

A NEW EQUATION OF MOTION FOR CLASSICAL CHARGED PARTICLES

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We propose a new equation of motion for classical charged particles, which is free from the well-known difficulties of Dirac's equation, is intuitively sound, and predicts reasonable radiation damping.

1. Introduction

The equation of motion of a charged particle has been a subject of interest for many years¹. The equation now generally accepted was obtained by Dirac by decomposing the retarded self-field into a sum field that renormalizes mass and a difference field that gives reaction². An explanation and re-derivation based on an absorber mechanism was provided by Wheeler and Feynman³. However, as is well recognized, Dirac's equation has certain inherent difficulties. First, it involves the derivative of the acceleration and hence needs one extra condition, in addition to the Newtonian initial conditions, to determine the motion. Second, it gives runaway solutions which can be avoided only by artificially introducing a pre-acceleration. Third, in certain cases it implies that the external energy supplied to the particle goes only into kinetic energy, and radiation is created from an acceleration energy which is negative and unphysical. It is the purpose of this work to obtain a new equation that is free from these difficulties and predicts reasonable results.

2. The New Equation

By following the old idea of expressing reaction only by the kinematical quantities of the particle, it is not possible to construct an equation that satisfactorily includes reaction. However, in classical electrodynamics in an inertial frame⁴ the only field that can accelerate a charged particle and make it radiate is the external electromagnetic field $F_{ext}^{\mu\nu}$. Accordingly, radiation reaction should be expressible by $F_{ext}^{\mu\nu}$ and the particle kinematics. On the other hand, since a charge e at rest experiences only an electric force $e\vec{E}$ and in motion experiences an additional magnetic force $e\vec{v} \times \vec{B}$, which together make up $e F_{ext}^{\mu\lambda} u_\lambda$, it is natural to assume that when accelerating a charge experiences still another force $e_1 F_{ext}^{\mu\lambda} \dot{u}_\lambda$ with e_1 a small constant (i. e., $e_1 F_{ext}^{\mu\lambda} \dot{u}_\lambda \ll e F_{ext}^{\mu\lambda} u_\lambda$ in most physical cases) and $\dot{u}^\lambda \equiv du^\lambda/ds$. Here we use geometrized unit $c=1$,

with signature $\eta_{\mu\nu} = (1, -1, -1, -1)$, $\{x^0, x^1, x^2, x^3\} \equiv \{t, x, y, z\}$, s the proper time, and u^μ the four-velocity.

Now, given the motion $u^\mu(s)$ of a charge the rate of radiated energy momentum $(-2e^2/3) \dot{u}_\lambda \dot{u}^\lambda u^\mu$ is obtained by integrating Poynting's vector on far-zone retarded sphere⁵. By using the radiation-neglected equation $m \dot{u}^\mu_{ret} = e F_{ext}^{\mu\lambda} u_\lambda$ this rate can be expressed as $(-2e^2/3m) F_{ext}^{\lambda\alpha} \dot{u}_\lambda u_\alpha u^\mu$ which is roughly the expression for radiation in terms of $F_{ext}^{\mu\nu}$. Equating the inertia and radiation to the forces the charge sees through $F_{ext}^{\mu\nu}$ we have the new equation of motion

$$m \dot{u}^\mu - \frac{2e^2}{3m} F_{ext}^{\lambda\alpha} \dot{u}_\lambda u_\alpha u^\mu = e F_{ext}^{\mu\lambda} u_\lambda + e_1 F_{ext}^{\mu\lambda} \dot{u}_\lambda \quad (1)$$

where the requirement that (1) be an identity after scalar multiplication by u_μ implies $e_1 = 2e^2/3m$ is indeed a small constant. For a system of charges, in (1) for the i -th charge $F_{ext}^{\mu\nu}(i)$ becomes $\sum_{j=1} F_{ret}^{\mu\nu}(j)$ where $F_{ret}^{\mu\nu}(j)$ is the retarded field of the j -th charge.

The general properties of (1) are:

1) Mass conservation; scalar multiplication by u_μ gives an identity and hence m is constant. 2) Self-evident radiation term; scalar multiplication by u_μ gives $(-2e^2/3m) F_{ext}^{\lambda\alpha} \dot{u}_\lambda u_\alpha u^\mu = (-2e^2/3) \dot{u}_\lambda \dot{u}^\lambda u^\mu$ which always represents radiation. This justifies the second term on the left of (1) as radiation reaction with u^μ determined by (1). 3) Newtonian motion; no more than the first derivative of velocity is involved and accordingly motion is determined by the initial velocity and position and by $F_{ext}^{\mu\nu}$. 4) No runaway solution (see below). 5) No pre-acceleration. 6) Additional effective external field; taking the radiation term to the right side and combining it with $e_1 F_{ext}^{\mu\lambda} \dot{u}_\lambda$ one can think of the total acceleration-dependent external force as derived from an effective field

$$f^{\mu\nu} = \frac{2e^2}{3m} \dot{u}_\lambda (F_{ext}^{\mu\lambda} u^\nu - F_{ext}^{\nu\lambda} u^\mu) \quad (2)$$

in addition to $F_{ext}^{\mu\nu}$ which the charge sees through the Lorentz force.

