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ON INTERPOLATION BY A HOMOGENEOUS POLYNOMIALS IN $\mathbb{R}^2$

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Abstract. In this paper, we study bivariate homogeneous interpolation polynomials. We show that the homogeneous Lagrange interpolation polynomial of a sufficiently smooth function converges to a homogeneous Hermite interpolation polynomial when the interpolation points coalesce.

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1. Introduction

The aim of this paper is to study bivariate homogeneous interpolation polynomials. Let $\mathcal{H}_n(\mathbb{R}^2)$ be the space of all homogeneous polynomials of degree n in $\mathbb{R}^2$. It is well-known that $\{x^n, x^{n-1}y, \ldots, y^n\}$ is a basis for $\mathcal{H}_n(\mathbb{R}^2)$ and $\dim \mathcal{H}_n(\mathbb{R}^2) = n+1$. A set $X = \{\mathbf{x}_i = (a_i, b_i) : i = 0, \ldots, n\} \subset \mathbb{R}^2 \setminus \{0\}$ is said to be pairwise projectively distinct (PPD for short) if $\mathbf{x}_i$ and $\mathbf{x}_j$ are linearly independent, or equivalently $a_i b_j - a_j b_i \neq 0$, for every $0 \leq i < j \leq n$. Let f be a function defined on X. Bialas-Ciez and Calvi [1] pointed out that there exists a unique polynomial $h \in \mathcal{H}_d(\mathbb{R}^2)$ interpolating f at X, that is,

$$h(\mathbf{x}_i) = f(\mathbf{x}_i), \quad 0 \leq i \leq n.$$

Moreover, the following equality holds:

$$h(\mathbf{x}) = \sum_{i=0}^{n} f(\mathbf{x}_i) \prod_{j=0, j\neq i}^{n} \frac{x b_j - y a_j}{a_i b_j - b_i a_j}, \quad \mathbf{x} = (x, y).$$

The polynomial h is called the homogeneous Lagrange interpolation polynomial of f at X, and is denoted by $\mathbf{L}^{\mathrm{h}}[X; f]$. Here we are concerned with the following problem.

Problem. Let $\{X^k\}$ be a sequence of PPD sets in $\mathbb{R}^2$ with $\mathrm{card}(X^k) = n + 1$ for $k \geq 1$. Assume that X^k coalesces to a set A when $k \to \infty$. The natural questions are the following.

(1) Does the sequence $\{\mathbf{L}^{\mathrm{h}}[X^k; f]\}$ converge for every suitably defined function f?

(2) If yes, what is the limit? can it be understood as a Hermite-type interpolation polynomial of f at A?

Positive answers to the above questions can be expected since Hermite interpolation is usually the result of the collapsing of points in Lagrange interpolation. Note that analogous problems have been studied by many authors. For instance, in [2, 5], the authors found sufficient conditions that guarantee the convergence of multivariate Lagrange interpolation polynomials of sufficiently smooth functions to the Taylor polynomials. Phung [8] showed that the limit of bivariate Lagrange interpolation polynomials at Bos configurations on circles is a Hermite interpolation polynomial at the center of the circles when the radii of the circles tend to 0. In [6], based on a beautiful result of Bos and Calvi [4], Calvi and Phung proved that the limit of Lagrange projectors at Bos configurations on the irreducible algebraic curves in $\mathbb{C}^2$ are the Hermite projectors introduced by Bos and Calvi [4].

To investigate the above problem, we restrict our attention to the case where X^k lies on the right half of the unit circle, that is,

$$X^k \subset \mathcal{C}^+ = \{\mathbf{x} = (x, y) : \|\mathbf{x}\| = 1, x > 0\}, \quad \|\mathbf{x}\| = \sqrt{x^2 + y^2}.$$

This assumption does not lose of generality since $h(t\mathbf{x}) = t^n h(\mathbf{x})$ for $h \in \mathcal{H}_n(\mathbb{R}^2)$, $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^2$.

In Theorem 3.1 below, we give a Hermite type interpolation scheme for $\mathcal{H}_n(\mathbb{R}^2)$ in which the interpolation points lie on $\mathcal{C}^+$. From this we infer that the Hermite interpolation polynomial of a suitably defined function exists uniquely. In Theorem 4.1, we prove that when f is of class C^n, then the homogeneous Hermite interpolation polynomial of f at A is the limit of the sequence of polynomials $\{\mathbf{L}^{\mathrm{h}}[X^k; f]\}$ when X^k coalesces to A. Finally, we establish a continuity property of homogeneous Hermite interpolation with respect to the interpolation points.

