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GABOR DUALS FOR OPERATOR-VALUED GABOR FRAMES ON LOCALLY COMPACT ABELIAN GROUPS

Y. HU, P. LI

Nanjing University of Aeronautics and Astronautics, Nanjing, China
Anqing Normal University, Anqing, China[*]
E-mails: *ymhu712@126.com; pengtongli@nuaa.edu.cn; pengtonglee@sina.com*

Abstract. Motivated by the ordinary Gabor frames in $L^2(\mathbb{R}^d)$ and operator-valued frames on abstract Hilbert spaces, we investigate operator- valued Gabor frames associated with locally compact Abelian groups. Necessary and sufficient conditions for an operator-valued Gabor frame to admit a Parseval/tight Gabor dual are given. In particular, we consider a special case, which includes the case of ordinary Gabor frames.

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1. Introduction

The Gabor system was first proposed by D. Gabor [8] for the purpose of applications in signal processing. It is a collection of functions generated by a window function $g \in L^2(\mathbb{R})$ and by translations and modulations:

$$\mathcal{G}(g, \alpha, \beta) = \{e^{2\pi i m\alpha x} g(x - n\beta) : m, n \in \mathbb{Z}\},$$

where α and β are two positive parameters. In [9], K. Gröchenig generalized the notion of Gabor systems to the locally compact Abelian groups. To ensure stable reconstruction of signals, the Gabor system needs to be a frame, a concept introduced by R. Duffin and A. Schaeffer [4] as a generalization of the Riesz bases. In recent years, the Gabor frames were one of the extensively studied research topics in the frame theory (see [3, 5, 7, 10, 13, 15, 16, 17]). The early Gabor frames were mainly studied by using classical Fourier/harmonic analysis methods. Meanwhile, as it was indicated by a number researches, more abstract tools from other fields of pure mathematics, such as operator algebras and group representations, can be used in the study of Gabor frames (see [2, 11, 12, 13, 16] for some recent significant results). Gabor analysis actually has roots in the theory of von Neumann algebras, which can

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be traced back to the von Neumann's work [21] in 1930s about the von Neumann lattices, and the related work by M. Rieffel [20] in 1980s about the incompleteness property of Gabor families.

In the frame theory, the tight frames play an important role due to their simplicity (e.g., the canonical dual of a tight frame is a scalar multiple of the tight frame itself), and due to some other useful features in applications (e.g., tight frames are optimal for erasures). Note that when a frame itself is not a tight frame, the canonical dual frame cannot be tight. However, it is possible that tight dual frames exist even when a given frame is not a tight one. This problem has been deeply investigated by D. Han in several papers. For instance, in [12], it was characterized the existence of tight/Parseval dual frames with the same structure for non-tight Gabor frames in $L^2(\mathbb{R}^d)$.

As a generalized version of ordinary frames, the operator-valued frame was introduced and studied in Kaftal et al [19]. This new type of frames can be used in the quantum communication (see [1]) and in the packet network, and so it becomes an attractive object of study.

In this paper, motivated by the ordinary Gabor frames in $L^2(\mathbb{R}^d)$ and by the operator-valued frames on abstract Hilbert spaces, we consider the so-called operator-valued Gabor frames associated with locally compact abelian groups. For simplicity, the abbreviations "OPV" and "LCA" will be used for "operator-valued" and "locally compact Abelian", respectively.

The paper is structured as follows. In Section 2 we give some preliminaries for OPV-Gabor frames. The main results of this paper are stated and proved in Section 3. Necessary and sufficient conditions for an OPV-Gabor frame associated with an LCA group to admit a Parseval (respectively tight) Gabor dual are given in Theorems 3.1 and 3.2. In Corollary 3.1 and Proposition 3.1, we consider a special case including the case of ordinary Gabor frames and partially generalize the results of D. Han from [12].

2. Preliminaries for OPV-Gabor frames

Throughout the paper, G will denote an LCA group and $\widehat{G}$ will denote the dual group of G, which consists of all characters, that is, all continuous homomorphisms from G into the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Note that under pointwise multiplication and equipped with an appropriate topology, $\widehat{G}$ is also an LCA group. Considering the well-known *Haar measure* on the LCA group G, which is unique up to a positive constant, we have the Hilbert space $L^2(G)$ in the usual way. One then defines the *translation operator* $T_\lambda, \lambda \in G$, as

$$T_\lambda : L^2(G) \to L^2(G), \quad (T_\lambda f)(x) = f(x\lambda^{-1}), \quad x \in G,$$

and the *modulation operator* $E_\gamma, \gamma \in \widehat{G}$, as

$$E_\gamma : L^2(G) \to L^2(G), \quad (E_\gamma f)(x) = \gamma(x) f(x), \quad x \in G.$$

Clearly, both T_λ and E_λ are unitary operators.

Recall that a closed subgroup Λ of $G \times \widehat{G}$ is called a *lattice* if it is discrete and co-compact, that is, the quotient group $G \times \widehat{G}/\Lambda$ is compact. Given such a lattice Λ, we write $l^2(\Lambda)$ for the usual Hilbert space consisting of all scalar functions x on Λ such that $x(\nu) = 0$ for all but a countable number of ν and $\sum_{\nu \in \Lambda} |x(\nu)|^2 < \infty$.

In what follows we use the following notation. By $B(L^2(G))$ we denote the algebra of all bounded linear operators on $L^2(G)$ and by $\{\chi_\nu\}_{\nu \in \Lambda}$ we denote the standard orthonormal basis of $l^2(\Lambda)$, where χ_ν is the characteristic function at the single point set $\{\nu\}$. By I and I_0 we denote the identity operators in $L^2(G)$ and $l^2(\Lambda)$, respectively. We always write Λ for a fixed lattice in $G \times \widehat{G}$ and e for the group unit of Λ. Also, for a bounded linear operator T on a Hilbert space, its adjoint operator is denoted by T^*.

The central object of this paper is the so-called *OPV-Gabor system* in $L^2(G)$ with modulation and translation along a lattice Λ of $G \times \widehat{G}$, generated by an operator $A \in B(L^2(G))$. This is a collection of operators of the following form:

$$\mathcal{G}(A, \Lambda) = \{A\pi(\nu) : \pi(\nu) = E_\gamma T_\lambda \text{ for } \nu = (\lambda, \gamma) \in \Lambda\}.$$

If an OPV-Gabor system is also an OPV-frame in the sense of [19], then it is called an OPV-Gabor frame. The explicit definition is as follows.