3. Special Cases Compared with Dirac's Equation

Now we shall examine the implications of (1) for certain basic physical situations and compare the results with those⁶ of the Dirac equation

$$m \ddot{u}^\mu = e F_{ext}^{\mu\lambda} u_\lambda + \frac{2e^2}{3} (\ddot{u}^\mu + \dot{u}_\lambda \dot{u}^\lambda u^\mu). \quad (3)$$

a) No external field, $F_{ext}^{\mu\nu} = 0$; (1) directly gives $u^\mu = \text{constant}$, but for (3) this solution has to be „physically“ singled out from the infinity of runaway solutions.

b) Constant uniform electric field, $\vec{E} = \vec{e}_x E$; the new equation (1) gives $u^\mu = (\cosh \eta, \sinh \eta, 0, 0)$, where $\eta \equiv C_1 + eEs/m$, for initial velocity $\vec{v} = \vec{e}_x \tanh C_1$; the Dirac equation (3) gives $u^\mu = (\cosh \xi, \sinh \xi, 0, 0)$, where $\xi \equiv K_1 + K_2 \exp(s/\tau) + eEs/m$ and $\tau \equiv 2e^2/3m$, which with the physical requirement $\ddot{u}^\mu = 0$ when $E=0$ implies $K_2 \equiv 0$. Thus (1) yields the same solution as (3), but (1) works all by itself. Also from (1) the radiation $(-2e^2/3)\ddot{u}_\lambda \dot{u}^\lambda u^\mu = (2e^4 E^2/3m^2) u^\mu$ is supplied by the external force $e_1 F_{ext}^{\mu\lambda} \dot{u}_\lambda$, but from (3) it is supplied by the negative acceleration energy term $(2e^2/3)\ddot{u}^\mu$.

c) Incident rectangular pulse $\vec{E} = \vec{e}_x E$ for $0 < s < s_1$; (1) gives

$$u_{(N)}^\mu = \begin{cases} (\cosh C_1, \sinh C_1, 0, 0) & s < 0, \\ (\cosh \psi, \sinh \psi, 0, 0) & 0 < s < s_1, \\ (\cosh \zeta, \sinh \zeta, 0, 0) & s_1 < s, \end{cases} \quad (4)$$

where $\psi \equiv eEs/m + C_1$ and $\zeta \equiv eEs_1/m + C_1$. But (3) with $\dot{u}^\mu(\infty) = 0$ gives

$$u_{(D)}^\mu = \begin{cases} (\cosh \alpha, \sinh \alpha, 0, 0) & s < 0, \\ (\cosh \sigma, \sinh \sigma, 0, 0) & 0 < s < s_1, \\ (\cosh \zeta, \sinh \zeta, 0, 0) & s_1 < s, \end{cases} \quad (5)$$

where $\alpha \equiv C_1 + (eE\tau/m)(1 - \exp(-s_1/\tau)) \exp(s/\tau)$,

$\sigma \equiv C_1 + (eE\tau/m)(1 - \exp[(s-s_1)/\tau]) + eEs/m$. Thus (5) represents pre-acceleration whereas (4) shows that the electron does not respond until the pulse hits it. The limiting case of a delta pulse² is easily obtained by letting $s_1 \rightarrow 0$ and keeping Es_1 constant. For this limit (4) gives simply a jump in velocity which is due only to idealizing the incident wave as a delta function, whereas (5) gives a purely pre-acceleration motion.

