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ON THE ALMOST EVERYWHERE CONVERGENCE OF MULTIPLE FOURIER-HAAR SERIES

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Abstract. The paper deals with the question of convergence of multiple Fourier-Haar series with partial sums taken over homothetic copies of a given convex bounded set $W \subset \mathbb{R}_+^n$ containing the intersection of some neighborhood of the origin with $\mathbb{R}_+^n$. It is proved that for this type sets W with symmetric structure it is guaranteed almost everywhere convergence of Fourier-Haar series of any function from the class $L(\ln^+ L)^{n-1}$.

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1. Definitions and notation

We use the following notation: $\mathbb{I}$ is the unit interval $[0,1]$; $\mathbb{Z}_0$ is the set of all nonnegative integers; Δ_k^i and $\widehat{\Delta}_k^i$ ($k \in \mathbb{Z}$, $i \in \mathbb{Z}$) are dyadic intervals $(\frac{i-1}{2^k}, \frac{i}{2^k})$ and $[\frac{i-1}{2^k}, \frac{i}{2^k}]$, respectively; $\overline{p,q}$ ($p, q \in \mathbb{Z}$, $p \le q$) is the set $[p,q] \cap \mathbb{Z}$.

Recall (see [1]) that the Haar orthonormal system $h = (h_m)_{m \in \mathbb{N}}$ consists of the functions defined on $\mathbb{I}$ in the following way: $h_1(x) = 1$ ($x \in \mathbb{I}$); if $m = 2^k + i$ ($k \in \mathbb{Z}_0$, $i \in \overline{1, 2^k}$), then $h_m(x) = 2^{k/2}$ when $x \in \Delta_{k+1}^{2i-1}$, $h_m(x) = -2^{k/2}$ when $x \in \Delta_{k+1}^{2i}$, $h_m(x) = 0$ when $x \notin \widehat{\Delta}_k^i$, at the inner points of discontinuity h_m is defined as the average of the limits from the right and from the left, and at the endpoints of $\mathbb{I}$ as the limits from inside of the interval.

In what follows, if something else is not said, we will assume that the dimension n is greater than 1. Let $\theta^{(1)} = (\theta_m^{(1)})_{m \in \mathbb{N}}, \dots, \theta^{(n)} = (\theta_m^{(n)})_{m \in \mathbb{N}}$ be systems of functions on $\mathbb{I}$. Their product $\theta^{(1)} \times \cdots \times \theta^{(n)}$ is defined as the system of functions $\theta_{\mathbf{m}}(\mathbf{x}) = \theta_{m_1}^{(1)}(x_1) \dots \theta_{m_n}^{(n)}(x_n)$, where $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{I}^n$. The multiple Haar system is defined as the product $h \times \cdots \times h$.

Let $E \subset \mathbb{N}$ and $\lambda > 1$. A set E is said to be *λ-lacunar* if for every $m, m^* \in E$ with $m < m^*$ we have $m^*/m \ge \lambda$. A set $E \subset \mathbb{N}^n$ is called *λ-lacunar* if there are one-dimensional λ-lacunar sets $E_1, \dots, E_n \subset \mathbb{N}$ such that $E \subset E_1 \times \cdots \times E_n$. A

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set $E \subset \mathbb{N}^n$ is said to be *lacunar* if E is λ-lacunar for some $\lambda > 1$. A sequence $(a_{\mathbf{m}})_{\mathbf{m}\in\mathbb{N}^n}$ or a series $\sum_{\mathbf{m}\in\mathbb{N}^n} a_{\mathbf{m}}$ is said to be *lacunar* (resp. λ-*lacunar*) if the set $E = \{\mathbf{m} \in \mathbb{N}^n : a_{\mathbf{m}} \neq 0\}$ is lacunar (resp. is λ-lacunar).

By $H(\mathbf{x})$ ($\mathbf{x} \in \mathbb{I}^n$) we denote the *spectrum* of the multiple Haar system at a point $\mathbf{x} \in \mathbb{I}^n$, that is, the set $\{\mathbf{m} \in \mathbb{N}^n : h_{\mathbf{m}}(\mathbf{x}) \neq 0\}$. By $\mathbb{I}_{\mathrm{d}}$ we denote the set of all dyadic-irrational numbers of $\mathbb{I}$.

From the definition of Haar system it easily follows that: *if $x \in \mathbb{I}_{\mathrm{d}}$, then $H(x)$ is a 3/2-lacunar set*, and hence, taking into account that $H(\mathbf{x}) = H(x_1)\times\cdots\times H(x_n)$, we have that *$H(\mathbf{x})$ is 3/2-lacunar at every $\mathbf{x} \in \mathbb{I}_{\mathrm{d}}^n$*.

Let $k \in \mathbb{N}$ and $\lambda > 1$. A set $E \subset \mathbb{N}$ we call (k,λ)-*sparse* if there are disjoint λ-lacunar sets $E_1,\dots,E_k \subset \mathbb{N}$ such that $E = E_1 \cup\cdots\cup E_k$. Obviously, the notion of $(1,\lambda)$-sparse set coincides with that of λ-lacunar set. A set $E \subset \mathbb{N}^n$ we call (k,λ)-*sparse* if there are (k,λ)-*sparse* one-dimensional sets $E_1,\dots,E_n \subset \mathbb{N}$ such that $E \subset E_1\times\cdots\times E_n$. A set $E \subset \mathbb{N}^n$ we call *sparse* if it is (k,λ)-*sparse* for some $k \in \mathbb{N}$ and $\lambda > 1$.