Definition 2.1. For an OPV-Gabor system $\mathcal{G}(A, \Lambda)$ for $L^2(G)$, if there exist two constants $C, D > 0$ such that

$$CI \leq \sum_{\nu \in \Lambda} (A\pi(\nu))^*(A\pi(\nu)) \leq DI, \tag{2.1}$$

where the series converges in the strong operator topology, then $\mathcal{G}(A, \Lambda)$ is called an *OPV-Gabor frame* for $L^2(G)$. The optimal constants (maximal for C and minimal for D) are called the *lower* and the *upper frame bounds*, respectively. An OPV-Gabor frame $\mathcal{G}(A, \Lambda)$ is called *tight* if $C = D$, and is called *Parseval* if $C = D = 1$. If we only require the second inequality in (2.1), then $\mathcal{G}(A, \Lambda)$ is called a *Bessel system.*

It is easy to verify that the condition (2.1) is satisfied if and only if there exist two constants $C, D > 0$ such that

$$C \parallel f \parallel^2 \leq \sum_{\nu\in\Lambda} \parallel A\pi(\nu)f \parallel^2 \leq D \parallel f \parallel^2 \quad \text{for all } f \in L^2(G). \tag{2.2}$$

A closed subspace M of $L^2(G)$ is called Λ-*shift invariant* if it is π-invariant, that is, $\pi(\nu)M \subseteq M$ for all $\nu \in \Lambda$. In [12], a subspace Gabor frame for the ordinary (vector) case was introduced. Similarly, we can also define the subspace Gabor frame for the operator-valued case. If an OPV-Gabor system $\mathcal{G}(A,\Lambda)$ satisfies the condition (2.2) only for $f \in M$, then we say that $\mathcal{G}(A,\Lambda)$ is a *subspace OPV-Gabor frame* for M.

As we know, the analysis operators and the frame operators play an important role in the study of frame theory. Let $\mathcal{G}(A,\Lambda)$ be a Bessel OPV-Gabor system for $L^2(G)$. Following [14, 19], the *analysis operator* θ_A for $\mathcal{G}(A,\Lambda)$ is an operator from $L^2(G)$ into the tensor product space $l^2(\Lambda) \otimes L^2(G)$, defined by

$$\theta_A(f) = \sum_{\nu\in\Lambda} \chi_\nu \otimes A\pi(\nu)(f) \quad \text{for } f \in L^2(G).$$

Clearly, for the adjoint of θ_A we have $\theta_A^*(\chi_\nu \otimes f) = (A\pi(\nu))^*(f)$ for $\nu \in \Lambda, f \in L^2(G)$. The operator $S_A = \theta_A^*\theta_A = \sum_{\nu\in\Lambda}(A\pi(\nu))^*(A\pi(\nu))$ is called the *frame operator* of $\mathcal{G}(A,\Lambda)$. As in [19], for every $\nu \in \Lambda$ we define the partial isometry: $L_\nu : L^2(G) \to l^2(\Lambda) \otimes L^2(G)$, $L_\nu(f) = \chi_\nu \otimes f$. Then, we have

$$L_\omega^* L_\nu = \begin{cases} I & \text{if } \omega = \nu, \\ 0 & \text{if } \omega \neq \nu, \end{cases} \quad \text{and} \quad \sum_{\nu\in\Lambda} L_\nu L_\nu^* = I_0 \otimes I, \tag{2.3}$$

where the convergence is in the strong operator topology.

If $\mathcal{G}(A,\Lambda)$ is a Bessel OPV-Gabor system, then we have

$$\theta_A = \sum_{\nu\in\Lambda} L_\nu A\pi(\nu) \quad \text{and} \quad \theta_A^* = \sum_{\nu\in\Lambda} (A\pi(\nu))^* L_\nu^*. \tag{2.4}$$

We collect several simple and useful facts for analysis operators as a lemma, which are also true for general frames on Hilbert spaces.

Lemma 2.1. *Let $\mathcal{G}(A,\Lambda)$ be a Bessel OPV-Gabor system for $L^2(G)$. Then the following assertions hold:*

(i) *$\mathcal{G}(A,\Lambda)$ is an OPV-Gabor frame for $L^2(G)$ if and only if θ_A is injective and has closed range.*

(ii) *$\mathcal{G}(A,\Lambda)$ is a subspace OPV-Gabor frame for a Λ-shift invariant subspace M if and only if $\theta_A^*\theta_A$ is an invertible bounded operator when restricted to M.*

(iii) *$\mathcal{G}(A,\Lambda)$ is a Parseval subspace OPV-Gabor frame if and only if $\theta_A^*\theta_A$ (or equivalently, $\theta_A\theta_A^*$) is an orthogonal projection. In particular, $\mathcal{G}(A,\Lambda)$ is a Parseval OPV-Gabor frame for $L^2(G)$ if and only if $\theta_A^*\theta_A = I$.*

In the case where $\mathcal{G}(A,\Lambda)$ is an OPV-Gabor frame for $L^2(G)$, by using Lemma 3.1(iii), we can conclude that $\mathcal{G}(AS_A^{-1},\Lambda)$ is also an OPV-Gabor frame for $L^2(G)$ and $\theta_A S_A^{-1} = \theta_{AS_A^{-1}}$. Thus we obtain the following reconstruction formula:

$$\theta^*_{AS_A^{-1}}\theta_A = \theta^*_A\theta_{AS_A^{-1}} = I.$$

The frame $\mathcal{G}(AS_A^{-1},\Lambda)$ is called the *canonical OPV-Gabor dual* of $\mathcal{G}(A,\Lambda)$. In general, if a Bessel OPV-Gabor system $\mathcal{G}(B,\Lambda)$ for $L^2(G)$ satisfies

$$\theta^*_B\theta_A = \theta^*_A\theta_B = I,$$

then $\mathcal{G}(B,\Lambda)$ is called an *alternate OPV-Gabor dual* of $\mathcal{G}(A,\Lambda)$ (cf. [19]). In this case, $\mathcal{G}(B,\Lambda)$ must be an OPV-Gabor frame for $L^2(G)$. The canonical and alternate OPV-Gabor duals are simply referred to OPV-Gabor duals.

Definition 2.2. Let $\mathcal{G}(A,\Lambda)$ be an OPV-Gabor frame for $L^2(G)$. A Bessel OPV-Gabor system $\mathcal{G}(B,\Lambda)$ for $L^2(G)$ is called a *Parseval (*respectively *tight) OPV-Gabor dual* for $\mathcal{G}(A,\Lambda)$ if it is an OPV-Gabor dual of $\mathcal{G}(A,\Lambda)$ (that is, $\theta^*_B\theta_A = \theta^*_A\theta_B = I$), and at the same time it is also a Parseval (respectively tight) OPV-Gabor frame for $L^2(G)$.

For OPV-Gabor frames, we are interested in the existence of their Parseval or tight OPV-Gabor duals. More precisely, our goal is to find conditions under which a Parseval (tight) OPV-Gabor dual exists for a given OPV-Gabor frame. This topic will be discussed in the next section, and we will need the following lemma.