d) Motion perpendicular to uniform magnetic field $\vec{B} = \vec{e}_x B$; in this case exact analytic solutions cannot be found for (1) and (3), but a perturbation method can be used to obtain and compare their total correctional forces which spiral the circular orbit inward as a result of synchrotron radiation⁷. Now the first-order corrections,

$$f_{(N)}^\mu \equiv \frac{2e^3}{3m} \dot{u}_\lambda \left(F_{ext}^{\mu\lambda} + F_{ext}^{\lambda\alpha} u_\alpha u^\mu \right) = -\frac{2e^4 B^2}{3m^2} \frac{\beta}{\sqrt{1-\beta^2}} \left\{ \frac{\beta}{\sqrt{1-\beta^2}} u^\mu + \right. \\ \left. + \left(0, 0, \sin \frac{eBs}{m}, \cos \frac{eBs}{m} \right) \right\} = \frac{2e^2}{3} \left(\ddot{u}_{(1)}^\mu + \dot{u}_{(1)}^\lambda u_{(1)}^\lambda u_{(1)}^\mu \right) \equiv f_{(D)}^\mu, \quad (6)$$

are equal. Here $\dot{m}u^{\mu} \equiv eF_{ext}^{\mu\lambda} u_{\lambda}$ and u^{μ} represent circular motion without radiation perturbation. The second-order corrections are

$$f^{\mu} \equiv \frac{2e^3}{3m} \left[\dot{u}_{\lambda} \left(F_{ext}^{\mu\lambda} + F_{ext}^{\lambda\alpha} u_{\alpha} u^{\mu} \right) + \dot{u}_{\lambda} F_{ext}^{\lambda\alpha} \left(u_{\alpha} u^{\mu} + u_{\alpha} u^{\mu} \right) \right] =$$

$$= \frac{2e^3}{3m} F_{ext}^{\mu\lambda} \dot{u}_{\lambda} u_{\alpha} u^{\mu} + \frac{4}{9} \frac{e^7 B^3}{m^4} \frac{\beta}{(1-\beta^2)^{3/2}} \frac{eBs}{m} \left\{ \frac{2\beta}{\sqrt{1-\beta^2}} u^{\mu} + \left(0, 0, \sin \frac{eBs}{m}, \right. \right.$$

$$\left. \left. \cos \frac{eBs}{m} + \frac{m}{eBs} \sin \frac{eBs}{m} \right) \right\}, \quad (7-a)$$

$$f^{\mu} \equiv \frac{2e^2}{3} \left[\ddot{u}^{\mu} + 2 \dot{u}_{\lambda} \dot{u}^{\lambda} u^{\mu} + \dot{u}_{\lambda} \dot{u}^{\lambda} u^{\mu} \right] = \frac{2e^2}{3} \dot{u}_{\lambda} \dot{u}^{\lambda} u^{\mu} +$$

$$+ \frac{4}{9} \frac{e^7 B^3}{m^4} \frac{\beta}{(1-\beta^2)^{3/2}} \frac{eBs}{m} \left\{ \frac{2\beta}{\sqrt{1-\beta^2}} \left(1 + \frac{m}{2eBs} \sin \frac{2eBs}{m} \right) u^{\mu} + \right.$$

$$\left. + \left(0, 0, \sin \frac{eBs}{m} - \frac{2m}{eBs} \cos \frac{eBs}{m}, \cos \frac{eBs}{m} + \frac{m}{eBs} \sin \frac{eBs}{m} \right) \right\}, \quad (7-b)$$

Here the second-order total solution $u^{\mu} \equiv u^{\mu}_{(1)} + u^{\mu}_{(2)}$ which satisfies

$\dot{m}\ddot{u}^{\mu} = eF_{ext}^{\mu\lambda} \dot{u}_{\lambda} + f^{\mu}$ is the same for (1) and (3). Comparing the differences of second-order forces $\Delta f^{\mu} \equiv f^{\mu}_{(2)} - f^{\mu}_{(N2)}$ and the first-order correction force $f^{\mu}_{(1)}$ to the main force $\dot{m}\dot{u}^{\mu}$ we get

$\Delta f^{\mu}_{(2)}$	$f^{\mu}_{(1)}$	$\dot{m}\dot{u}^{\mu}_{(1)}$	
$\frac{\beta^2}{(1-\beta^2)^2} \left(\frac{r_c}{r_1} \right)^2$	$\frac{\beta}{\sqrt{1-\beta^2}} \left(\frac{r_c}{r_1} \right)$	1	for $\frac{\beta^2}{1-\beta^2} \lesssim 1$
$\frac{\beta^4}{(1-\beta^2)^3} \left(\frac{r_c}{r_1} \right)^2$	$\frac{\beta^3}{(1-\beta^2)^{3/2}} \left(\frac{r_c}{r_1} \right)$	1	for $\frac{\beta^2}{1-\beta^2} \gtrsim 1$