It is easy to see that *if $x \in \mathbb{I}\backslash\mathbb{I}_{\mathrm{d}}$, then $H(x)$ is a $(2,3/2)$-sparse set*. Consequently, taking into account that $H(\mathbf{x}) = H(x_1)\times\cdots\times H(x_n)$, we have that *$H(\mathbf{x})$ is $(2,3/2)$-sparse at every $\mathbf{x} \in \mathbb{I}^n \setminus \mathbb{I}_{\mathrm{d}}^n$*. Thus, for arbitrary point $\mathbf{x} \in \mathbb{I}^n$ it is guaranteed $(2,3/2)$-sparseness of the spectrum $H(\mathbf{x})$.

A point $\mathbf{x} \in \mathbb{R}^n$ we call *dyadic-irrational* if $\mathbf{x} \in \mathbb{I}_{\mathrm{d}}^n$, that is, if each coordinate of $\mathbf{x}$ is a dyadic-irrational number.

A sequence $(a_{\mathbf{m}})_{\mathbf{m}\in\mathbb{N}^n}$ or a series $\sum_{\mathbf{m}\in\mathbb{N}^n} a_{\mathbf{m}}$ we will call *sparse* (resp. (k,λ)-*sparse*) if the set $E = \{\mathbf{m} \in \mathbb{N}^n : a_{\mathbf{m}} \neq 0\}$ is sparse (resp. is (k,λ)-sparse).

Let $W \subset \mathbb{R}_+^n$, where $\mathbb{R}_+ = [0,\infty)$. For a series $\sigma = \sum_{\mathbf{m}\in\mathbb{N}^n} a_{\mathbf{m}}$ by $S_W(\sigma)$ we denote its *partial sum by the set W*, that is, $S_W(\sigma) = \sum_{\mathbf{m}\in W} a_{\mathbf{m}}$. Note that the sum by empty set of indices we assume to be 0.

The convergence of partial sums $S_{rW}(\sigma)$ as $r \to \infty$ will be referred as W-*convergence* of the series σ. Here rW denotes the homothetic copy of the set W by a coefficient $r > 0$, that is, $rW = \{r\mathbf{x} : \mathbf{x} \in W\}$.

For the cases $W = \mathbb{I}^n$ and $W = \{\mathbf{x} \in \mathbb{R}_+^n : x_1^2+\cdots+x_n^2 \leq 1\}$, the W-convergence is called *cubical convergence* and *spherical convergence*, respectively.

A set $W \subset \mathbb{R}_+^n$ we call *standard* if it is bounded and contains an intersection of some neighborhood of the origin with $\mathbb{R}_+^n$.

A set $E \subset \mathbb{R}^n$ we call *symmetric with respect to k-th variable* if E is symmetric with respect to hyperplane $\{\mathbf{x} \in \mathbb{R}^n : x_k = 0\}$.

We will say that a standard convex set $W \subset \mathbb{R}^n_+$ is of *symmetric type* if there exists a symmetric with respect to each variable convex set $E \subset \mathbb{R}^n$ for which $W = E \cap \mathbb{R}^n_+$.

Recall that a sequence $(a_{\mathbf{m}})_{\mathbf{m}\in\mathbb{N}^n}$ is called *convergent* if $a_{\mathbf{m}}$ tends to a limit as $\min(m_1, \dots, m_n) \to \infty$, and a series $\sigma = \sum_{\mathbf{m}\in\mathbb{N}^n} a_{\mathbf{m}}$ is called *convergent in the Pringsheim sense* if the sequence of its rectangular partial sums $S_{\mathbf{m}}(\sigma) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} a_{\mathbf{i}}$ $(\mathbf{m} \in \mathbb{N}^n)$ is convergent.

By a *section of a multiple sequence* $(a_{\mathbf{m}})_{\mathbf{m}\in\mathbb{N}^n}$ we shall mean the sequence obtained from $(a_{\mathbf{m}})$ by fixing some coordinates of the index $\mathbf{m}$, and by a *section of a series* $\sigma = \sum_{\mathbf{m}\in\mathbb{N}^n} a_{\mathbf{m}}$ we shall mean a series composed by some section of the sequence $(a_{\mathbf{m}})$.

A multiple numerical series is said to *converge regularly* to a number s if it converges to s in the Pringsheim sense and if each of its sections is convergent in the Pringsheim sense (for one-dimensional sections ordinary convergence is considered). This type of convergence for double series was studied by Hardy [2] and Moricz [3]. For the briefness of formulations, for one-dimensional series the regular convergence will be identified with the ordinary convergence.