By $\mathcal{P}_n(\mathbb{R})$ we denote the vector space of all univariate polynomials of degree less than or equal to n. Each vector space $\mathcal{P}_n(\mathbb{R})$, $\mathcal{H}_n(\mathbb{R}^2)$ is endowed with a norm. The set of all natural numbers is denoted by $\mathbb{N}$.

2. Univariate Hermite interpolation

Let $t_1, \ldots, t_\lambda$ be distinct real numbers, and let $\nu_1, \ldots, \nu_\lambda$ be positive integers such that $n + 1 = \nu_1 + \cdots + \nu_\lambda$. The following theorem is well-known.

Theorem 2.1. *Given a function f for which $f^{(\nu_i - 1)}(t_i)$ exists for $i = 1, \ldots, \lambda$. Then there exists a unique $p \in \mathcal{P}_n(\mathbb{R})$ such that*

$$p^{(j)}(t_i) = f^{(j)}(t_i), \quad \forall 1 \le i \le \lambda, \ 0 \le j \le \nu_i - 1.$$

The polynomial p in Theorem 2.1 is denoted by $\mathrm{H}[\{(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)\}; f]$ and is called the Hermite interpolation polynomial. The coefficient of t^n in $\mathrm{H}[\{(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)\}; f]$, denoted by $f[(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)]$, is called the divided difference. A formula for Hermite interpolation polynomial can be found in [3, Theorem 1.1]. Using this, we can prove the following factorization property of generalized Vandermonde determinants (see [8] for details of the proof.)

Lemma 2.1. *Let $t_1,\dots,t_\lambda$ be pairwise distinct real numbers, and let $\nu_1,\dots,\nu_\lambda$ be positive integers such that $n+1=\nu_1+\cdots+\nu_\lambda$. Let $T=\{(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)\}$, and let $\mathcal{F}=\{f_0,\dots,f_n\}$ be a set of sufficiently differentiable functions at the t_j's. We denote by $\mathrm{VDM}(\mathcal{F};T)$ the determinant of the generalized Vandermonde matrix:*

$$\mathrm{V}(\mathcal{F};T)=\begin{bmatrix} f_0(t_1) & f_1(t_1) & \cdots & f_{n-1}(t_1) & f_n(t_1)\\ f_0'(t_1) & f_1'(t_1) & \cdots & f_{n-1}'(t_1) & f_n'(t_1)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ f_0^{(\nu_1-1)}(t_1) & f_1^{(\nu_1-1)}(t_1) & \cdots & f_{n-1}^{(\nu_1-1)}(t_1) & f_n^{(\nu_1-1)}(t_1)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ f_0(t_\lambda) & f_1(t_\lambda) & \cdots & f_{n-1}(t_\lambda) & f_n(t_\lambda)\\ f_0'(t_\lambda) & f_1'(t_\lambda) & \cdots & f_{n-1}'(t_\lambda) & f_n'(t_\lambda)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ f_0^{(\nu_\lambda-1)}(t_\lambda) & f_1^{(\nu_\lambda-1)}(t_\lambda) & \cdots & f_{n-1}^{(\nu_\lambda-1)}(t_\lambda) & f_n^{(\nu_\lambda-1)}(t_\lambda) \end{bmatrix}.$$

Then we have

$$\mathrm{VDM}(\mathcal{F};T)=\Big(\prod_{k=1}^{\lambda}\prod_{i=0}^{\nu_k-1} i!\Big)\prod_{1\le i<j\le\lambda}(t_j-t_i)^{\nu_i\nu_j}\,\mathrm{D}(\mathcal{F};T),$$

where

$$\mathrm{D}(\mathcal{F};T)=\begin{vmatrix} f_0[t_1] & f_1[t_1] & \cdots & f_n[t_1]\\ f_0[(t_1,2)] & f_1[(t_1,2)] & \cdots & f_n[(t_1,2)]\\ \vdots & \vdots & \ddots & \vdots\\ f_0[(t_1,\nu_1)] & f_1[(t_1,\nu_1)] & \cdots & f_n[(t_1,\nu_1)]\\ f_0[(t_1,\nu_1),(t_2,1)] & f_1[(t_1,\nu_1),(t_2,1)] & \cdots & f_n[(t_1,\nu_1),(t_2,1)]\\ \vdots & \vdots & \ddots & \vdots\\ f_0[(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)] & f_1[(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)] & \cdots & f_n[(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)] \end{vmatrix}.$$

Here the factor $\prod_{1\le i<j\le\lambda}(t_j-t_i)^{\nu_i\nu_j}$ does not appear when $\lambda=1$.

The following result gives a useful formula for Hermite interpolation polynomial which contains the terms $\mathrm{D}(\cdot;\cdot)$.