Here r_1 is the radius of the circular orbit for u^{μ} and r_c is the classical radius of the charged particle. Thus the new equation (1) predicts a faster inward spiraling than does Dirac's (3) by the deviation $\Delta f^{\mu}_{(2)}/\dot{m}\dot{u}^{\mu}_{(1)}$

compared to the main unperturbed orbit. For a typical electron synchrotron of 5 Bev, $r_1 \sim 5$ meters, $r_c = 2.8 \times 10^{-15}$ meters this deviation is 10^{-8} — far below the quantum fluctuation of synchrotron photon emission⁸. However, for highly energetic charged particles in a very strong electromagnetic field, as in astrophysical applications⁹ where $(1-\beta^2)^{-1}(e^2/m\varepsilon_0 c^2) \times (mc/eB)^{-1} \gtrsim 1$ (e.g. $b+2n \gtrsim 10$ for electrons of energy 10^n Bev in $B=10^b$ gauss) the deviation is large. In such strong fields the new equation (1) predicts orbits quite different from Dirac's (3).

e) Motion in coulomb field $\vec{E} = (q/r^2) \vec{e}_r$; by perturbation method as above the first-order corrections are

$$\begin{aligned} f_{(N1)}^\mu &= \frac{2m^2}{3q^2} \frac{\beta^3}{(1-\beta^2)^{5/2}} \left\{ (1, 0, 0, 0) - \frac{1}{\sqrt{1-\beta^2}} \cdot \frac{u^\mu}{(1)} \right\}, \\ f_{(D1)}^\mu &= \frac{2m^2}{3q^2} \frac{\beta^3}{(1-\beta^2)^{5/2}} \left\{ (0, -\sin \Omega_s, \cos \Omega_s, 0) - \frac{\beta}{\sqrt{1-\beta^2}} \frac{u^\mu}{(1)} \right\} \end{aligned} \quad (9)$$

where $\Omega \equiv -m\beta^3/eq(1-\beta^2)$. It follows that $f_{(D1)}^\mu$ has more backward tangential correction than $f_{(N1)}^\mu$ and thus the Dirac orbit collapses faster than the new orbit by

$$\frac{\Delta f_{(1)}^\mu}{\mu u_{(1)}^\mu} \equiv \frac{(2e^3/3m) F_{ext}^{\mu\lambda} u_{(1)\lambda}}{\mu u_{(1)}^\mu} \sim \frac{e}{q} \frac{\beta^3}{(1-\beta^2)} \quad (10)$$

There is no experimental data on this deviation.

f) Oscillating electric field¹⁰ $\vec{E} = \vec{e}_x E \cos \omega t$; for initial velocity zero the new equation (1) gives exactly

$$u_{(N)}^\mu = \left(\left(1 + \left(\frac{eE}{m\omega} \sin \omega t \right)^2 \right)^{1/2}, \frac{eE}{m\omega} \sin \omega t, 0, 0 \right) \quad (11)$$

which shows no damping because of the continuous supply of energy from the oscillating fields. Also the motion (11) is the same as that obtained from the radiation-neglected Newton's equation $\mu \dot{u}^\mu = e F^{\mu\lambda} u_\lambda$ because in this special case the radiation $(2e^4 E^2/3m^2) \cos^2 \omega t u^\mu$ is completely supplied by the additional external power-force $e_1 F_{ext}^{\mu\lambda} \dot{u}_\lambda$. This result agrees with the usual Thomson scattering¹¹ and says the latter is exact up to the order of neglecting the magnetic force from the incident wave. For Dirac's eq. (3), a perturbation force

$$f_{(D1)}^\mu = -\frac{2e^3 E}{3m} \sin \omega t \frac{u^\mu}{(N)} \frac{u^\nu}{(N)} \frac{u^\rho}{(N)} (0, 0, 0, 0) \quad (12)$$

shifts the oscillation phase forward and decreases the amplitude¹² which deviates the motion from $u_{(N)}^\mu$ when $\omega\tau \gtrsim 1$ ($\tau \sim 10^{-24}$ sec for e^-). But this cannot be checked experimentally because such high energetic Compton scattering must be treated quantum-electrodynamically.

4. Conclusion

The fact that (1) overcomes all former difficulties and predicts results not experimentally distinguishable from Dirac's in laboratory cases of basic importance, and the intuitive soundness of the new ideas on which it is based lead us to suggest that the new equation (1) correct-

ly accounts for radiation reaction in the motion of classical charged particles and should replace the celebrated Dirac equation of motion (3).

The new equation can manifest itself by predicting different motion and radiation rate for high energetic charges in very strong electromagnetic field, e. g., as in astrophysical cases for electrons with 10^6 Bev in 10^6 gauss that $b+2n \geq 10$. At present it seems not trivial to find an action integral for (1). However, there is no rigorously valid action integral¹³ that leads to (3).