2. Results

A rectangular partial sum of a multiple Fourier-Haar series at a dyadic-irrational point $\mathbf{x} \in \mathbb{I}^n$ is represented by an integral mean over an appropriate dyadic interval containing $\mathbf{x}$. From this connection and the well-known theorems by Lebesgue, Jessen, Marcinkiewicz and Zygmund (see [4, Ch. 2]) it follows that:

1) For every function $f \in L(\mathbb{I}^n)$, the Fourier-Haar series of f cubically converges to $f(\mathbf{x})$ at almost every point $\mathbf{x} \in \mathbb{I}^n$;

2) For every function $f \in L(\ln^+ L)^{n-1}(\mathbb{I}^n)$, the Fourier-Haar series of f converges in the Pringsheim sense to $f(\mathbf{x})$ at almost every point $\mathbf{x} \in \mathbb{I}^n$.

The optimality of the class $L(\ln^+ L)^{n-1}(\mathbb{I}^n)$ in the last assertion was shown by Zerekidze in [5], where it was proved that in any integral class $\varphi(L)(\mathbb{I}^n)$, wider than $L(\ln^+ L)^{n-1}(\mathbb{I}^n)$, there exists a function f with almost everywhere divergent Fourier-Haar series in the Pringsheim sense. As it was proved by Karagulyan [6] (see also [7]) a similar result is valid for Fourier series with respect to arbitrary product-system $\theta \times \cdots \times \theta$, where θ is a complete orthonormal system on $\mathbb{I}$ consisting of bounded functions.

Kemkhadze [8] proved that for every function $f \in L(\ln^+ L)^{n-1}(\mathbb{I}^n)$, the Fourier-Haar series of f spherically converges to $f(\mathbf{x})$ at almost every point $\mathbf{x} \in \mathbb{I}^n$. The optimality of the class $L(\ln^+ L)^{n-1}(\mathbb{I}^n)$ in this result was established in [9] (for $n = 2$) and in [10] (for arbitrary $n \geq 2$).

The following theorem shows that similar to spherical partial sums, almost everywhere W-convergence of Fourier-Haar series in the class $L(\ln^+ L)^{n-1}(\mathbb{I}^n)$ is valid for quite general type sets W.

Theorem 2.1. *Let $W \subset \mathbb{R}^n_+$ be a standard convex set of symmetric type. Then for every function $f \in L(\ln^+ L)^{n-1}(\mathbb{I}^n)$ the Fourier-Haar series of f is W-convergent to $f(\mathbf{x})$ at almost every point $\mathbf{x} \in \mathbb{I}^n$.*

We obtain Theorem 2.1 from the following two results.

Theorem 2.2 ([11])**.** *For every function $f \in L(\ln^+ L)^{n-1}(\mathbb{I}^n)$ the Fourier-Haar series of f is regularly convergent to $f(\mathbf{x})$ at almost every point $\mathbf{x} \in \mathbb{I}^n$. Furthermore, if $f \in L(\ln^+ L)^k(\mathbb{I}^n)$, where $0 \leq k \leq n-2$, then each $(k+1)$-dimensional section of Fourier-Haar series of f is regularly convergent at almost every point $\mathbf{x} \in \mathbb{I}^n$.*

Theorem 2.3. *Let $W \subset \mathbb{R}^n_+$ be a standard convex set of symmetric type. Then for an arbitrary function $f \in L(\mathbb{I}^n)$ and a point $\mathbf{x} \in \mathbb{I}^n$ the following implication holds: (the Fourier-Haar series of f regularly converges to $f(\mathbf{x})$ at the point $\mathbf{x}$) $\Rightarrow$ (the Fourier-Haar series of f W-converges to $f(\mathbf{x})$ at the point $\mathbf{x}$).*

The next assertion is a corollary of Theorems 2.2 and 2.3.

Theorem 2.4. *Let $W \subset \mathbb{R}^n_+$ be a standard convex set of symmetric type. Then for an arbitrary function $f \in L(\ln^+ L)^{n-2}(\mathbb{I}^n)$ the following implication holds: (the Fourier-Haar series of f converges in the Pringsheim sense to $f(\mathbf{x})$ at every point $\mathbf{x}$ from a set E) $\Rightarrow$ (the Fourier-Haar series of f W-converges to $f(\mathbf{x})$ at almost every point $\mathbf{x}$ from E).*

Taking into account sparseness of Fourier-Haar series at every point $\mathbf{x} \in \mathbb{I}^n$, we obtain Theorem 2.3 from the following result.

Theorem 2.5. *Let $W \subset \mathbb{R}^n_+$ be a standard convex set of symmetric type. Then for an arbitrary sparse numerical series $\sigma = \sum_{\mathbf{m} \in \mathbb{N}^n} a_{\mathbf{m}}$ the following implication holds: (σ is regularly convergent to a number s) $\Rightarrow$ (σ is W-convergent to s).*

Remark 2.1. For the case of lacunar series and spherical convergence, Theorem 2.4 was proved in [11]. For two-dimensional case, more complete results were obtained in [12].