Lemma 2.2. *Let $\mathcal{G}(A,\Lambda)$ be an OPV-Gabor frame for $L^2(G)$ and let S_A be its frame operator. If $\mathcal{G}(B,\Lambda)$ is an OPV-Gabor dual of $\mathcal{G}(A,\Lambda)$ with upper frame bound b, then $\|S_A^{-1}\| \le b$.*

Proof. By the hypotheses, we have $\theta^*_B\theta_B \le bI$, and hence $\theta_B\theta^*_B \le bI_0 \otimes I$. Thus

$$I = \theta^*_A\theta_B\theta^*_B\theta_A \le b\theta^*_A\theta_A = bS_A,$$

whence $S_A^{-1} \le bI$, or equivalently, $\|S_A^{-1}\| \le b$. □

In particular, Lemma 2.2 implies that a necessary condition for an OPV-Gabor frame $\mathcal{G}(A,\Lambda)$ to have a Parseval Gabor dual is that the optimal lower frame bound is greater than or equal to one, that is, $\|S_A^{-1}\| \le 1$.

3. The existence of Parseval and tight Gabor duals for OPV-Gabor frames

In this section, we give complete characterizations for OPV-Gabor frames for $L^2(G)$ to admit Parseval (respectively tight) OPV-Gabor duals. For this purpose, we need to recall a few concepts and notation, which can be found in [18]. Let $\mathcal{A}$ be a von Neumann algebra, that is, it is a $*$-algebra of bounded linear operators on a Hilbert space such that the identity operator $I \in \mathcal{A}$ and $\mathcal{A}$ is closed in the weak operator topology. Call $\mathcal{A}$ *finite* if every isometry in $\mathcal{A}$ is unitary. Two orthogonal projections P and Q in $\mathcal{A}$ are said to be *equivalent* in the sense of Murray-von Neumann, if there exists a partial isometry $V \in \mathcal{A}$ such that $VV^* = P$ and $V^*V = Q$. In this case we write $P \sim Q$. We use the notation $P \precsim Q$ if P is equivalent to a subprojection of Q in $\mathcal{A}$. A *trace* "tr" on $\mathcal{A}$ is a positive linear functional satisfying $\operatorname{tr}(T^*T) = \operatorname{tr}(TT^*)$ for all $T \in \mathcal{A}$. A *faithful normal trace* on $\mathcal{A}$ is a trace that is continuous in the weak operator topology and satisfies the condition that $\operatorname{tr}(T) > 0$ whenever $T \in \mathcal{A}$ is a nonzero positive operator. Denote by $\mathcal{A}'$ the commutant of $\mathcal{A}$. A *center-valued trace* τ on $\mathcal{A}$ is a linear mapping from $\mathcal{A}$ to its center $\mathcal{A} \cap \mathcal{A}'$ satisfying the following conditions:

(i) $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{A}$;
(ii) $\tau(C) = C$ for all $C \in \mathcal{A} \cap \mathcal{A}'$;
(iii) $\tau(A)$ is a nonzero positive whenever $A \in \mathcal{A}$ is a nonzero positive operator;
(iv) $\tau(CA) = C\tau(A)$ for all $A \in \mathcal{A}, C \in \mathcal{A} \cap \mathcal{A}'$.

We remark that if $\mathcal{A}$ is a finite von Neumann algebra, then $\mathcal{A}$ must have a unique center-valued trace τ. Moreover, if "tr" is a faithful normal trace on $\mathcal{A}$, then we have $\operatorname{tr}(A) = \operatorname{tr}(\tau(A))$ for all $A \in \mathcal{A}$.

Let Λ be a lattice in $G \times \widehat{G}$ and $\nu = (\lambda, \gamma) \in \Lambda$. It is clear that $\pi(\nu) = E_\gamma T_\lambda$ is a unitary operator on $L^2(G)$. The commutator relation

$$T_\lambda E_\gamma = \overline{\gamma(\lambda)} E_\gamma T_\lambda$$

leads to the following useful identities

$$\pi(\nu)^* = \overline{\gamma(\lambda)}\pi(\nu^{-1}), \quad \pi(\nu_1)\pi(\nu_2) = \mu(\nu_1, \nu_2)\pi(\nu_1\nu_2),$$

where $\mu(\nu_1, \nu_2) = \overline{\gamma_2(\lambda_1)}$ belongs to the circle group $\mathbb{T}$ and $\nu_i = (\lambda_i, \gamma_i) \in \Lambda$ ($i = 1, 2$). Following [11, 14], the mapping π is called a *projective unitary representation* of Λ on $L^2(G)$, and the mapping $(\nu_1, \nu_2) \to \mu(\nu_1, \nu_2)$ is called a *multiplier* of π. It follows from the results of [14] that

(i) $\mu(\nu_1, \nu_2\nu_3)\mu(\nu_2, \nu_3) = \mu(\nu_1\nu_2, \nu_3)\mu(\nu_1, \nu_2)$ for all $\nu_1, \nu_2, \nu_3 \in \Lambda$;
(ii) $\mu(\nu, e) = \mu(e, \nu) = 1$ for all $\nu \in \Lambda$;
(iii) $\mu(\nu, \nu^{-1}) = \mu(\nu^{-1}, \nu)$ for all $\nu \in \Lambda$.

Following [14], there exists an associated *right regular μ-projective representation* r of Λ on the Hilbert space $l^2(\Lambda)$ defined by $r(\nu)(\chi_\omega) = \overline{\mu(\omega, \nu^{-1})}\chi_{\omega\nu^{-1}}$, $\nu, \omega \in \Lambda$.

We can check that

$$r(\nu_1)r(\nu_2) = \overline{\mu(\nu_2^{-1}, \nu_1^{-1})}r(\nu_1\nu_2) = \overline{\mu(\nu_2, \nu_1)}r(\nu_1\nu_2)$$

for all $\nu_1, \nu_2 \in \Lambda$. Clearly, every $r(\nu)$ is unitary and r is a projective unitary representation of Λ with multiplier $\overline{\mu(\nu_2, \nu_1)}$. We also introduce another projective unitary representation:

$$\widetilde{r} : \Lambda \to B(l^2(\Lambda) \otimes L^2(G)), \quad \widetilde{r}(\nu) = r(\nu) \otimes I.$$

Of course, there exists an associated *left regular μ-projective representation* λ of Λ on the Hilbert space $l^2(\Lambda)$ defined by

$$\lambda(\nu)(\chi_\omega) = \overline{\mu(\nu, \omega)}\chi_{\nu\omega}, \quad \nu, \omega \in \Lambda.$$

Since Λ is an Abelian group, they are essentially the same.

We will need the following two lemmas.