Proposition 2.1. *Let $t_1,\dots,t_\lambda$ be pairwise distinct real numbers, and let $\nu_1,\dots,\nu_\lambda$ be positive integers such that $n+1=\nu_1+\cdots+\nu_\lambda$. Let $T=\{(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)\}$,*

$\mathcal{M} = \{1, t, \dots, t^n\}$, *and let* f *be a well-defined function. Then*

$$\mathrm{H}[\{(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)\}; f](t) = \sum_{i=0}^{n} \mathrm{D}(\mathcal{M}[t^i \leftarrow f(t)]; T) t^i.$$

Here $\mathcal{M}[t^i \leftarrow f(t)]$ *means that we substitute* $f(t)$ *for* t^i *in* $\mathcal{M}$.

Proof. For convenience, we set $f_i(t) = t^i$ for $0 \le i \le n$, and write

$$\mathrm{H}[\{(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)\}; f](t) = \sum_{i=0}^{n} c_i f_i(t).$$

It follows from the interpolation conditions that

$$\sum_{i=0}^{n} c_i f_i^{(k)}(t_j) = f^{(k)}(t_j), \quad 1 \le j \le \lambda,\ 0 \le k \le \nu_j - 1. \tag{2.1}$$

The determinant of the matrix of coefficients corresponding to the above system of linear equations is $\mathrm{VDM}(\mathcal{M}; T)$. A result in [3, p. 3] shows that

$$\mathrm{VDM}(\mathcal{M}; T) = \Big(\prod_{k=1}^{\lambda} \prod_{i=0}^{\nu_k - 1} i!\Big) \prod_{1 \le i < j \le \lambda} (t_j - t_i)^{\nu_i \nu_j}.$$

Using the Cramer rule in (2.1) and Lemma 2.1, we obtain

$$c_i = \frac{\mathrm{VDM}(\mathcal{M}[f_i \leftarrow f]; T)}{\mathrm{VDM}(\mathcal{M}; T)} = \mathrm{D}(\mathcal{M}[f_i \leftarrow f]; T), \quad 0 \le i \le n,$$

The proof is complete. □

When working with Hermite interpolation and the divided difference, it is convenient to use interpolation sets in which elements may be repeated. For example, if $A = \{1, -2, 3, -2, 1, -4, 1\}$, then we can write $A = \{(1,3), (-2,2), (3,1), (-4,1)\}$. More generally, any set $\{s_0, \dots, s_n\}$ can be identified with $\{(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)\}$. Here, t_i are pairwise distinct and the notation (t_i, ν_i) means that t_i is repeated ν_i times. Hence, we can write $\mathrm{H}[\{s_0,\dots,s_n\}; f]$ (resp. $f[s_0,\dots,s_n]$) for $\mathrm{H}[\{(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)\}; f]$ (resp. $f[(t_1,\nu_1),\dots,(t_\lambda,\nu_\lambda)]$). It is important to note that the divided difference is continuous with respect to interpolation points (see [3, Corollary 1.5]).

Lemma 2.2. *Let* $I \subset \mathbb{R}$ *be an interval and* $g \in \mathrm{C}^n(I)$. *Then the map*

$$(s_0, \dots, s_n) \in I^{n+1} \longmapsto g[s_0, \dots, s_n]$$

is continuous.

3. Bivariate homogeneous Hermite interpolation

In this section, we first give some results concerning vanishing of derivatives of functions, which are used to prove that the homogeneous Hermite interpolation problem in $\mathbb{R}^2$ has a unique solution.

3.1. **Vanishing of derivatives of functions.**

Lemma 3.1. *Let k be a natural number, and let g and h be k-times differentiable functions at $t_0 \in \mathbb{R}$. If $g(t_0) \neq 0$ and $(gh)^{(i)}(t_0) = 0$ for $i = 0, \ldots, k$, then $h^{(i)}(t_0) = 0$ for $i = 0, \ldots, k$.*

Proof. Since by assumption $g(t_0)h(t_0) = 0$ and $g(t_0) \neq 0$, we have $h(t_0) = 0$. Assume that the assertion holds for $i = 0, \ldots, j-1$ with $j \leq k$, and prove it for j. By the Leibnitz formula, we obtain

$$0 = (gh)^{(j)}(t_0) = g(t_0)h^{(j)}(t_0) + \sum_{i=1}^{j} \binom{j}{i} g^{(i)}(t_0)h^{(j-i)}(t_0) = g(t_0)h^{(j)}(t_0).$$