3. Proof of Theorem 2.5

Let $W \subset \mathbb{R}^n_+$ be a standard set. Denote by $t_i(W)$ $(i \in \overline{1,n})$ the supremum of the i-th coordinates of those points of W which belong to the axes Ox_i. Obviously, $t_i(W) > 0$. Let us consider two intervals $I(W)$ and $J(W)$ associated with W, defined as follows:

$$I(W) = [0, t_1(W)] \times \cdots \times [0, t_n(W)],$$
$$J(W) = \left[0, \frac{t_1(W)}{2n}\right] \times \cdots \times \left[0, \frac{t_n(W)}{2n}\right].$$

Lemma 3.1. *Let $W \subset \mathbb{R}^n_+$ be a standard convex set of symmetric type. Then $J(W) \subset W \subset I(W)$.*

Proof. We first prove the inclusion $W \subset I(W)$. Assume the opposite, that is, $W \setminus I(W) \neq \emptyset$. Let E be a convex set which is symmetric with respect to each variable and such that $E \cap \mathbb{R}^n_+ = W$. Observe that for each point $\mathbf{x} = (x_1, \dots, x_n)$ from the set $W \setminus I(W)$ there is $i \in \overline{1,n}$ for which $x_i > t_i(W)$. Without loss of generality, we can assume that

$$x_n > t_n(W). \tag{3.1}$$

Taking into account the symmetry of E, we have $(-x_1, \dots, -x_{n-1}, x_n) \in E$. The point $(0, \dots, 0, x_n)$ is a midpoint of the segment joining $\mathbf{x} = (x_1, \dots, x_{n-1}, x_n)$ and $(-x_1, \dots, -x_{n-1}, x_n)$. Therefore, by convexity of E we conclude that $(0, \dots, 0, x_n) \in E$, and consequently, we have

$$(0, \dots, 0, x_n) \in W. \tag{3.2}$$

The relations (3.1) and (3.2) contradict the definition of the number $t_n(W)$, and the obtained contradiction proves the inclusion $W \subset I(W)$.

Now, we prove the second inclusion $J(W) \subset W$. For every $i \in \overline{1,n}$, by $\mathbf{x}_i$ we denote the point lying on the axes Ox_i and having i-th coordinate equal to the number $t_i(W)/2$. From the properties of W it follows that all points $O, \mathbf{x}_1, \dots, \mathbf{x}_n$ belong to W (here O denotes the origin). Then we consider the convex hull of the points $O, \mathbf{x}_1, \dots, \mathbf{x}_n$ which we denote by $\mathrm{Conv}(O, \mathbf{x}_1, \dots, \mathbf{x}_n)$. From the convexity of W it follows that

$$\mathrm{Conv}(O, \mathbf{x}_1, \dots, \mathbf{x}_n) \subset W. \tag{3.3}$$

As it is well-known, the convex hull of points $\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_m$ has the following representation $\{\sum_{i=0}^m \lambda_i \mathbf{y}_i : \lambda_0, \dots, \lambda_m \geq 0, \sum_{i=0}^m \lambda_i = 1\}$. Consequently, we have

$$\mathrm{Conv}(O, \mathbf{x}_1, \dots, \mathbf{x}_n) = \left\{ \sum_{i=1}^n \lambda_i \mathbf{x}_i : \lambda_1, \dots, \lambda_n \geq 0, \ \sum_{i=1}^n \lambda_i \leq 1 \right\}. \tag{3.4}$$

Let $\mathbf{x}$ be an arbitrary point from the interval $J(W)$. For each $i \in \overline{1,n}$, we take the number λ_i equal to the ratio of x_i and $t_i(W)/2$. Then we have

$$\sum_{i=1}^{n} \lambda_i \leq \sum_{i=1}^{n} \left[\frac{t_i(W)}{2n} \div \frac{t_i(W)}{2}\right] = \sum_{i=1}^{n} \frac{1}{n} = 1, \quad \mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i.$$

From (3.3) and (3.4) we conclude that $\mathbf{x} \in W$. Consequently, $J(W) \subset W$. $\square$

Let $W \subset \mathbb{R}^n$, $i \in \overline{1,n}$ and $t \in \mathbb{R}$. Consider the section of W by hyperplane $\{\mathbf{x} \in \mathbb{R}^n : x_i = t\}$, that is, the set $W[i,t] = W \cap \{\mathbf{x} \in \mathbb{R}^n : x_i = t\})$. Denote by $W(i,t)$ the projection of $W[i,t]$ onto $\mathbb{R}^{n-1}$ taken by the variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$. We will refer the sets $W(i,t)$ as sections of W.

Lemma 3.2. *Let $n \geq 3$ and $W \subset \mathbb{R}^n_+$ be a standard convex set of symmetric type. Then for every $i \in \overline{1,n}$ and $t \in (0, t_i(W))$ the section $W(i,t) \subset \mathbb{R}^{n-1}_+$ is an $(n-1)$-dimensional standard convex set of symmetric type.*

Proof. Without loss of generality we assume that $i = n$. Let $(\mathbf{e}_i)_{i=1}^n$ be the standard algebraic basis in $\mathbb{R}^n$. Suppose $\mathbf{x} = t\mathbf{e}_n$, $\mathbf{x}_i = \frac{t_i(W)}{2}\mathbf{e}_i$ $(i = 1, \dots, n-1)$ and $\mathbf{x}_n = t^*\mathbf{e}_n$, where t^* is some number from the interval $(t, t_n(W))$. From the properties of W it follows that the points $\mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_n$ belong to W.