Lemma 3.1. *Let $\mathcal{G}(A, \Lambda)$ be a Bessel OPV-Gabor system for $L^2(G)$. Then*

(i) $L_e^*\widetilde{r}(\nu) = \overline{\mu(\nu, \nu^{-1})}L_\nu^*$ *for all* $\nu \in \Lambda$*;*
(ii) $\theta_A\pi(\nu) = \mu(\nu, \nu^{-1})\widetilde{r}(\nu)\theta_A$ *for all* $\nu \in \Lambda$*;*
(iii) $S_A\pi(\nu) = \pi(\nu)S_A$ *for all* $\nu \in \Lambda$.

Proof. (i) Let $\nu \in \Lambda$ and $f \in L^2(G)$. Then we have

$$\widetilde{r}(\nu^{-1})L_e(f) = (r(\nu^{-1}) \otimes I)(\chi_e \otimes f) = \overline{\mu(e, \nu)}\chi_\nu \otimes f = \chi_\nu \otimes f = L_\nu(f).$$

This means that $\widetilde{r}(\nu^{-1})L_e = L_\nu$, and hence $L_e^*\widetilde{r}(\nu^{-1})^* = L_\nu^*$. Noting that

$$r(\nu)r(\nu^{-1}) = \overline{\mu(\nu^{-1}, \nu)}r(\nu\nu^{-1}) = \overline{\mu(\nu, \nu^{-1})}I_0,$$

we have $r(\nu^{-1})^* = \mu(\nu, \nu^{-1})r(\nu)$. Thus, $\widetilde{r}(\nu^{-1})^* = \mu(\nu, \nu^{-1})\widetilde{r}(\nu)$ and $\mu(\nu, \nu^{-1})L_e^*\widetilde{r}(\nu) = L_\nu^*$. Therefore, $L_e^*\widetilde{r}(\nu) = \overline{\mu(\nu, \nu^{-1})}L_\nu^*$ for all $\nu \in \Lambda$.

(ii) For all $\nu \in \Lambda, f \in L^2(G)$, by (2.4), we have

$$\begin{aligned}
\theta_A\pi(\nu)(f) &= \sum_{\omega\in\Lambda} L_\omega A\pi(\omega)\pi(\nu)(f) = \sum_{\omega\in\Lambda} \mu(\omega, \nu)L_\omega A\pi(\omega\nu)(f) \\
&= \sum_{\omega\in\Lambda} \mu(\omega\nu^{-1}, \nu)L_{\omega\nu^{-1}}A\pi(\omega)(f) = \mu(\nu, \nu^{-1})\sum_{\omega\in\Lambda} \overline{\mu(\omega, \nu^{-1})}L_{\omega\nu^{-1}}A\pi(\omega)(f) \\
&= \mu(\nu, \nu^{-1})\sum_{\omega\in\Lambda} \overline{\mu(\omega, \nu^{-1})}\chi_{\omega\nu^{-1}} \otimes A\pi(\omega)(f) \\
&= \mu(\nu, \nu^{-1})\widetilde{r}(\nu)\sum_{\omega\in\Lambda} \chi_\omega \otimes A\pi(\omega)(f) = \mu(\nu, \nu^{-1})\widetilde{r}(\nu)\theta_A(f),
\end{aligned}$$

where the identity $\mu(\omega\nu^{-1}, \nu) = \mu(\nu, \nu^{-1})\overline{\mu(\omega, \nu^{-1})}$ follows from the properties (i) and (ii) of μ listed above. This shows that $\theta_A\pi(\nu) = \mu(\nu, \nu^{-1})\widetilde{r}(\nu)\theta_A$ for all $\nu \in \Lambda$.

(iii) For every $\nu \in \Lambda$, we have

$$S_A\pi(\nu) = \sum_{\omega\in\Lambda}(A\pi(\omega))^*(A\pi(\omega))\pi(\nu) = \sum_{\omega\in\Lambda}\pi(\nu)\pi(\nu)^*(A\pi(\omega))^*(A\pi(\omega))\pi(\nu)$$
$$= \pi(\nu)\sum_{\omega\in\Lambda}(A\pi(\omega)\pi(\nu))^*(A\pi(\omega)\pi(\nu)) = \pi(\nu)\sum_{\omega\in\Lambda}\overline{\mu(\omega,\nu)}\mu(\omega,\nu)(A\pi(\omega\nu))^*(A\pi(\omega\nu))$$
$$= \pi(\nu)\sum_{\omega\in\Lambda}(A\pi(\omega\nu))^*(A\pi(\omega\nu)) = \pi(\nu)S_A,$$

as required. Lemma 3.1 is proved. □

Lemma 3.2. *Let $\mathcal{G}(A,\Lambda)$ be a Bessel OPV-Gabor system for $L^2(G)$ with the analysis operator θ_A, and let $M = \overline{\mathrm{Range}(\theta_A^*\theta_A)}$. Then there exists an operator $T \in B(L^2(G))$ such that:*

(i) *$\mathcal{G}(T,\Lambda)$ is a subspace Parseval OPV-Gabor frame for M;*

(ii) $\mathrm{Range}(\theta_T) = \overline{\mathrm{Range}(\theta_A)}$.

Proof. By the polar decomposition theorem, there is a partial isometry $V : L^2(G) \to l^2(\Lambda)\otimes L^2(G)$ with the initial space

$$M = \overline{\mathrm{Range}(\theta_A^*\theta_A)}\ \left(= \overline{\mathrm{Range}(\theta_A^*\theta_A)^{\frac{1}{2}}} = \overline{\mathrm{Range}(\theta_A^*)}\right)$$

and the final space $K = \overline{\mathrm{Range}(\theta_A)}$, such that $\theta_A = V(\theta_A^*\theta_A)^{\frac{1}{2}}$. It follows from Lemma 3.1 (iii) that M is π-invariant and

$$\mu(\nu,\nu^{-1})\widetilde{r}(\nu)V(\theta_A^*\theta_A)^{\frac{1}{2}} = \mu(\nu,\nu^{-1})\widetilde{r}(\nu)\theta_A = \theta_A\pi(\nu)$$
$$= V(\theta_A^*\theta_A)^{\frac{1}{2}}\pi(\nu) = V\pi(\nu)(\theta_A^*\theta_A)^{\frac{1}{2}}$$

for all $\nu \in \Lambda$. Define a projective unitary representation:

$$\widetilde{R} : \Lambda \to B(l^2(\Lambda)\otimes L^2(G)), \quad \widetilde{R}(\nu) = \mu(\nu,\nu^{-1})\widetilde{r}(\nu).$$

Then $\widetilde{R}(\nu)V(\theta_A^*\theta_A)^{\frac{1}{2}} = V\pi(\nu)(\theta_A^*\theta_A)^{\frac{1}{2}}$ for every $\nu \in \Lambda$. Also, by Lemma 3.1 (ii), for $\nu \in \Lambda, f \in L^2(G)$, we have

$$\widetilde{R}(\nu)\theta_A(f) = \mu(\nu,\nu^{-1})\widetilde{r}(\nu)\theta_A(f) = \theta_A\pi(\nu)(f).$$

Hence $\widetilde{R}(\nu)\theta_A = \theta_A\pi(\nu)$ and K is $\widetilde{R}$-invariant. So, the operator V induces a unitary equivalence between the two sub-representations $\widetilde{R}|_K$ and $\pi|_M$.

