int_{\boldsymbol{\alpha}}^{\boldsymbol{t}_2} F_1^p(\boldsymbol{t}_1)\prod_{j=2}^m F_j^p(\boldsymbol{t}_j-\boldsymbol{t}_{j-1})d\boldsymbol{t}_1\\
&\geq\int_{\boldsymbol{\alpha}}^{\boldsymbol{T}_1} F_1^p(\boldsymbol{x})d\boldsymbol{x}\prod_{j=2}^m\int_{\boldsymbol{0}}^{\boldsymbol{T}_j} F_j^p(\boldsymbol{x})d\boldsymbol{x}.
\end{aligned}$$

Combining with (3.15), we have the desired inequality (3.13). □

We next obtain the inequalities for the $(\varphi+\psi)-$convolution.

Theorem 3.6 *Let functions $\rho_j,\ j=1,2,...,m,$ be positive on D such that the convolution $\prod_{j=1}^m *_{\varphi+\psi}\rho_j$ exists. For positive functions F_j satisfying*

$$0<m_j^{\frac{1}{p}}\leq F_j\leq M_j^{\frac{1}{p}}<\infty \tag{3.16}$$

on the set $D,\ j=1,2,...,m,$ and for $p>1$,

$$\begin{aligned}
&\left\|\left(\prod_{j=1}^m *_{\varphi+\psi}(F_j\rho_j)(\boldsymbol{\xi}_m)\right)\left(\prod_{j=1}^m *_{\varphi+\psi}\rho_j(\boldsymbol{\xi}_m)\right)^{\frac{1}{p}-1}\right\|_{L_p(D)}\\
&\geq 2^{\frac{m-1}{p}}\prod_{i=2}^m\left\{A_{p,q}^{-n}\left(\prod_{j=1}^i\frac{m_j}{M_j}\right)\right\}\prod_{j=2}^m A_{p,q}\left(\frac{m_j}{M_j}\right)\prod_{j=1}^m\|F_j\|_{L_p(D,\rho_j)}.
\end{aligned} \tag{3.17}$$

Proof. From Lemma 2.9 and by induction on m, we have

$$\left(\prod_{j=1}^{m} *_{\varphi+\psi}(F_j\rho_j)(\boldsymbol{\xi}_m)\right)^{p}\left(\prod_{j=1}^{m} *_{\varphi+\psi}\rho_j(\boldsymbol{\xi}_m)\right)^{1-p}$$

$$\geq \prod_{i=2}^{m}\left\{A_{p,q}^{-np}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}\prod_{j=2}^{m}A_{p,q}^{p}\left(\frac{m_j}{M_j}\right)\prod_{j=1}^{m} *_{\varphi+\psi}(F_j^p\rho_j)(\boldsymbol{\xi}_m).$$

Therefore,

$$\int_D\left(\prod_{j=1}^{m} *_{\varphi+\psi}(F_j\rho_j)(\boldsymbol{\xi}_m)\right)^{p}\left(\prod_{j=1}^{m} *_{\varphi+\psi}\rho_j(\boldsymbol{\xi}_m)\right)^{1-p} d\boldsymbol{\xi}_m$$

$$\geq \prod_{i=2}^{m}\left\{A_{p,q}^{-np}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}\prod_{j=2}^{m}A_{p,q}^{p}\left(\frac{m_j}{M_j}\right)\int_D\prod_{j=1}^{m} *_{\varphi+\psi}(F_j^p\rho_j)(\boldsymbol{\xi}_m)d\boldsymbol{\xi}_m$$

$$= 2^{m-1}\prod_{i=2}^{m}\left\{A_{p,q}^{-np}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}\prod_{j=2}^{m}A_{p,q}^{p}\left(\frac{m_j}{M_j}\right)\prod_{j=1}^{m}\int_D F_j^p(\boldsymbol{x})\rho_j(\boldsymbol{x})d\boldsymbol{x}.$$

The Theorem is thus proved. □

¿From Theorem 3.6, we have the following inequalities:

Corollary 3.7 *Let $F_j (j = 1, 2, ..., m)$ be positive functions satisfying*

$$0 < m_j^{\frac{1}{p}} \leq F_j \leq M_j^{\frac{1}{p}} < \infty \tag{3.18}$$

on the set $\mathbb{R}_+^n$. Then for any positive continuous functions ρ_j on $\mathbb{R}_+^n$, and for the iterated Fourier cosin convolution $\prod_{j=1}^m *_{\mathfrak{F}_c}$, we have the inequality

$$\left\|\left(\prod_{j=1}^{m} *_{\mathfrak{F}_c}(F_j\rho_j)\right)\left(\prod_{j=1}^{m} *_{\mathfrak{F}_c}\rho_j\right)^{\frac{1}{p}-1}\right\|_{L_p(\mathbb{R}_+^n)}$$
$$\geq \left[\frac{2}{\sqrt{2\pi}^n}\right]^{\frac{m-1}{p}}\prod_{i=2}^{m}\left\{A_{p,q}^{-n}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}\prod_{j=2}^{m}A_{p,q}\left(\frac{m_j}{M_j}\right)\prod_{j=1}^{m}\|F_j\|_{L_p(\mathbb{R}_+^n,\rho_j)}. \tag{3.19}$$

Corollary 3.8 *Let functions $\rho_{j,r}$ $(j = 1, 2, ..., m;\ r = 1, 2, ..., s)$ be positive on D such that the convolution $\prod_{j=1}^m *_{\varphi,\psi}\rho_{j,r}$ exists. For some positive functions $F_{j,r}$ satisfying*

$$0 < m_{j,r}^{\frac{1}{p}} \leq F_{j,r} \leq M_{j,r}^{\frac{1}{p}} < \infty \tag{3.20}$$

on the set D $(j = 1, 2, ..., m;\ r = 1, 2, ..., s)$, and for $p > 1$,

$$\left\|\left(\sum_{r=1}^{s}\prod_{j=1}^{m} *_{\varphi,\psi}(F_{j,r}\rho_{j,r})\right)\left(\sum_{r=1}^{s}\prod_{j=1}^{m} *_{\varphi,\psi}\rho_{j,r}\right)^{\frac{1}{p}-1}\right\|_{L_p(D)}^{p}$$
$$\geq 2^{m-1}A_{p,q}^{-p}\left(\frac{m_0}{M_0}\right) \tag{3.21}$$
$$\times\sum_{r=1}^{s}\left\{\prod_{j=2}^{m}A_{p,q}\left(\frac{m_{j,r}}{M_{j,r}}\right)\prod_{i=2}^{m}A_{p,q}^{-np}\left(\prod_{j=1}^{i}\frac{m_{j,r}}{M_{j,r}}\right)\prod_{j=1}^{m}\|F_{j,r}\|_{L_p(D,\rho_{j,r})}^{p}\right\},$$

where

$$m_0 = \min_r \left\{ \prod_{j=1}^{m} m_{j,r} \right\}, \quad M_0 = \max_r \left\{ \prod_{j=1}^{m} M_{j,r} \right\}.$$

4 Applications

4.1 The Bernoulli-Euler Beam Equation

We consider the vertical deflection $u(x)$ of an infinite beam on an elastic foundation under the action of a prescribed vertical load $W(x)$. The deflection $u(x)$ satisfies the ordinary differential equation

$$EI\frac{d^4u}{dx^4} + \kappa u = W(x), \quad -\infty < x < \infty, \tag{4.1}$$

where EI is the flexural rigidity and κ is the foundation modulus of the beam. We find the solution assuming that $W(x)$ has a compact support and u, u', u'', u''' all tend to zero as $|x| \to \infty$. Put

$$a^4 = \frac{\kappa}{EI}, \quad F(x)\rho(x) = \frac{W(x)}{EI}.$$

By using the Fourier transform, we obtain the solution ([2, pp. 63-64])

$$u(x) = \frac{1}{2a^3}\int_{-\infty}^{\infty} F(\xi)\rho(\xi)G(x-\xi)d\xi, \tag{4.2}$$

where

$$G(\xi) = \exp\left(-\frac{a}{\sqrt{2}}|\xi|\right)\sin\left(\frac{a\xi}{\sqrt{2}} + \frac{\pi}{4}\right). \tag{4.3}$$

Let $b, c \in \mathbb{R}$ and

$$x \in [-b, b], \quad \xi \in [-c, c], \quad -\frac{\sqrt{2}}{4a}\pi < -b-c < b+c < \pi\frac{3\sqrt{2}}{4a}.$$

Clearly,

$$0 < \alpha := \sin\left(\frac{\pi}{4} - \frac{a(b+c)}{\sqrt{2}}\right) \le \sin\left(\frac{a(x-\xi)}{\sqrt{2}} + \frac{\pi}{4}\right).$$