Hence, $h^{(j)}(t_0) = 0$, and the result follows. □

Lemma 3.2. *Let g be a k-times differentiable functions at $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then*

$$\frac{d^i}{d\theta^i} g(\tan\theta)\Big|_{\theta=\theta_0} = 0, \quad \forall i = 0, \ldots, k$$

if and only if

$$g^{(i)}(\tan\theta_0) = 0 \quad \forall i = 0, \ldots, k.$$

Proof. We first assume that $\frac{d^i}{d\theta^i} g(\tan\theta)\Big|_{\theta=\theta_0} = 0$ for every $i = 0, \ldots, k$. We prove the lemma by induction on k. The assertion is trivial for $k = 0$. Assuming that the assertion holds for $k-1$, we prove it for k. For convenience, we set $\varphi(\theta) = \tan\theta$. Using Faa di Bruno's formula from [9], we obtain

$$\frac{d^k}{d\theta^k} g(\varphi(\theta))\Big|_{\theta=\theta_0} = \sum \frac{k!}{n_1! \cdots n_k!} g^{(n)}(\varphi(\theta_0)) \prod_{j=1}^{k} \Big(\frac{\varphi^{(j)}(\theta_0)}{j!}\Big)^{n_j}. \tag{3.1}$$

where $n = n_1 + \cdots + n_k$ and the sum is over all values of $n_1, \ldots, n_k \in \mathbb{N}$ such that $n_1 + 2n_2 + \cdots + kn_k = k$. Note that $n \leq k$ and $n = k$ only if $n_1 = n = k$ and $n_2 = \cdots = n_k = 0$. Hence, it follows from the induction hypothesis and (3.1) that

$$0 = \frac{d^k}{d\theta^k} g(\varphi(\theta))\Big|_{\theta=\theta_0} = g^{(k)}(\tan\theta_0) \frac{1}{\cos^{2k}(\theta_0)}.$$

Thus, $g^{(k)}(\tan\theta_0) = 0$.

Conversely, from (3.1) we see that $\frac{d^k}{d\theta^k} g(\varphi(\theta))\Big|_{\theta=\theta_0}$ is a linear combination of $k+1$ derivatives $g(\varphi(\theta_0)), g'(\varphi(\theta_0)), \ldots, g^{(k)}(\varphi(\theta_0))$. Hence, if these $k+1$ numbers are equal to 0, then $\frac{d^i}{d\theta^i} g(\tan\theta)\Big|_{\theta=\theta_0} = 0$ for every $i = 0, \ldots, k$. □

Corollary 3.1. *Let f and g be k-times differentiable functions at $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then*

$$\frac{d^i}{d\theta^i} f(\tan\theta)\Big|_{\theta=\theta_0} = \frac{d^i}{d\theta^i} g(\tan\theta)\Big|_{\theta=\theta_0}, \quad \forall i = 0, \ldots, k,$$

if and only if

$$f^{(i)}(\tan\theta_0) = g^{(i)}(\tan\theta_0), \quad \forall i = 0, \dots, k.$$

3.2. **Homogeneous Hermite interpolation.**

Theorem 3.1. *Let $n, \nu_1, \dots, \nu_\lambda$ be positive integers such that $\nu_1 + \cdots + \nu_\lambda = n+1$, and let $\{\mathbf{n}_1, \dots, \mathbf{n}_\lambda\}$ be a set of distinct points on $\mathcal{C}^+$ with $\mathbf{n}_i = (\cos\alpha_i, \sin\alpha_i)$, $\alpha_j \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then, for any given data $\{c_{ik}\}$, there exists a unique polynomial $h \in \mathcal{H}_n(\mathbb{R}^2)$ such that*

$$\frac{d^k}{d\alpha^k} h(\cos\alpha, \sin\alpha)\Big|_{\alpha=\alpha_i} = c_{ik}, \quad 1 \le i \le \lambda, \;\; 0 \le k \le \nu_i - 1. \tag{3.2}$$

Proof. Since the relation (3.2) consists of $n+1$ interpolation conditions and $\dim \mathcal{H}_n(\mathbb{R}^2) = n+1$, it suffices to show that if $h \in \mathcal{H}_n(\mathbb{R}^2)$ satisfies the following conditions

$$\frac{d^k}{d\alpha^k} h(\cos\alpha, \sin\alpha)\Big|_{\alpha=\alpha_i} = 0, \quad 1 \le i \le \lambda, \;\; 0 \le k \le \nu_i - 1, \tag{3.3}$$

then $h = 0$. We write

$$h(\cos\alpha, \sin\alpha) = (\cos^n\alpha) h(1, \tan\alpha) = (\cos^n\alpha) q(\tan\alpha),$$

where $q(t) = h(1,t) \in \mathcal{P}_n(\mathbb{R})$. From (3.3), we have

$$\frac{d^k}{d\alpha^k}\Big((\cos^n\alpha) q(\tan\alpha)\Big)\Big|_{\alpha=\alpha_i} = 0, \quad 1 \le i \le \lambda, \;\; 0 \le k \le \nu_i - 1.$$