Let $T = L_e^*V$. For all $\nu \in \Lambda$, by Lemma 3.1(i), we have

$$T\pi|_M(\nu) = L_e^*V\pi|_M(\nu) = L_e^*\widetilde{R}(\nu)V|_M = \mu(\nu,\nu^{-1})L_e^*\widetilde{r}(\nu)V|_M$$
$$= \mu(\nu,\nu^{-1})\overline{\mu(\nu,\nu^{-1})}L_\nu^*V|_M = L_\nu^*V|_M.$$

It follows from (**??**) and (2.4) that

$$\sum_{\nu\in\Lambda}(T\pi|_M(\nu))^*(T\pi|_M(\nu)) = \sum_{\nu\in\Lambda}V^*L_\nu L_\nu^*V|_M = V^*(\sum_{\nu\in\Lambda}L_\nu L_\nu^*)V|_M = V^*V|_M$$

and

$$\theta_T = \sum_{\nu\in\Lambda} L_\nu T\pi|_M(\nu) = \sum_{\nu\in\Lambda} L_\nu L_\nu^* V|_M = V|_M.$$

Since V^*V is an orthogonal projection on M, it follows that $\mathcal{G}(T,\Lambda)$ is a subspace Parseval OPV-Gabor frame for M and $\mathrm{Range}(\theta_T) = \mathrm{Range}(V) = \overline{\mathrm{Range}(\theta_A)}$. $\square$

We remark that in the case when $\mathcal{G}(A,\Lambda)$ is an OPV-Gabor frame for $L^2(G)$, that is, S_A is invertible on $L^2(G)$, then the partial isometry V is already well-known to be the analysis operator of the associated Parseval OPV-Gabor frame $\mathcal{G}(AS_A^{-\frac{1}{2}},\Lambda)$.

Given a Bessel OPV-Gabor system $\mathcal{G}(A,\Lambda)$ for $L^2(G)$, and let θ_A be its analysis operator. It follows from Lemma 3.1(ii) that the norm closure $\overline{\mathrm{Range}(\theta_A)}$ is invariant under $\widetilde{r}(\nu)$ for every $\nu \in \Lambda$. So, if we use P_A to denote the orthogonal projection of $l^2(\Lambda)\otimes L^2(G)$ onto $\overline{\mathrm{Range}(\theta_A)}$, then P_A belongs to the commutant of $\widetilde{r}(\Lambda)$, that is, $P_A \in \widetilde{r}(\Lambda)' = r(\Lambda)' \otimes B(L^2(G))$. These results are extensions to the projective unitary representations of some results stated in Lemma 6.4 of [19].

Now we are ready to state and prove our first main result, which generalizes Theorem 2.2 of [12], but the proof turns out to be more complicated.

Theorem 3.1. *Let G be an LCA group and Λ be a lattice of $G\times\widehat{G}$. Assume that $\mathcal{G}(A,\Lambda)$ is an OPV-Gabor frame for $L^2(G)$ whose frame operator S_A satisfies the condition $\|S_A^{-1}\| \le 1$, and denote $M = \overline{\mathrm{Range}(I - S_A^{-1})}$. Then $\mathcal{G}(A,\Lambda)$ has a Parseval OPV-Gabor dual if and only if there exists a subspace Parseval OPV-Gabor frame $\mathcal{G}(T,\Lambda)$ for M such that $P_T \precsim I_0\otimes I - P_A$ in the von Neumann algebra $\widetilde{r}(\Lambda)'$.*

Proof. We first assume that $\mathcal{G}(A,\Lambda)$ has a Parseval OPV-Gabor dual $\mathcal{G}(B,\Lambda)$. Let

$$C = B - AS_A^{-1}.$$

Since both $\mathcal{G}(B,\Lambda)$ and $\mathcal{G}(AS_A^{-1},\Lambda)$ are OPV-Gabor frames, we have that $\mathcal{G}(C,\Lambda)$ is a Bessel OPV-Gabor system. Moreover, $\theta_A S_A^{-1} = \theta_{AS_A^{-1}}$. So, we can write

$$\theta_A^*\theta_C = \theta_A^*(\theta_B - \theta_A S_A^{-1}) = I - S_A S_A^{-1} = 0 \tag{3.1}$$

and

$$\begin{aligned}\theta_C^*\theta_C &= (\theta_B^* - S_A^{-1}\theta_A^*)(\theta_B - \theta_A S_A^{-1})\\ &= \theta_B^*\theta_B + S_A^{-1}S_AS_A^{-1} - \theta_B^*\theta_AS_A^{-1} - S_A^{-1}\theta_A^*\theta_B = I - S_A^{-1},\end{aligned}$$

implying that

$$M = \overline{\mathrm{Range}(I - S_A^{-1})} = \overline{\mathrm{Range}(\theta_C^*\theta_C)}.$$

Next, it follows from Lemma 3.2 that there exists a subspace Parseval OPV-Gabor frame $\mathcal{G}(T,\Lambda)$ for M such that $\mathrm{Range}(\theta_T) = \overline{\mathrm{Range}(\theta_C)}$. By (3.1), we have $\theta_A^*\theta_T =$

0. Thus, $P_A \perp P_T$, which implies that $P_T \le I_0 \otimes I - P_A$. Noting that $P_T, P_A \in \widetilde{r}(\Lambda)'$, we have $P_T \precsim I_0 \otimes I - P_A$ in the von Neumann algebra $\widetilde{r}(\Lambda)'$.