For each $i = 1, \dots, n-1$ let us consider the point $\mathbf{y}_i = \alpha\mathbf{x}_n + (1-\alpha)\mathbf{x}_i$, where $\alpha = t/t^*$. Using convexity of W we have $\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_{n-1} \in W$. Besides, the n-th coordinate of each point $\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_{n-1}$ is equal to t. From these facts it follows that the section $W(n,t)$ is a standard set in $\mathbb{R}^{n-1}$.

Observe that the set $W \cap \{\mathbf{x} \in \mathbb{R}^n : x_n = t\}$ is convex as an intersection of two convex sets. Consequently, $W(n,t)$ is a convex subset of $\mathbb{R}^{n-1}$.

Let E be the convex set that is symmetric with respect to each variable for which $E \cap \mathbb{R}^n_+ = W$. It is easy to see that $E \cap \{\mathbf{x} \in \mathbb{R}^n : x_n = t\}$ is symmetric with respect to variables $x_1, \dots, x_{n-1}$. Consequently, $E(n,t)$ is a subset of $\mathbb{R}^{n-1}$ that is symmetric with respect to each variable. Now, taking into account the equality $E(n,t) \cap \mathbb{R}^{n-1}_+ = W(n,t)$, we conclude that $W(n,t)$ is a standard convex set of symmetric type. $\square$

Lemma 3.3. *Let $W \subset \mathbb{R}^n_+$ be a standard convex set of symmetric type. Then for every $i = 1, \dots, n-1$ and $t_1, \dots, t_i > 0$, the set*

$$W \cap ([0,t_1] \times \cdots \times [0,t_i] \times \mathbb{R}^{n-i}_+)$$

is also a standard convex set of symmetric type.

Proof. Denote $V = [0,t_1] \times \cdots \times [0,t_i] \times \mathbb{R}^{n-i}_+$. It is easy to see that V is a convex set of symmetric type and the intersection of two convex sets of symmetric type is also similar one. Consequently, $W \cap V$ is a convex set of symmetric type. On the other hand, taking into account that W is standard we can conclude that $W \cap V$ is a standard set. $\square$

Let us introduce the following notation. We denote:
by $\mathbb{M}_n$ the class of all subsets of $\overline{1,n}$;
by $|M|$ the number of elements of a set M;
by $\pi(n,M)$ $(M \in \mathbb{M}_n)$ the bijection $\pi(n,M) : \overline{1,n} \to \overline{1,n}$ with the following properties:

- $\pi(n,M)$ is increasing on the set $\overline{1,|M|}$ and maps this set onto M,
- $\pi(n,M)$ is increasing on $\overline{|M|+1,n}$ and maps this set onto $\overline{1,n} \setminus M$;

by $(\mathbf{t},\mathbf{h},M)$ $(M \in \mathbb{M}_n, \mathbf{t} \in \mathbb{R}^{|M|}, \mathbf{h} \in \mathbb{R}^{n-|M|})$ the point $\mathbf{x}$ of $\mathbb{R}^n$ such that $x_{\pi(n,M)(i)} = t_i$ if $i \in \overline{1,|M|}$ and $x_{\pi(n,M)(i)} = h_{i-|M|}$ if $i \in \overline{|M|+1,n}$;
by $A \times^M B$ $(M \in \mathbb{M}_n, A \subset \mathbb{R}^{|M|}, B \subset \mathbb{R}^{n-|M|})$ the product of the sets A and B corresponding to the set M, that is, the set $\{(\mathbf{t},\mathbf{h},M) : \mathbf{t} \in A, \mathbf{h} \in B\}$ (it is clear that $A \times^M B = B \times^{\overline{1,n}\setminus M} A$);
by $\boldsymbol{\Delta}(n,M)$ $(M \in \mathbb{M}_n, 1 \le |M| < n)$ the class of "M-dimensional" intervals Δ of type $([0,p_1] \times \ldots [0,p_{|M|}]) \times^M \{(q_1,\ldots,q_{n-|M|})\}$, where $p_1,\ldots,p_{|M|},q_1,\ldots, q_{n-|M|} \in \mathbb{N}$; and by $l(\Delta)$ the largest among numbers q_i from the definition of an interval $\Delta \in \boldsymbol{\Delta}(n,M)$.

By $C(a_1,\ldots,a_m)$ will be denoted positive constants depending on parameters $a_1,\ldots,a_m$. For a standard set $W \subset \mathbb{R}^n_+$ we denote $t(W) = \min\{t_1(W),\ldots,t_n(W)\}$. Obviously, we have $t(W) > 0$.

Lemma 3.4. *Let $k \in \mathbb{N}$, $\lambda > 1$, $E \subset \mathbb{N}^n$ be a (k,λ)-sparse set, and let $W \subset \mathbb{R}^n_+$ be a standard convex set of symmetric type. Then the set $E \cap W$ may be decomposed in the following way:*

$$E \cap W = (E \cap J(W)) \cup \bigcup_{\Delta \in \boldsymbol{\Delta}} (E \cap \Delta),$$

where $\boldsymbol{\Delta} \subset \bigcup\{\boldsymbol{\Delta}(n,M) : M \in \mathbb{M}_n, 1 \le |M| < n\}$, $|\boldsymbol{\Delta}| \le C(n,k,\lambda)$, the intervals $\Delta \in \boldsymbol{\Delta}$ are disjoint and they do not intersect $J(W)$, $\Delta \subset W$ and $l(\Delta) > t(W)/2n$ for every $\Delta \in \boldsymbol{\Delta}$.