Since $\alpha_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$, Lemma 3.1 gives

$$\frac{d^k}{d\alpha^k} q(\tan\alpha)\Big|_{\alpha=\alpha_i} = 0, \quad 1 \le i \le \lambda, \;\; 0 \le k \le \nu_i - 1.$$

Using Lemma 3.2, we obtain

$$q^{(k)}(\tan\alpha_i) = 0, \quad 1 \le i \le \lambda, \;\; 0 \le k \le \nu_i - 1.$$

The uniqueness of univariate Hermite interpolation in Theorem 2.1 implies that $q = 0$ since $\deg p \le n$. Hence, $h(x,y) = x^n q(\frac{y}{x}) = 0$. □

Corollary 3.2. *Under the assumptions of Theorem 3.1, if f is a function defined on $\mathcal{C}^+$ such that the function $\alpha \mapsto f(\cos\alpha, \sin\alpha)$ is $(\nu_i - 1)$-times differentiable at α_i for $i = 1, \dots, \lambda$, then there exists a unique polynomial $h = \mathbf{H}^{\mathrm{h}}[\{(\mathbf{n}_1, \nu_1), \dots, (\mathbf{n}_\lambda, \nu_\lambda)\}; f] \in \mathcal{H}_n(\mathbb{R}^2)$ satisfying*

$$\frac{d^k}{d\alpha^k} h(\cos\alpha, \sin\alpha)\Big|_{\alpha=\alpha_i} = \frac{d^k}{d\alpha^k} f(\cos\alpha, \sin\alpha)\Big|_{\alpha=\alpha_i}, \quad 1 \le i \le \lambda, \;\; 0 \le k \le \nu_i - 1.$$

4. Limit of homogeneous Lagrange interpolation polynomials

In this section, we give a formula for homogeneous Lagrange interpolation polynomials, and then use it to respond the proposed problem.

Proposition 4.1. *Let $X = \{\mathbf{x}_0, \ldots, \mathbf{x}_n\}$ be a set of distinct points on $\mathcal{C}^+$ with $\mathbf{x}_i = (\cos\alpha_i, \sin\alpha_i)$, $\alpha_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and let f be a function defined on X. Then*

$$\mathbf{L}^{\mathrm{h}}[X; f](\mathbf{x}) = \sum_{i=0}^{n} \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T) x^{n-i} y^i,$$

where $\mathcal{M} = \{1, t, \ldots, t^n\}$, $T = \{\tan\alpha_0, \ldots, \tan\alpha_n\}$ and the function $\hat{f}$ is defined by $\hat{f}(\tan\alpha) = \frac{f(\cos\alpha, \sin\alpha)}{\cos^n\alpha}$.

Proof. Since $\mathbf{L}^{\mathrm{h}}[X; f] \in \mathcal{H}_d(\mathbb{R}^2)$, we can write $\mathbf{L}^{\mathrm{h}}[X; f](\mathbf{x}) = \sum_{i=0}^{n} d_i x^{n-i} y^i$. By definition, we have

$$\sum_{i=0}^{n} d_i \cos^{n-i}\alpha_k \sin^i \alpha_k = f(\cos\alpha_k, \sin\alpha_k), \quad k = 0, \ldots, n.$$

Dividing both sides by $\cos^n \alpha_k$, we obtain

$$\sum_{i=0}^{n} d_i \tan^i \alpha_k = \frac{f(\cos\alpha_k, \sin\alpha_k)}{\cos^n \alpha_k} = \hat{f}(\tan\alpha_k), \quad k = 0, \ldots, n.$$

By Cramer's rule, the coefficients d_i's are given by

$$d_i = \frac{\mathrm{VDM}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T)}{\mathrm{VDM}(\mathcal{M}, T)} = \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T), \quad 0 \le i \le n,$$

where Lemma 2.1 is used in the case of pairwise distinct points to reduce the last fraction. □

Theorem 4.1. *Let $n, \nu_1, \ldots, \nu_\lambda$ be positive integers such that $\nu_1 + \cdots + \nu_\lambda = n+1$, and let $X^k = \{\mathbf{x}_0^k, \ldots, \mathbf{x}_n^k\}$, $k \ge 1$, be sets of distinct points on $\mathcal{C}^+$ such that*

$$\lim_{k\to\infty} \mathbf{x}_j^k = \mathbf{n}_1 \quad \textit{for} \quad 0 \le j \le \nu_1 - 1$$

and

$$\lim_{k\to\infty} \mathbf{x}_j^k = \mathbf{n}_m \quad \textit{for} \quad \nu_1 + \cdots + \nu_{m-1} \le j \le \nu_1 + \cdots + \nu_m - 1,\ 2 \le m \le \lambda,$$

where $\mathbf{n}_i$'s are pairwise distinct points on $\mathcal{C}^+$. Then, for any function f of class C^n on $\mathcal{C}^+$, we have

$$\lim_{k\to\infty} \mathbf{L}^{\mathrm{h}}[X^k; f] = \mathbf{H}^{\mathrm{h}}[\{(\mathbf{n}_1, \nu_1), \ldots, (\mathbf{n}_\lambda, \nu_\lambda)\}; f].$$