Also it can be shown that for $m=0$ equation (1) gives $\dot{u}^\mu=0$ and $u_\mu u^\mu \equiv 0$ independent of $F_{ext}^{\mu\nu}$. Thus a massless particle follows a null geodesic and cannot interact with the electromagnetic field whether it be charged or not. This might add a new degree of freedom to the charge conservation law. The additional force (see (2)) appearing in (1) alters the conventional interaction $-\int u_\mu A_{ext}^\mu$. Thus this work is a first step in including radiation reaction in curved spacetime¹⁴ and may possibly lead to changes in quantum theory.

A c k n o w l e d g m e n t

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10. Suggested by R. P. Feynman; also suggested by P. A. M. Dirac in private communication for basic test of time-dependent field.
11. Ref. 1, Landau. Sec. 78.
12. Ref. 6, Plass, Eq. (92-95).

13. By adding $A_{(-)}^{\mu} J_{\mu}$, where $A_{(-)}^{\mu} \equiv \frac{1}{2} (A_{ret}^{\mu} - A_{adv}^{\mu})$ and $J^{\mu} \equiv \rho u^{\mu}$, in the Lagrangian integrand and varying u^{μ} with A^{μ} fixed, then finally evaluating $F^{\mu\nu}$ at the charge to obtain Dirac's (3) is not a correct variation principle—see example F. Rohrlich, Ph. Rev. Letters 12, 575 (1964); since A^{μ} is a function of u^{μ} .
14. For a work following Dirac's idea, see B. S. DeWitt and R. W. Brehme, Ann. Phys. (N. Y.) 9, 220 (1960).

ՇԱՐԺՄԱՆ ՆՈՐ ԴԱՍԱԿԱՆ ՀԱՎԱՍԱՐՈՒՄ ԼԻՑԲԱՎՈՐՎԱԾ ՄԱՍՆԻԿՆԵՐԻ ՀԱՄԱՐ

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Շարժման Դիրակի հավասարումը (3) իր գրման օրից (1938 թ.) հանդիսանում է էլեկտրամագնիսական դաշտերում լիցքավորված մասնիկի շարժման համար հանրաճանաչ հավասարում։ Սակայն բաշ հայտնի է, որ նա ունի հետևյալ անցանկալի հատկությունները.

ա) պարունակում է արագացման ածանցյալը և, հետևապես, շարժման միակ որոշման համար պահանջում է բացի նյութոսնի սկզբնական պայմաններից ևս մեկ լրացուցիչ պայման.

բ) Ունի «ինքնարագացող» լուծումներ, որոնցից կարելի է ազատվել միայն գերարագացման դադարի մոտենալով։

գ) Որոշ դեպքերում հավասարումը բերում է նրան, որ արտաքին էներգիան լրիվ փոխանցվում է կինետիկ էներգիայի, իսկ ճառագայթումը առաջանում է ի հաշիվ արագացման էներգիայի, որը բացասական է և հետևաբար դուրի է ֆիզիկական իմաստից։

Սույն աշխատանքում առաջարկվում է լիցքավորված մասնիկի շարժման նոր դասական հավասարում, որը ազատ է վերոհիշյալ թերություններից և կանխադրված է խելամիտ արդյունքներ կարևոր էքսպերիմենտալ դեպքերի համար։ Ինտուիտիվ դատողությունները հանգեցնում են այն մտքին, որ նոր հավասարումը կոռեկտ կերպով է նկարագրում ճառագայթման և էներգիան և զալիս է փոխարինելու Դիրակի հնացած հավասարումը։

НОВОЕ КЛАССИЧЕСКОЕ УРАВНЕНИЕ ДВИЖЕНИЯ ДЛЯ ЗАРЯЖЕННЫХ ЧАСТИЦ

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Уравнение движения Дирака (3) со времени его написания в 1938 г. является общепринятым классическим уравнением движения заряженной частицы в электромагнитных полях. Хорошо известно, однако, что оно обладает следующими нежелательными свойствами:

а) содержит производную ускорения и следовательно для однозначного определения движения нуждается, дополнительно к ньютоновским начальным условиям в еще одном условии,

б) содержит „самоускоряющиеся“ решения, которые могут быть исключены лишь введением понятия о сверхускорении,

в) в определенных случаях уравнение приводит к тому, что внешняя энергия целиком переходит в кинетическую энергию частицы, излучение же происходит за счет энергии ускорения, которая отрицательна и следовательно не имеет физического смысла.

В настоящей работе предлагается новое классическое уравнение движения для заряженной частицы, которое свободно от вышеуказанных недостатков и предсказывает разумные результаты для важных экспериментальных случаев. Интуитивные соображения наводят на мысль, что новое уравнение корректно учитывают реакцию излучения и должно заменить устаревшее уравнение движения Дирака.