Proof. For $i \in \overline{1,n}$, $t \in \mathbb{R}$, an one-dimensional interval I and a standard convex set V, we denote

$$\Gamma_i(t) = \{\mathbf{x} \in \mathbb{R}^n : x_i = t\}, \quad \Gamma_i(I) = \{\mathbf{x} \in \mathbb{R}^n : x_i \in I\},$$
$$Q_i(V) = \left\{q \in \left(\frac{t_i(V)}{2n}, t_i(V)\right] \cap \mathbb{N} : E \cap \Gamma_i(q) \ne \emptyset\right\}.$$

Then we have the decomposition:

$$V = \left(V \cap \Gamma_i\left(\left[0, \frac{t_i(V)}{2n}\right]\right)\right) \cup \left(V \cap \Gamma_i\left(\left(\frac{t_i(V)}{2n}, t_i(V)\right]\right)\right). \tag{3.5}$$

Also, by virtue of (k, λ)-sparseness of the set E, the following inequality holds:

$$|Q_i(V)| \le k(1 + \log_\lambda(2n)). \tag{3.6}$$

Let us introduce the following sets

$$W_i = W \cap \left(\left[0, \frac{t_1(W)}{2n}\right] \times \cdots \times \left[0, \frac{t_i(W)}{2n}\right] \times \mathbb{R}_+^{n-i}\right) \quad (1 \le i \le n-1).$$

Also, we define $W_0 = W$ and $W_n = J(W)$. Obviously, we have $W = W_0 \supset W_1 \supset W_2 \supset \cdots \supset W_{n-1} \supset W_n = J(W)$. Observe that by Lemma 3.3 each W_i is a standard convex set of symmetric type.

Taking into account (3.5), it is easy to see that for the cases $V = W, k = 1;\ V = W_1, k = 2; \ldots, V = W_{n-1}, k = n$, the following decompositions hold:

$$E \cap W = (E \cap W_1) \cup \bigcup_{q \in Q_1(W)} (E \cap W \cap \Gamma_1(q)), \tag{3.7_1}$$

$$E \cap W_1 = (E \cap W_2) \cup \bigcup_{q \in Q_2(W_1)} (E \cap W_1 \cap \Gamma_2(q)), \tag{3.7_2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$E \cap W_{n-1} = (E \cap J(W)) \cup \bigcup_{q \in Q_n(W_{n-1})} (E \cap W_{n-1} \cap \Gamma_n(q)). \tag{3.7_n}$$

Consequently, we have

$$E \cap W = (E \cap J(W)) \cup \bigcup_{i=1}^{n} \bigcup_{q \in Q_i(W_{i-1})} (E \cap W_{i-1} \cap \Gamma_i(q)). \tag{3.8}$$

In each of decompositions (3.7_i) the components $W_{i-1} \cap \Gamma_i(q)$ $(q \in Q_i(W_{i-1}))$ and W_i are disjoint, and hence, we conclude that:

$$\begin{gathered}\text{The components } J(W) \text{ and } W_{i-1} \cap \Gamma_i(q) \quad (i \in \overline{1, n},\ \ q \in Q_i(W_{i-1}))\\ \text{in decomposition (3.8) are disjoint.}\end{gathered} \tag{3.9}$$

By (3.6) for every $i \in \overline{1, n}$ we have

$$|Q_i(W_{i-1})| \le k(1 + \log_\lambda(2n)). \tag{3.10}$$

It is easy to see that for every $i \in \overline{1, n}$

$$t_i(W_{i-1}) = t_i(W), \ \ \ldots, \ \ t_n(W_{i-1}) = t_n(W).$$

Consequently, for every $i \in \overline{1, n}$ and $q \in Q_i(W_{i-1})$ we have

$$q > \frac{t_i(W_{i-1})}{2n} = \frac{t_i(W)}{2n} \ge \frac{t(W)}{2n}. \tag{3.11}$$

The representation (3.8) gives a possibility to prove the lemma by induction with respect to n.

Taking into account that W_i are standard convex sets of symmetric type and using the properties (3.9)–(3.11), we easily conclude the validity of the lemma in the case $n = 2$.

Let us perform the induction step from $n - 1$ to n.