Proof. Let us define $\alpha_i^k = \arg(\mathbf{x}_i^k)$ for $0 \le i \le n$ and $\alpha_i = \arg(\mathbf{n}_i)$ for $1 \le i \le \lambda$. The hypothesis gives

$$\lim_{k\to\infty} \alpha_j^k = \alpha_1 \quad \text{for} \quad 0 \le j \le \nu_1 - 1 \tag{4.1}$$

and

$$\lim_{k\to\infty} \alpha_j^k = \alpha_m \quad \text{for} \quad \nu_1 + \cdots + \nu_{m-1} \le j \le \nu_1 + \cdots + \nu_m - 1,\ 2 \le m \le \lambda. \tag{4.2}$$

Without loss of generality, we can assume that $-\frac{\pi}{2} < \alpha_1 < \cdots < \alpha_\lambda < \frac{\pi}{2}$. Since the interpolation polynomial is independent of the ordering of the interpolation points, the relations (4.1) and (4.2) allow us to assume that $-\frac{\pi}{2} < \alpha_0^k < \alpha_1^k < \cdots < \alpha_n^k < \frac{\pi}{2}$. Evidently, the setting $\hat{f}(\tan\alpha) := \frac{f(\cos\alpha,\sin\alpha)}{\cos^n\alpha}$ with $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is equivalent to

$$\hat{f}(t) = (t^2+1)^{\frac{n}{2}} f\Big(\frac{1}{\sqrt{t^2+1}}, \frac{t}{\sqrt{t^2+1}}\Big), \quad t \in \mathbb{R}.$$

Hence, $\hat{f} \in \mathrm{C}^n(\mathbb{R})$. By Proposition 4.1, we can write

$$\mathbf{L}^{\mathrm{h}}[X^k; f](\mathbf{x}) = \sum_{i=0}^{n} d_i^k x^{n-i} y^i, \tag{4.3}$$

where

$$d_i^k = \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T^k), \quad 0 \le i \le n,$$

and $\mathcal{M} = \{1, t, \ldots, t^n\}$, $T^k = \{\tan\alpha_0^k, \ldots, \tan\alpha_n^k\}$. Let us set

$$T = \{\big(\tan\alpha_1, \nu_1\big), \ldots, \big(\tan\alpha_\lambda, \nu_\lambda\big)\} = \{\tan\beta_0, \tan\beta_1, \ldots, \tan\beta_n\}, \quad \beta_0 \le \cdots \le \beta_n.$$

By relations (4.1) and (4.2), we have

$$\lim_{k\to\infty} \tan\alpha_i^k = \tan\beta_i, \quad 0 \le i \le n.$$

From the formula for $\mathrm{D}(\cdot;\cdot)$ in Lemma 2.1, we see that the (l,m)-entries of the matrices corresponding to $\mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T^k)$ and $\mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T)$ are

$$g[\tan\alpha_0^k, \ldots, \tan\alpha_{l-1}^k] \quad \text{and} \quad g[\tan\beta_0, \ldots, \tan\beta_{l-1}],$$

where $g \in \{1, \ldots, t^{i-1}, \hat{f}, t^{i+1}, \ldots, t^n\}$. It follows from Lemma 2.2 that

$$\lim_{k\to\infty} g[\tan\alpha_0^k, \ldots, \tan\alpha_{l-1}^k] = g[\tan\beta_0, \ldots, \tan\beta_{l-1}].$$

Therefore, we have

$$\lim_{k\to\infty} \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T^k) = \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T),$$

and consequently,

$$\lim_{k\to\infty} \mathbf{L}^{\mathrm{h}}[X^k; f](\mathbf{x}) = \sum_{i=0}^{n} \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T) x^{n-i} y^i =: H(x,y).$$