Conversely, assume that there exists a subspace Parseval OPV-Gabor frame $\mathcal{G}(T,\Lambda)$ for $M = \overline{\mathrm{Range}(I - S_A^{-1})}$ such that $P_T \precsim I_0 \otimes I - P_A$ in the von Neumann algebra $\widetilde{r}(\Lambda)'$. Then there exists a subprojection $Q \le I_0 \otimes I - P_A$ such that $P_T \sim Q$ in the von Neumann algebra $\widetilde{r}(\Lambda)'$. Let $V \in \widetilde{r}(\Lambda)'$ be the partial isometry such that $VV^* = P_T$ and $V^*V = Q$. Set $E = L_e^* V^* \theta_T$. Then for all $\nu \in \Lambda$, by Lemma 3.1(i) and (ii), we obtain

$$\begin{aligned} E\pi|_M(\nu) &= L_e^* V^* \theta_T \pi|_M(\nu) = \mu(\nu,\nu^{-1}) L_e^* V^* \widetilde{r}(\nu)\theta_T \\ &= \mu(\nu,\nu^{-1}) L_e^* \widetilde{r}(\nu) V^* \theta_T = \mu(\nu,\nu^{-1})\overline{\mu(\nu,\nu^{-1})} L_\nu^* V^* \theta_T = L_\nu^* V^* \theta_T. \end{aligned}$$

So, by (2.4), we have

$$\theta_E = \sum_{\nu\in\Lambda} L_\nu E\pi|_M(\nu) = \sum_{\nu\in\Lambda} L_\nu L_\nu^* V^* \theta_T = V^*\theta_T.$$

Therefore

$$\theta_E^*\theta_E = (V^*\theta_T)^*(V^*\theta_T) = \theta_T^*(VV^*)\theta_T = \theta_T^* P_T \theta_T = \theta_T^*\theta_T = I_M.$$

It follows that $\mathcal{G}(E,\Lambda)$ is also a subspace Parseval OPV-Gabor frame for $M = \overline{\mathrm{Range}(I - S_A^{-1})}$. On the other hand, we have

$$\mathrm{Range}(V^*) = \mathrm{Range}(Q) \subseteq \mathrm{Range}(I_0 \otimes I - P_A) = \mathrm{Range}(\theta_A)^\perp = \ker(\theta_A^*).$$

So, we have $\theta_A^*\theta_E = \theta_A^* V^* \theta_T = 0$, and hence

$$\theta_E^*\theta_A = \theta_A^*\theta_E = 0. \tag{3.2}$$

Write $D = \sqrt{I - S_A^{-1}}$, and apply Lemma 3.1 (iii), to obtain $D\pi(\nu) = \pi(\nu)D$ for all $\nu \in \Lambda$, from which we can see that $\mathcal{G}(ED,\Lambda)$ is a Bessel OPV-Gabor system for $L^2(G)$ and $\theta_{ED} = \theta_E D$. Taking into account that D is self-adjoint, we get

$$M = \overline{\mathrm{Range}(I - S_A^{-1})} = \overline{\mathrm{Range}(D)}.$$

Since $\mathcal{G}(E,\Lambda)$ is also a subspace Parseval OPV-Gabor frame for M, we have

$$I - S_A^{-1} = D^2 = D(\theta_E^*\theta_E)D = (\theta_E D)^*(\theta_E D) = \theta_{ED}^*\theta_{ED}.$$

Observing that $S_A^{-1} = \theta^*_{AS_A^{-1}}\theta_{AS_A^{-1}}$, we obtain

$$\theta^*_{AS_A^{-1}}\theta_{AS_A^{-1}} + \theta_{ED}^*\theta_{ED} = I. \tag{3.3}$$

Let $B = AS_A^{-1} + ED$. Then $\theta_B = \theta_A S_A^{-1} + \theta_E D$, and by (3.2) and (3.3), we obtain

$$\begin{aligned} \theta_B^*\theta_B &= (\theta_A S_A^{-1} + \theta_E D)^*(\theta_A S_A^{-1} + \theta_E D) = S_A^{-1}\theta_A^*\theta_A S_A^{-1} + S_A^{-1}(\theta_A^*\theta_E)D \\ &+ D(\theta_E^*\theta_A)S_A^{-1} + D\theta_E^*\theta_E D = \theta^*_{AS_A^{-1}}\theta_{AS_A^{-1}} + \theta_{ED}^*\theta_{ED} = I \end{aligned}$$

and

$$\theta_B^*\theta_A = (\theta_A S_A^{-1} + \theta_E D)^*\theta_A = S_A^{-1}\theta_A^*\theta_A + D\theta_E^*\theta_A = I.$$

The above arguments show that $\mathcal{G}(B,\Lambda)$ is a Parseval OPV-Gabor dual frame of $\mathcal{G}(A,\Lambda)$. This completes the proof. □

We next consider the Parseval OPV-Gabor duals in certain special case. Let $\mathcal{G}(A,\Lambda)$ be an OPV-Gabor frame for $L^2(G)$. If $B = TA$, where $T \in B(L^2(G))$ is an invertible operator, then $\mathcal{G}(B,\Lambda)$ is also an OPV-Gabor frame for $L^2(G)$. In this case, we say that that $\mathcal{G}(B,\Lambda)$ is *left-similar* to $\mathcal{G}(A,\Lambda)$. By an appropriate modification of the arguments used in Lemma 6.4 of [19], we obtain $P_A \sim P_B$ in $\widetilde{r}(\Lambda)'$. Since $r(\Lambda)'$ is a finite von Neumann algebra (cf. [6, 18]), so is $r(\Lambda)' \otimes I$. Keeping this fact in mind, an (OPV)-Gabor frame $\mathcal{G}(A,\Lambda)$ is said to satisfy a *finite von Neumann algebra condition*, or simply *F-condition*, if $P_A \in r(\Lambda)' \otimes I$. Moreover, we say that a lattice Λ of $G\times\widehat{G}$ is an *F-lattice* if every OPV-Gabor frame (including subspace OPV-Gabor frame) $\mathcal{G}(A,\Lambda)$ satisfies the F-condition.

A natural problem is whether such F-lattices exist or not. In [19], the authors discussed the OPV-frames associated with discrete (not necessarily countable) group representations on abstract Hilbert spaces. In particular, Corollary 7.4 of [19] contains a necessary and sufficient condition for all the OPV-frame generators to be left-similar, which is generalized in Corollary 3.14 of [13] for the case of vector frames. Given a lattice Λ of $G \times \widehat{G}$ and an OPV-Gabor frame $\mathcal{G}(A,\Lambda)$. If all the OPV-Gabor frames are left-similar to $\mathcal{G}(A,\Lambda)$, then there are no projections in $\widetilde{r}(\Lambda)'$ that are different but Murray-von Neumann equivalent to it. It follows from Corollary 7.4 of [19] that P_A belongs to the center $\widetilde{r}(\Lambda)' \cap \widetilde{r}(\Lambda)''$. Denote by $w^*(r(\Lambda))$ and $w^*(\lambda(\Lambda))$ the von Neumann algebras generated by $r(\Lambda)$ and $\lambda(\Lambda)$, respectively. It is well known that $r(\Lambda)' = w^*(\lambda(\Lambda))$ and $\lambda(\Lambda)' = w^*(r(\Lambda))$ (cf. [6, 14]). Since Λ is an Abelian group, we have $w^*(r(\Lambda)) = w^*(\lambda(\Lambda))$, and hence $\widetilde{r}(\Lambda)' \cap \widetilde{r}(\Lambda)'' = r(\Lambda)' \otimes I$. The above discussion tells us that if all the OPV-Gabor frames $\mathcal{G}(A,\Lambda)$ are left-similar, then $P_A \in r(\Lambda)' \otimes I$, which means that Λ is an F-lattice.