Consider the projections of the sets $E \cap \Gamma_i(q)$ and $W_{i-1} \cap \Gamma_i(q)$ ($i \in \overline{1,n}, q \in Q_i(W_{i-1})$) to the space $\mathbb{R}^{n-1}$, taken with respect to variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$. These projections are denoted by $E(i,q)$ and $W_{i-1}(i,q)$, respectively. It is easy to see that $E(i,q)$ is a (k,λ)-sparse subset of $\mathbb{N}^{n-1}$. On the other hand, by Lemma 3.2, $W_{i-1}(i,q)$ is an $(n-1)$-dimensional standard convex set of symmetric type. Using the induction hypothesis for the sets $E(i,q)$ and $W_{i-1}(i,q)$, we obtain a decomposition of the set $E(i,q) \cap W_{i-1}(i,q)$ by means of the family $\mathbf{\Delta}(i,q) \subset \bigcup\{\mathbf{\Delta}(n-1,M) : M \in \mathbb{M}_{n-1}, 1 \le |M| < n-1\}$ of lower-dimensional intervals, that is,

$$E(i,q) \cap W_{i-1}(i,q) = (E \cap J(W_{i-1}(i,q))) \cup \bigcup_{\Delta \in \mathbf{\Delta}(i,q)} (E(i,q) \cap \Delta),$$

where the family $\mathbf{\Delta}(i,q)$ has the properties stated in the lemma.

Next, for every $i \in \overline{1,n}$ and $q \in Q_i(W_{i-1})$, let us consider the family $\widetilde{\mathbf{\Delta}}(i,q)$ of the intervals $\{q\} \times^{\{i\}} \Delta$, where $\Delta \in \mathbf{\Delta}(i,q)$ or $\Delta = J(W_{i-1}(i,q))$. Then we have

$$E \cap W = (E \cap J(W)) \cup \bigcup_{i=1}^{n} \bigcup_{q \in Q_i(W_{i-1})} \bigcup_{\Delta \in \widetilde{\mathbf{\Delta}}(i,q)} (E \cap \Delta).$$

Finally, taking into account the properties (3.9)–(3.11), we easily see that the family

$$\mathbf{\Delta} = \bigcup_{i=1}^{n} \bigcup_{q \in Q_i(W_{i-1})} \widetilde{\mathbf{\Delta}}(i,q)$$

possesses all the properties of the desired decomposition of the set $E \cap W$. $\square$

Remark 3.1. If for every number $r > 0$ we use Lemma 3.4 for E and rW, then we can conclude that the intervals Δ from the decomposition of $E \cap rW$ satisfy the inequality $l(\Delta) > rt(W)/2n$. To prove this we have to take into account the following evident equality $t(rW) = rt(W)$.

For $\mathbf{x} \in \mathbb{R}^n$ denote $||\mathbf{x}|| = \sum_{i=1}^{n} |x_i|$.

The following lemma was proved in [9] (see [9], Lemma 2).

Lemma 3.5. *Let* $\sigma = \sum_{\mathbf{i} \in \mathbb{N}^n} a_{\mathbf{i}}$ *be a numerical series,* $M \in \mathbb{M}_n$, $1 \le |M| < n$, $\mathbf{p} \in \mathbb{N}^{|M|}$, $\mathbf{q} \in \mathbb{N}^{n-|M|}$ *and* $\Delta = ([0,p_1] \times \dots [0,p_{|M|}]) \times^M \{\mathbf{q}\}$. *Then*

$$S_\Delta(\sigma) = \sum_{\mathbf{d} \in \{0,1\}^{n-|M|}} (-1)^{||\mathbf{d}||} S_{(\mathbf{p},\mathbf{q}-\mathbf{d},M)}(\sigma).$$

Remark 3.2. For the general term of a series $\sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}$ the following well-known representation holds: $a_{\mathbf{i}} = \sum_{\mathbf{d}\in\{0,1\}^n}(-1)^{||\mathbf{d}||} S_{\mathbf{m}-\mathbf{d}}(\sigma)$.

For any $n \in \mathbb{N}$ assume that $\mathbf{\Delta}(n,\emptyset) = \{\{\mathbf{q}\} : \mathbf{q} \in \mathbb{N}^n\}$, and denote

$$\mathbf{\Delta}(n) = \bigcup_{M\in\mathbb{M}_n, |M|<n} \mathbf{\Delta}(n, M).$$

Also, for $\Delta = \{\mathbf{q}\} \in \mathbf{\Delta}(n,\emptyset)$ by $l(\Delta)$ we denote the maximal among the coordinates of $\mathbf{q}$.

Lemma 3.6. *Let $n \in \mathbb{N}$ and $\sigma = \sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}$ be a regularly convergent numerical series. Then*

$$\lim_{\Delta\in\mathbf{\Delta}(n),\, l(\Delta)\to\infty} S_\Delta(\sigma) = 0.$$

Proof. For the one-dimensional case the lemma is obvious. Let us perform the induction step from $n-1$ to n.

For an arbitrary given $\varepsilon > 0$ we must find a natural number N such that

$$|S_\Delta(\sigma)| < \varepsilon \tag{3.12}$$

for every $\Delta \in \mathbf{\Delta}(n)$ with $l(\Delta) \geq N$.

Taking into account convergence of σ in the Pringsheim sense, we can find a natural number N_1 such that

$$|S_{\mathbf{m}}(\sigma) - s| < \varepsilon/2^n \tag{3.13}$$

for every $\mathbf{m} \in \mathbb{N}^n$ having all coordinates not less than N_1. Here s denotes the sum of the series σ.