To complete the proof, it remains to verify that $H = \mathbf{H}^{\mathrm{h}}[\{(\mathbf{n}_1, \nu_1), \ldots, (\mathbf{n}_\lambda, \nu_\lambda)\}; f]$. We have

$$H(1,t) = \sum_{i=0}^{n} \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T) t^i = \mathbf{H}^{\mathrm{H}}\big[\{\big(\tan\alpha_1, \nu_1\big), \ldots, \big(\tan\alpha_\lambda, \nu_\lambda\big)\}; \hat{f}\big](t),$$

where we use Proposition 2.1 in the second relation. It follows that, for $1 \le i \le \lambda$, $0 \le k \le \nu_i - 1$,

$$\begin{aligned} \frac{d^k}{dt^k} H(1,t)\Big|_{t=\tan\alpha_i} &= \frac{d^k}{dt^k}\Big(\mathbf{H}^{\mathrm{H}}\big[\{\big(\tan\alpha_1,\nu_1\big),\dots,\big(\tan\alpha_\lambda,\nu_\lambda\big)\};\hat{f}\big](t)\Big)\Big|_{t=\tan\alpha_i} \\ &= \frac{d^k}{dt^k}\hat{f}(t)\Big|_{t=\tan\alpha_i}. \end{aligned}$$

Corollary 3.1 now yields

$$\frac{d^k}{d\alpha^k} H(1,\tan\alpha)\Big|_{\alpha=\alpha_i} = \frac{d^k}{d\alpha^k}\hat{f}(\tan\alpha)\Big|_{\alpha=\alpha_i}, \quad 1 \le i \le \lambda,\ \ 0 \le k \le \nu_i - 1.$$

In other words,

$$\frac{d^k}{d\alpha^k}\frac{H(\cos\alpha,\sin\alpha)}{\cos^n\alpha}\Big|_{\alpha=\alpha_i} = \frac{d^k}{d\alpha^k}\frac{f(\cos\alpha,\sin\alpha)}{\cos^n\alpha}\Big|_{\alpha=\alpha_i}, \quad 1 \le i \le \lambda,\ \ 0 \le k \le \nu_i - 1.$$

Using Lemma 3.1, we get

$$\frac{d^k}{d\alpha^k} H(\cos\alpha,\sin\alpha)\Big|_{\alpha=\alpha_i} = \frac{d^k}{d\alpha^k} f(\cos\alpha,\sin\alpha)\Big|_{\alpha=\alpha_i}, \quad 1 \le i \le \lambda,\ \ 0 \le k \le \nu_i - 1.$$

It follows that H must be identical with $\mathbf{H}^{\mathrm{h}}[\{(\mathbf{n}_1,\nu_1),\dots,(\mathbf{n}_\lambda,\nu_\lambda)\};f]$. □

The next example shows that Theorem 4.1 does not hold when $f \notin \mathrm{C}^n(\mathcal{C}^+)$.

Example 4.1. For simplicity, we work with $n = 1$. The equation (4.3) gives

$$\mathbf{L}^{\mathrm{h}}[X^k;f](\mathbf{x}) = \mathrm{D}(\mathcal{M}[1 \leftarrow \hat{f}(t)];T^k)x + \mathrm{D}(\mathcal{M}[t \leftarrow \hat{f}(t)];T^k)y,$$

where $\mathcal{M} = \{1,t\}$, $T^k = \{\tan\alpha_0^k, \tan\alpha_1^k\}$. Let us define $f : \mathcal{C}^+ \to \mathbb{R}$ by

$$f\Big(\frac{1}{\sqrt{t^2+1}}, \frac{t}{\sqrt{t^2+1}}\Big) = \frac{\hat{f}(t)}{(t^2+1)^{\frac{1}{2}}}, \quad t \in \mathbb{R},$$

with

$$\hat{f}(t) = \begin{cases} \sqrt[3]{t^2}\sin\frac{1}{t} & \text{if} \quad t \ne 0 \\ 0 & \text{if} \quad t = 0. \end{cases}$$

Since $\hat{f}$ is continuous in $\mathbb{R}$ but not differentiable at 0, the function f does not belong to $\mathrm{C}^1(\mathcal{C}^+)$. If we take $\alpha_0^k = \arctan(-s_k)$ and $\alpha_1^k = \arctan(s_k)$ with $s_k = \frac{1}{\pi/2+2k\pi}$, then $\mathbf{x}_0^k$ and $\mathbf{x}_1^k$ tend to $(1,0)$. For the coefficient of y in $\mathbf{L}^{\mathrm{h}}[X^k;f]$, we have

$$\hat{f}[\tan\alpha_0^k, \tan\alpha_1^k] = \frac{\hat{f}(s_k) - \hat{f}(-s_k)}{s_k - (-s_k)} = \frac{1}{\sqrt[3]{s_k}} \to +\infty.$$

Let $X = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ be a set of not necessarily distinct points on $\mathcal{C}^+$. Then we can write $X = \{(\mathbf{n}_1,\nu_1),\dots,(\mathbf{n}_\lambda,\nu_\lambda)\}$ with $\nu_1 + \dots + \nu_\lambda = n+1$. Hence we can identify $\mathbf{H}^{\mathrm{h}}[X;f]$ with $\mathbf{H}^{\mathrm{h}}[\{(\mathbf{n}_1,\nu_1),\dots,(\mathbf{n}_\lambda,\nu_\lambda)\};f]$. Note that $\mathbf{H}^{\mathrm{h}}[X;f]$ does not depend on the ordering of points in X.