In [18] it was shown that there exists a unique center-valued trace τ on the von Neumann algebra $r(\Lambda)' \otimes I$, and for all orthogonal projections $P, Q \in r(\Lambda)' \otimes I$, $P \precsim Q$ in $r(\Lambda)' \otimes I$ if and only if $\tau(P) \leq \tau(Q)$. So, in the case where Λ is an F-lattice of $G \times \widehat{G}$, we can obtain the following corollary of Theorem 3.1.

Corollary 3.1. *Let G be an LCA group and Λ be an F-lattice of $G \times \widehat{G}$. Assume that $\mathcal{G}(A,\Lambda)$ is an OPV-Gabor frame for $L^2(G)$ whose frame operator S_A satisfies the condition $\|S_A^{-1}\| \leq 1$, and denote $M = \overline{\mathrm{Range}(I - S_A^{-1})}$. Then $\mathcal{G}(A,\Lambda)$ has a*

Parseval OPV-Gabor dual if and only if there exists a subspace Parseval OPV-Gabor frame $\mathcal{G}(T,\Lambda)$ *for* M *such that* $\tau(P_T) \leq I_0 \otimes I - \tau(P_A)$*, where* τ *is the center-valued trace on the von Neumann algebra* $r(\Lambda)' \otimes I$*.*

Proof. Assume that $\mathcal{G}(A,\Lambda)$ has a Parseval OPV-Gabor dual $\mathcal{G}(B,\Lambda)$. In the proof of the "only if" part of Theorem 3.1, in fact we have $P_T, P_A \in r(\Lambda)' \otimes I$. Hence $P_T \precsim I_0 \otimes I - P_A$ in $r(\Lambda)' \otimes I$, meaning that $\tau(P_T) \leq I_0 \otimes I - \tau(P_A)$.

Conversely, by the hypotheses we have $P_T \precsim I_0 \otimes I - P_A$ in $r(\Lambda)' \otimes I$. Since $r(\Lambda)' \otimes I$ is a subalgebra of $\widetilde{r}(\Lambda)'$, we have $P_T \precsim I_0 \otimes I - P_A$ in $\widetilde{r}(\Lambda)'$. So, we can apply Theorem 3.1 to conclude that $\mathcal{G}(A,\Lambda)$ has a Parseval OPV-Gabor dual. $\square$

Example 3.1. In the case $G = (\mathbb{R}^d, +)$, with the identification $x \in \mathbb{R}^d \leftrightarrow \gamma_x \in \widehat{G}$, we have $\widehat{G} = G$, where $\gamma_x(y) = e^{2\pi i \langle x,y\rangle}$. Let $g, f_0 \in L^2(\mathbb{R}^d)$ with $\|f_0\| = 1$, and let M_1 and M_2 be two non-singular $d \times d$ real matrices. Denote by A the rank one operator given by $Af = \langle f, g\rangle f_0$ for $f \in L^2(\mathbb{R}^d)$, and write $\Lambda = M_1\mathbb{Z}^d \times M_2\mathbb{Z}^d$, which is the so-called time-frequency lattice and plays an important role in time-frequency analysis. Then $\mathcal{G}(A,\Lambda)$ is an OPV-Gabor frame for $L^2(\mathbb{R}^d)$ if and only if there exist two constants $C, D > 0$ such that

$$C \parallel f \parallel^2 \leq \sum_{\nu\in\Lambda} |\langle f, g_\nu\rangle|^2 \leq D \parallel f \parallel^2 \quad \text{for all } f \in L^2(\mathbb{R}^d),$$

where $g_\nu(x) = e^{2\pi i\langle l,x\rangle} g(x-k)$ for $\nu = (k,l) \in \Lambda$. Hence, in this case, an OPV-Gabor frame is indeed an ordinary (vector) Gabor frame. The associated analysis operator θ_A is an operator from $L^2(\mathbb{R}^d)$ to $l^2(\Lambda) \otimes L^2(\mathbb{R}^d)$ defined by

$$\theta_A(f) = \sum_{\nu\in\Lambda} \chi_\nu \otimes \langle f, g_\nu\rangle f_0 \quad \text{for } f \in L^2(\mathbb{R}^d),$$

which leads to the orthogonal projection $P_A \in r(\Lambda)' \otimes I$. So the F-condition holds, and moreover, Λ is an F-lattice in this case. Thus, Corollary 3.1 holds for ordinary Gabor frames, and hence Theorem 2.2 of [12] is a special case of Corollary 3.1.

It is well known that the equation $\mathrm{tr}(X) = \langle X\chi_e, \chi_e\rangle$ for $X \in r(\Lambda)'$, defines a faithful normalized trace on $r(\Lambda)'$ (cf. [6]). Denote by ρ the corresponding map:

$$\rho : r(\Lambda)' \otimes I \to \mathbb{C}I, \quad \rho(X \otimes I) = \mathrm{tr}(X)I \tag{3.4}$$

for every $X \in r(\Lambda)'$. Then, by Lemma 8.3 of [19], we have

$$\rho(\Phi) = L_e^*\Phi L_e \quad \text{for all } \Phi \in r(\Lambda)' \otimes I. \tag{3.5}$$

The next proposition provides a characterization in the case where $r(\Lambda)'$ is a factor von Neumann algebra.

Proposition 3.1. *Let G be an LCA group, and let Λ be an F-lattice of $G\times\widehat{G}$ such that $r(\Lambda)'$ is a factor von Neumann algebra. Assume that $\mathcal{G}(A,\Lambda)$ is an OPV-Gabor frame for $L^2(G)$ whose frame operator S_A satisfies the condition $\|S_A^{-1}\|\le 1$, and denote $M=\overline{\mathrm{Range}(I-S_A^{-1})}$. Then $\mathcal{G}(A,\Lambda)$ has a Parseval OPV-Gabor dual if and only if there exists a subspace Parseval OPV-Gabor frame $\mathcal{G}(T,\Lambda)$ for M such that $(T|_M)(T|_M)^*\le I-AS_A^{-1}A^*$.*

Proof. Since Λ is an F-lattice of $G\times\widehat{G}$, by Corollary 3.1, $\mathcal{G}(A,\Lambda)$ has a Parseval OPV-Gabor dual if and only if there exists a subspace Parseval OPV-Gabor frame $\mathcal{G}(T,\Lambda)$ for M such that $\tau(P_T)\le I_0\otimes I-\tau(P_A)$, where τ is the center-valued trace on the finite von Neumann algebra $r(\Lambda)'\otimes I$. Noting that $P_A,P_T\in r(\Lambda)'\otimes I$, we can assume that $P_A=P_1\otimes I$, $P_T=P_2\otimes I$, where P_1,P_2 are two orthogonal projections in the finite von Neumann algebra $r(\Lambda)'$. Let τ_Λ be the center-valued trace on $r(\Lambda)'$. Then $\tau(P_T)\le I_0\otimes I-\tau(P_A)$ if and only if $\tau_\Lambda(P_2)\le I_0-\tau_\Lambda(P_1)$. Also, since $r(\Lambda)'$ is a factor, we have that

$$\tau_\Lambda(P_1)=\mathrm{tr}(P_1)I_0,\quad \tau_\Lambda(P_2)=\mathrm{tr}(P_2)I_0.$$