For every $k \in \overline{1,n}$ and $t \in \overline{1,N_1}$ let us consider the section $\sigma(k,t)$ of the series $\sigma = \sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}$ which we derive by n-tuples $\mathbf{i} = (i_1,\ldots,i_n)$ having k-th coordinate equal to t. Using induction hypothesis for each $(n-1)$-dimensional series $\sigma(k,t)$ $(k \in \overline{1,n},\ t \in \overline{1,N_1})$ we can find a natural number $N(k,t)$ such that

$$|S_\Delta(\sigma(k,t))| < \varepsilon/N_1 \tag{3.14}$$

for every $\Delta \in \mathbf{\Delta}(n-1)$ with $l(\Delta) \geq N(k,t)$.

Let N_2 be the maximal among the numbers $N(k,t)$ $(k \in \overline{1,n},\ t \in \overline{1,N_1})$. Define the number N as follows $N = N_1 + N_2$.

Now, we proceed to prove the inequality (3.12). Suppose, $\Delta \in \mathbf{\Delta}(n)$ and $l(\Delta) \geq N$. Note that: 1) for the case $\Delta \in \mathbf{\Delta}(n,M)$, $1 \leq |M| < n$, Δ has the form: $([0,p_1]\times\ldots[0,p_{|M|}]) \times^M \{(q_1,\ldots,q_{n-|M|})\}$; 2) for the case $\Delta \in \mathbf{\Delta}(n,\emptyset)$, Δ has the form: $\{(q_1,\ldots,q_n)\}$.

Case 1. Each among the numbers p_j and q_j from the definition of Δ is greater than N_1.

We use Lemma 3.5 and Remark 3.2 to estimate $|S_\Delta(\sigma)|$ by a sum of $|S_{\mathbf{m}}(\sigma) - S_{\mathbf{m}'}(\sigma)|$ type expressions, where all coordinates of $\mathbf{m}$ and $\mathbf{m}'$ are not less than N_1. Observe that the number of such expressions is not greater than 2^{n-1}. Hence, taking into account (3.13), we obtain $|S_\Delta(\sigma)| < 2^{n-1}(\varepsilon/2^n + \varepsilon/2^n) = \varepsilon$. Thus, in this case the inequality (3.12) is proved.

Case 2. At least one among the numbers p_j and q_j from the definition of Δ is not greater than N_1.

Suppose that for a k-th dimension the above mentioned inequality is fulfilled and that for a m-th dimension $l(\Delta) = q_m$. Obviously, $k \neq m$. The interval Δ will be decomposed by sections $\Delta[k,1], \dots, \Delta[k,N_1]$. Note that if a section $\Delta[k,t]$ is non-empty, then $\Delta(k,t) \in \mathbf{\Delta}(n-1)$ and $\Delta(k,t)$ is derived from Δ by omitting its k-th dimension. Consequently, taking into account that $k \neq m$, we have $l(\Delta(k,t)) = q_m = l(\Delta) \geq N_2$. From the last estimation, using (3.14) and the definition of the number N_2, for every $k \in \overline{1,n}$ and $t \in \overline{1,N_1}$ with $\Delta[k,t] \neq \emptyset$, we obtain $\left|\sum_{\mathbf{i}\in\Delta[k,t]} a_{\mathbf{i}}\right| < \varepsilon/N_1$. Consequently, we have

$$|S_\Delta(\sigma)| = \left|\sum_{\mathbf{i}\in\Delta} a_{\mathbf{i}}\right| \leq \sum_{t=1}^{N_1}\left|\sum_{\mathbf{i}\in\Delta[k,t]} a_{\mathbf{i}}\right| < N_1\frac{\varepsilon}{N_1} = \varepsilon.$$

This completes the proof of inequality (3.12). $\square$

Now, we proceed directly to the proof of Theorem 2.5.

By E denote the set $\{\mathbf{m} \in \mathbb{N}^n : a_{\mathbf{m}} \neq 0\}$. According to the condition of the theorem, the set E is (k,λ)-sparse for some $k \in \mathbb{N}$ and $\lambda > 1$.

Let $\mathbf{\Delta}_r \subset \mathbf{\Delta}(n)$ $(r > 0)$ be a family of lower-dimensional intervals constituting a decomposition of the set $E \cap rW$ according to Lemma 3.4. Then, in view of properties of $\mathbf{\Delta}_r$ (see Lemma 3.4), we have

$$S_{rW}(\sigma) = S_{rJ(W)}(\sigma) + \sum_{\Delta\in\mathbf{\Delta}_r} S_\Delta(\sigma),$$

$$|\mathbf{\Delta}_r| \leq C(n,k,\lambda), \quad l(\Delta) > rt(W)/2n.$$

From the last two estimates and Lemma 3.6 we obtain

$$\lim_{r\to\infty}\sum_{\Delta\in\mathbf{\Delta}_r} S_\Delta(\sigma) = 0.$$

On the other hand, from the convergence of σ in the Pringsheim sense it follows that $\lim\limits_{r\to\infty} S_{rJ(W)}(\sigma) = s$. Thus, $\lim\limits_{r\to\infty} S_{rW}(\sigma) = s$. Theorem 2.5 is proved. $\square$

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