Corollary 4.1. *Let n be a positive integer, and let $X^k = \{\mathbf{x}_0^k, \ldots, \mathbf{x}_n^k\}$, $k \geq 0$, be sets of not necessarily distinct point on $\mathcal{C}^+$ such that*

$$\lim_{k\to\infty} \|X^k - X^0\| = 0,$$

where

$$\|X^k - X\| = \max\{\|\mathbf{x}_i^k - \mathbf{x}_i^0\| : i = 0, \ldots, n\}.$$

Then, for any function f of class C^n on $\mathcal{C}^+$, we have

$$\lim_{k\to\infty} \mathbf{H}^{\mathrm{h}}[X^k; f] = \mathbf{H}^{\mathrm{h}}[X^0; f].$$

Proof. We can find α_i^k in $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\alpha_i^k = \arg(\mathbf{x}_i^k)$ for $0 \leq i \leq n$ and $k \geq 0$. By hypothesis, we have

$$\lim_{k\to\infty} \max\{|\alpha_i^k - \alpha_i^0| : i = 0, \ldots, n\} = 0. \tag{4.4}$$

Since the homogeneous Hermite interpolation polynomials are independent of the ordering of the nodes, we can rearrange α_i^k's, if necessary, to get the ordering

$$-\frac{\pi}{2} < \alpha_0^k \leq \alpha_1^k \leq \cdots \leq \alpha_n^k < \frac{\pi}{2}, \quad k \geq 0,$$

and (4.4) still holds true. This enables us to group consecutive identical angles so that the orderings in the $(\alpha_i^k)_{i=0}^m$ do not change. From the proof of Theorem 4.1, we have

$$\mathbf{H}^{\mathrm{h}}\left[X^k; f\right] = \sum_{i=0}^{n} \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T^k) x^{n-i} y^i, \quad T^k = \{\tan\alpha_0^k, \ldots, \tan\alpha_n^k\}.$$

Since $\lim_{k\to\infty} \tan\alpha_i^k = \tan\alpha_i^0$ for $i = 0, \ldots, n$, it follows from Lemma 2.2 that the (l, m)-entry of the matrix corresponding to $\mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T^k)$ converges to the (l, m)-entry of the matrix corresponding to $\mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T^0)$ for every $1 \leq l, m \leq n + 1$. Therefore, we have

$$\lim_{k\to\infty} \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T^k) = \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T^0),$$

and consequently,

$$\lim_{k\to\infty} \mathbf{H}^{\mathrm{h}}\left[X^k; f\right] = \sum_{i=0}^{n} \mathrm{D}(\mathcal{M}[t^i \leftarrow \hat{f}(t)]; T^0) x^{n-i} y^i = \mathbf{H}^{\mathrm{h}}\left[X^0; f\right].$$

The proof is complete. □

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Список литературы

[1] L. Bialas-Ciez and J.-P. Calvi, “Homogeneous minimal polynomials with prescribed interpolation conditions”, Trans. Amer. Math. Soc., **368**, 8383 – 8402 (2016).
[2] T. Bloom and J-.P. Calvi, “A continiuty property of Mulivariate Lagrange interpolation”, Math. Comp., **66**, 1561 – 1577 (1997).
[3] B. Bojanov, H. Hakopian and A. Sahakian, Spline Functions and Multivariate Interpolations, Springer-Verlag (1993).
[4] L. Bos and J.-P.Calvi, “Taylorian points of an algebraic curve and bivariate Hermite interpolation”, Ann. Scuola Norm. Sup. Pisa Cl. Sci, **7**, 545 – 577 (2008).
[5] J.-P. Calvi and Phung V. M., “On the continuity of multivariate Lagrange interpolation at natural lattices”, L.M.S. J. Comp. Math., **6**, 45 – 60 (2013).
[6] J.-P. Calvi and Phung V. M., “Can we define Taylor polynomials on algebraic curves?”, Ann. Polon. Math., **118**, 1 – 24 (2016).
[7] V. M. Phung, “On bivariate Hermite interpolation and the limit of certain bivariate Lagrange projectors”, Ann. Polon. Math., **115**, 1 – 21 (2015).
[8] V. M. Phung, “Harmonic interpolation of Hermite type based on Radon projections in two directions”, J. Math. Anal. Appl., **454** , 481 – 501 (2017).
[9] S. Roman, “The formula of Faa di Bruno”, Amer. Math. Monthly, **87**, 805 – 809 (1980).

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