Since

$$\alpha\exp\left\{-\frac{a(b+c)}{\sqrt{2}}\right\} \le G(x-\xi) \le \exp\left\{-\frac{a|b-c|}{\sqrt{2}}\right\}$$

we see that the condition

$$0 < m^{\frac{1}{p}} \le F(\xi)G(x-\xi) \le M^{\frac{1}{p}}, \tag{4.4}$$

holds if

$$0 < \frac{1}{\alpha}\exp\left\{\frac{a(b+c)}{\sqrt{2}}\right\}m^{\frac{1}{p}} \le F(\xi) \le M^{\frac{1}{p}}\exp\left\{\frac{a|b-c|}{\sqrt{2}}\right\}. \tag{4.5}$$

Thus, for $-b \le d < e \le b$, the formal solution $u(x)$ satisfies the inequality

$$\int_d^e u^p(x)dx \ge (e-d)\alpha^p\exp\left\{-\frac{pa(b+c)}{\sqrt{2}}\right\}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-p} \times \left(\int_d^e \rho(x)dx\right)^{p-1}\int_d^e F^p(x)\rho(x)dx. \tag{4.6}$$

4.2 The Helmholtz Equation

We consider the Dirichlet problem for the Helmholtz equation in a half space of $\mathbb{R}^3$, i.e. the determination of the bounded solution of

$$\Delta_3 u(x,y,t) + ku(x,y,t) = 0, \quad (x,y) \in \mathbb{R}^2, \quad t \in \mathbb{R}_+, \quad k \in \mathbb{R}_+, \tag{4.7}$$

under the boundary value condition

$$u(x,y,0) = F(x,y)\rho(x,y), \quad F(x,y)\rho(x,y) \in L_1(\mathbb{R}^2). \tag{4.8}$$

Its solution has the form ([1, pp. 75-76])

$$\begin{aligned} u(x,y,t) = 2t\left(\frac{k}{2\pi}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & \frac{F(\xi,\tau)\rho(\xi,\tau)}{(t^2+|\xi-x|^2+|\tau-y|^2)^{\frac{3}{2}}} \\ & \times K_{\frac{3}{2}}\left(k\sqrt{t^2+|\xi-x|^2+|\tau-y|^2}\right) d\xi d\tau, \end{aligned} \tag{4.9}$$

where $K_\nu(x)$ denotes the McDonald function and for $\nu = \frac{3}{2}$ (see [9, Supp. 10]):

$$K_{\frac{3}{2}}(x) = \sqrt{\frac{\pi}{2x}}\left(1+\frac{1}{x}\right)e^{-x}.$$

We rewrite (4.9) as

$$u(x,y,t) = \frac{t}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi,\tau)\rho(\xi,\tau)G(\xi-x,y-\tau)d\xi d\tau, \tag{4.10}$$

where

$$G(\xi,\tau) = \frac{1+k\sqrt{t^2+\xi^2+\tau^2}}{(t^2+\xi^2+\tau^2)^{\frac{9}{4}}}e^{-k\sqrt{t^2+\xi^2+\tau^2}}.$$

Let $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ be some real numbers and

$$x \in [a_1,a_2], \quad \xi \in [b_1,b_2], \quad y \in [c_1,c_2], \quad \tau \in [d_1,d_2].$$

Denote

$$\begin{aligned} \alpha &= \max\{|a_1-b_1|,|a_1-b_2|,|a_2-b_1|,|a_2-b_2|\}, \\ \beta &= \max\{|c_1-d_1|,|c_1-d_2|,|c_2-d_1|,|c_2-d_2|\}. \end{aligned}$$

We have

$$0 < \frac{1+kt}{(t+\alpha^2+\beta^2)^{\frac{9}{4}}}e^{-k\sqrt{t^2+\alpha^2+\beta^2}} \le G(\xi-x,y-\tau) \le \frac{1}{t^{\frac{9}{2}}}.$$

Thus

$$\int_{c_1-\tau}^{c_2-\tau} dy \int_{a_1-\xi}^{a_2-\xi} G^p(x,y)dx \ge \frac{(c_2-c_1)(a_2-a_1)(1+kt)^p}{(t+\alpha^2+\beta^2)^{\frac{9p}{4}}}e^{-pk\sqrt{t^2+\alpha^2+\beta^2}}.$$