Thus, $\tau_\Lambda(P_2)\le I_0-\tau_\Lambda(P_1)$ if and only if $\mathrm{tr}(P_2)\le 1-\mathrm{tr}(P_1)$. By (3.4) we have $\rho(P_A)=\mathrm{tr}(P_1)I$, $\rho(P_T)=\mathrm{tr}(P_2)I$. Hence $\mathrm{tr}(P_2)\le 1-\mathrm{tr}(P_1)$ if and only if $\rho(P_T)\le I-\rho(P_A)$. By using (2.3), (2.4), (**??**) and (3.5), we can write

$$\begin{aligned}\rho(P_T)&=L_e^*P_TL_e=L_e^*\theta_T\theta_T^*L_e=L_e^*\theta_T\sum_{\nu\in\Lambda}(T\pi|_M(\nu))^*L_\nu^*L_e=L_e^*\theta_T(T\pi|_M(e))^*\\&=L_e^*\theta_T(T|_M)^*=L_e^*\sum_{\nu\in\Lambda}L_\nu T\pi|_M(\nu)(T|_M)^*=T\pi|_M(e)(T|_M)^*=(T|_M)(T|_M)^*.\end{aligned}$$

Similarly it can be shown that $\rho(P_A)=\rho(P_{AS_A^{-1/2}})=(AS_A^{-\frac12})(AS_A^{-\frac12})^*=AS_A^{-1}A^*$. Therefore, $\rho(P_T)\le I-\rho(P_A)$ if and only if $(T|_M)(T|_M)^*\le I-AS_A^{-1}A^*$, and the result follows. Proposition 3.1 is proved. □

Finally, we give a necessary and sufficient condition for an OPV-Gabor frame for $L^2(G)$ to admit a tight OPV-Gabor dual.

Theorem 3.2. *Let G be an LCA group and Λ be a lattice of $G\times\widehat{G}$. Suppose that $\mathcal{G}(A,\Lambda)$ is an OPV-Gabor frame for $L^2(G)$ with the frame operator S_A. Then $\mathcal{G}(A,\Lambda)$ has a tight OPV-Gabor dual if and only if there exists a subspace Parseval OPV-Gabor frame $\mathcal{G}(T,\Lambda)$ for $M=\overline{\mathrm{Range}(\|S_A^{-1}\|I-S_A^{-1})}$ such that $P_T\precsim I_0\otimes I-P_A$ in the von Neumann algebra $\widetilde{r}(\Lambda)'=r(\Lambda)'\otimes B(L^2(G))$.*

Proof. Assume first that $\mathcal{G}(A,\Lambda)$ has a tight OPV-Gabor dual $\mathcal{G}(B,\Lambda)$ with frame bound b. From Lemma 2.2, we have $b\ge\|S_A^{-1}\|$, which implies that $\|(bS_A)^{-1}\|\le$

1. Observe that $\mathcal{G}(\frac{1}{\sqrt{b}}B,\Lambda)$ is a Parseval OPV-Gabor dual of $\mathcal{G}(\sqrt{b}A,\Lambda)$, and the frame operator for $\mathcal{G}(\sqrt{b}A,\Lambda)$ is bS_A. It follows from Theorem 3.1 that there exists a subspace Parseval OPV-Gabor frame $\mathcal{G}(C,\Lambda)$ for $N=\overline{\mathrm{Range}(I-\frac{1}{b}S_A^{-1})}$ such that $P_C\precsim I_0\otimes I-P_{\sqrt{b}A}$ in $\widetilde{r}(\Lambda)'$. Noting that if $b>\|S_A^{-1}\|$, then $bI-S_A^{-1}$ is invertible, we have

$$\ker(I-\frac{1}{b}S_A^{-1})=\ker(bI-S_A^{-1})=\{0\}.$$

Thus when $b\geq\|S_A^{-1}\|$, we have $\ker(\|S_A^{-1}\|I-S_A^{-1})^{\perp}\subseteq\ker(I-\frac{1}{b}S_A^{-1})^{\perp}$, which means that

$$M=\overline{\mathrm{Range}(\|S_A^{-1}\|I-S_A^{-1})}\subseteq\overline{\mathrm{Range}(I-\frac{1}{b}S_A^{-1})}=N.$$

Define an operator $T:=C|_M$. It is easy to check that $\mathcal{G}(T,\Lambda)$ is a subspace Parseval OPV-Gabor frame for M and $\mathrm{Range}(\theta_T)\subseteq\mathrm{Range}(\theta_C)$. Combining this with the fact that $\mathrm{Range}(\theta_{\sqrt{b}A})=\mathrm{Range}(\theta_A)$, we get

$$P_T\leq P_C\precsim I_0\otimes I-P_{\sqrt{b}A}=I_0\otimes I-P_A$$

in $\widetilde{r}(\Lambda)'$.

Conversely, assume that there exists a subspace Parseval OPV-Gabor frame $\mathcal{G}(T,\Lambda)$ for $M=\overline{\mathrm{Range}(\|S_A^{-1}\|I-S_A^{-1})}$ such that $P_T\precsim I_0\otimes I-P_A$ in $\widetilde{r}(\Lambda)'$. Observe that

$$M=\ker(\|S_A^{-1}\|I-S_A^{-1})^{\perp}=\ker\left(I-\frac{S_A^{-1}}{\|S_A^{-1}\|}\right)^{\perp},$$

and $\|S_A^{-1}\|S_A$ is the frame operator for OPV-Gabor frame $\mathcal{G}(\sqrt{\|S_A^{-1}\|}A,\Lambda)$ satisfying $\|(\|S_A^{-1}\|S_A)^{-1}\|=1$. Since $\mathrm{Range}(\theta_{\sqrt{\|S_A^{-1}\|}A})=\mathrm{Range}(\theta_A)$ implies that $P_{\sqrt{\|S_A^{-1}\|}A}=P_A$, by Theorem 3.1, $\mathcal{G}(\sqrt{\|S_A^{-1}\|}A,\Lambda)$ has a Parseval OPV-Gabor dual $\mathcal{G}(B,\Lambda)$. Therefore $\mathcal{G}(A,\Lambda)$ has a tight OPV-Gabor dual $\mathcal{G}(\sqrt{\|S_A^{-1}\|}B,\Lambda)$. The proof is complete. Theorem 3.2 is proved. □

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