Hence, for a function F satisfying

$$0 < \frac{(t+\alpha^2+\beta^2)^{\frac{9}{4}}}{1+kt} e^{k\sqrt{t^2+\alpha^2+\beta^2}} m^{\frac{1}{p}} \leq F(\xi,\tau) \leq M^{\frac{1}{p}} t^{\frac{9}{2}} \tag{4.11}$$

and for a positive continuous function ρ on $[b_1, b_2] \times [d_1, d_2]$, we obtain

$$\begin{aligned} &\int_{c_1}^{c_2} dy \int_{a_1}^{a_2} u^p(x,y,t)dx \\ &\geq (c_2-c_1)(a_2-a_1)\frac{t^p(1+kt)^p}{(2\pi)^p(t+\alpha^2+\beta^2)^{\frac{9p}{4}}} e^{-pk\sqrt{t^2+\alpha^2+\beta^2}} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p} \\ &\quad\times \left(\int_{d_1}^{d_2} d\tau \int_{b_1}^{b_2} \rho(\xi,\tau)d\xi\right)^{p-1} \int_{d_1}^{d_2} d\tau \int_{b_1}^{b_2} F^p(\xi,\tau)\rho(\xi,\tau)d\xi. \end{aligned} \tag{4.12}$$

4.3 The Cauchy Problem for the Inhomogeneous Heat Equation

The equation of heat conduction with sources is given by

$$u_t(t,\boldsymbol{x}) - c^2\Delta_n u(t,\boldsymbol{x}) = F(t,\boldsymbol{x})\rho(t,\boldsymbol{x}), \quad (t,\boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.13}$$

where $F\rho \in L_1$ for every $t \in \mathbb{R}_+$, under the initial value condition

$$u(0,\boldsymbol{x}) = 0. \tag{4.14}$$

Its solution has the form (see [1, pp. 58-59])

$$u(t,\boldsymbol{x}) = \frac{1}{(4\pi c^2)^{n/2}} \int_0^t d\tau \int_{\mathbb{R}^n} \frac{F(\tau,\boldsymbol{\xi})\rho(\tau,\boldsymbol{\xi})}{(t-\tau)^{n/2}} \exp\left\{-\frac{|\boldsymbol{\xi}-\boldsymbol{x}|^2}{4c^2(t-\tau)}\right\} d\boldsymbol{\xi}. \tag{4.15}$$

Take

$$G(\boldsymbol{\xi},\tau) = \frac{1}{\tau^{n/2}} \exp\left\{-\frac{|\boldsymbol{\xi}|^2}{4c^2\tau}\right\}.$$

Let

$$\boldsymbol{\xi} \in [-\boldsymbol{a},\boldsymbol{a}], \quad \boldsymbol{\xi} \in [-\boldsymbol{b},\boldsymbol{b}], \quad t \in [T_1,T_2], \quad T_1 > 0.$$

Since

$$\frac{1}{T_2^{n/2}} \exp\left\{-\frac{|\boldsymbol{a}+\boldsymbol{b}|^2}{4c^2T_1}\right\} < \frac{1}{(t-\tau)^{n/2}} \exp\left\{-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|^2}{4c^2\tau}\right\} < \frac{1}{T_1^{n/2}},$$

we see that the condition

$$0 < m^{\frac{1}{p}} < F(\boldsymbol{\xi},\tau)G(\boldsymbol{x}-\boldsymbol{\xi},t-\tau) < M^{\frac{1}{p}} \tag{4.16}$$

holds if

$$0 < T_2^{n/2} \exp\left\{\frac{|\boldsymbol{a}+\boldsymbol{b}|^2}{4c^2T_1}\right\} m^{\frac{1}{p}} < F(\boldsymbol{\xi},\tau) < M^{\frac{1}{p}} T_1^{n/2}. \tag{4.17}$$

Thus, for $-\boldsymbol{a} < \boldsymbol{d}, \boldsymbol{e} < \boldsymbol{a}$, the inequality (3.10) yields

$$\begin{aligned}
&\int_{T_1}^{T_2} dt \int_{\boldsymbol{d}}^{\boldsymbol{e}} u^p(\boldsymbol{x}, t) d\boldsymbol{x} \\
&\geq \frac{1}{(4\pi c^2)^{np/2}} \frac{(\boldsymbol{e} - \boldsymbol{d})}{T_2^{np/2}} \exp\left\{-p\frac{|\boldsymbol{a} + \boldsymbol{b}|^2}{4c^2 T_1}\right\} \left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-(n+1)p} \\
&\quad \times \left(\int_{T_1}^{T_2} dt \int_{\boldsymbol{d}}^{\boldsymbol{e}} \rho(\boldsymbol{x}, t) d\boldsymbol{x}\right)^{p-1} \int_{T_1}^{T_2} (T_2 - t) dt \int_{\boldsymbol{d}}^{\boldsymbol{e}} F^p(\boldsymbol{x}, t)\rho(\boldsymbol{x}, t) d\boldsymbol{x},
\end{aligned} \tag{4.18}$$

where ρ is a positive continuous function on $[\boldsymbol{d}, \boldsymbol{e}] \times [T_1, T_2]$, and F satisfies (4.17).

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