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MEROMORPHIC FUNCTIONS SHARING THREE POLYNOMIALS WITH THEIR DIFFERENCE OPERATORS

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Abstract. In this paper, we focus on a conjecture concerning uniqueness problem of meromorphic functions sharing three distinct polynomials with their difference operators, which is mentioned in Chen and Yi (Result Math v. 63, pp. 557-565, 2013), and prove that it is true for meromorphic functions of finite order. Also, a result of Zhang and Liao, obtained for entire functions (Sci China Math v. 57, pp. 2143-2152, 2014), we generalize to the case of meromorphic functions.

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In Nevanlinna theory, the study of relationship between two meromorphic functions that share several values CM or IM is an important topic, resulting from the Nevanlinna's famous five and four values theorems (see [5]). In 1976, Rubel and Yang [7] showed that if a non-constant entire function f and its first derivative f' share two distinct values CM, then they are identical. This result was extended by Mues and Steinmetz [4] in 1979 from sharing values CM to IM, and by Yang [8] in 1990 from first derivative to the k-th derivatives.

The difference analogues of Nevanlinna's theory have been studied more recently and become very popular (see [2]). In 2013, under the restriction on the order of meromorphic functions, Chen and Yi [1] deduced a uniqueness theorem of meromorphic functions sharing three distinct values with their difference operator $\Delta_c f = f(z+c) - f(z)$, where c is a non-zero constant. More precisely, in [1] was proved the following theorem.

Theorem A. *Let f be a transcendental meromorphic function such that its order of growth $\rho(f)$ is finite but is not an integer, and let $c(\neq 0) \in C$. If f and $\Delta_c f(\not\equiv 0)$ share three distinct values e_1, e_2, ∞ CM, then $f(z+c) = 2f(z)$.*

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In [1], Chen and Yi conjectured that the conclusion of Theorem A still holds if the restriction imposed on $\rho(f)$ in Theorem A is omitted. In 2014, Zhang and Liao [11] considered the difference analogue of the result by Rubel and Yang and proved that the conjecture is true if f is an entire function of finite order. They obtained the following result.

Theorem B. *Let f be a transcendental entire function of finite order, and let a, b be two distinct constants. If f and $\Delta f = f(z+1) - f(z)(\not\equiv 0)$ share a, b CM, then $\Delta f = f$.*

In 2016, Lü and Lü [3] proved that the above conjecture holds if the meromorphic function is of finite order.

Theorem C. *Let f be a transcendental meromorphic function of finite order, and let $c(\neq 0)$ be a finite number. If $\Delta_c f$ and f share three distinct values e_1, e_2, ∞ CM, then $f = \Delta_c f$.*

In this paper, we continue the study of the above conjecture for meromorphic functions of finite order, and show that it remains true if the constants e_1, e_2, ∞ are replaced by the polynomials P_1, P_2, ∞ .

The next theorem is the main result of this paper.

Theorem 1. *Let f be a transcendental meromorphic function of finite order, and let $c(\neq 0)$ be a finite number. If $\Delta_c f$ and f share three distinct polynomials P_1, P_2, ∞ CM, then $f = \Delta_c f$.*

Remark. Obviously, Theorem 1 is an improvement of Theorem C.

We assume that the reader is familiar with the standard notation of Nevanlinna theory (see [9, 10]). In this paper, for two meromorphic functions f and g, we use the notation $f - g \not\equiv 0$ to denote that $f - g$ is not the zero function.

Next, we recall Nevanlinna's Lemma, which plays an important role in the proof of Theorem 1.

Nevanlinna's lemma [6]. *Let $\varphi_1, \varphi_2, \ldots, \varphi_p$ be linearly independent meromorphic functions satisfying $\varphi_1 + \varphi_2 + \cdots + \varphi_p = 1$. Then, for $j = 1, 2, \cdots, p$, we have*

$$T(r, \varphi_j) \leq \sum_{k=1}^{p} N(r, \frac{1}{\varphi_k}) - \sum_{k=1, k\neq j}^{p} N(r, \varphi_k) + N(r, W) - N(r, \frac{1}{W}) + S(r),$$

where $W = W(\varphi_1, \varphi_2 \cdots, \varphi_p)$ is the Wronskian of $\varphi_1, \cdots, \varphi_p$, and

$$S(r) = O(\log r) + O(\log max_{1\leq k\leq p} T(r, \varphi_j)) \quad as \quad r \to \infty,\, r \notin E,$$

for a set $E \subset (0, \infty)$ of finite Lebesgue measure. If all φ_k have finite order, then E can be chosen to be the empty set.

Proof of Theorem 1. Observe first that if P_1, P_2 are constants, then the theorem becomes Theorem C above. So, below we assume that one of P_1, P_2 is not constant, and, without loss of generality, we assume that $\deg P_2 \geq \deg P_1$. Our proof of the theorem is based on an idea from [3].

Since f, $\Delta_c f$ share P_1, P_2, ∞ CM and f is of finite order, then there exist two polynomials α, β such that

$$\frac{f - P_1}{\Delta_c f - P_1} = e^{\alpha}, \quad \frac{f - P_2}{\Delta_c f - P_2} = e^{\beta}. \tag{1}$$

If $e^{\alpha} = 1$ or $e^{\beta} = 1$, then $f = \Delta_c f$. If $e^{\alpha} = e^{\beta}$, then

$$\frac{f - P_1}{\Delta_c f - P_1} = \frac{f - P_2}{\Delta_c f - P_2},$$

implying that $f = \Delta_c f$.

On the contrary, suppose that $f \not\equiv \Delta_c f$. Then

$$e^{\alpha} \not\equiv 1, \quad e^{\beta} \not\equiv 1, \quad e^{\alpha} \not\equiv e^{\beta}.$$

Our aim below is to get a contradiction.

By (1), one has

$$f = P_1 + (P_2 - P_1)\frac{e^{\beta} - 1}{e^{\gamma} - 1}, \quad \Delta_c f = P_2 + (P_2 - P_1)\frac{1 - e^{-\alpha}}{e^{\gamma} - 1}, \tag{2}$$

where $\gamma = \beta - \alpha$.

It follows from (2) that

$$T(r, f) \leq T(r, e^{\beta}) + T(r, e^{\gamma}) + S(r, f). \tag{3}$$

Since $\Delta_c f = f(z + c) - f(z)$, we can write

$$\begin{aligned} \Delta_c f = {} & P_2(z) + [P_2(z) - P_1(z)]\frac{1 - e^{\gamma(z) - \beta(z)}}{e^{\gamma(z)} - 1} \\ = {} & [P_1(z + c) - P_1(z)] + [P_2(z + c) - P_1(z + c)]\frac{\beta_1(z)e^{\beta(z)} - 1}{\gamma_1(z)e^{\gamma(z)} - 1} \\ & - [P_2(z) - P_1(z)]\frac{e^{\beta(z)} - 1}{e^{\gamma(z)} - 1}, \end{aligned} \tag{4}$$

where $\beta_1(z) = e^{\beta(z+c) - \beta(z)}$ and $\gamma_1(z) = e^{\gamma(z+c) - \gamma(z)}$.

Next, we prove that $\deg \beta = \deg \gamma$ by considering two cases.

Case 1. Assume that $\deg \beta < \deg \gamma$.

Then e^{β} is a small function of e^{γ}, and hence, we have

$$\deg[\beta(z + c) - \beta(z)] \leq \deg \beta(z) < \deg \gamma(z), \ \deg[\gamma(z + c) - \gamma(z)] < \deg \gamma(z),$$

implying that β_1, γ_1 are also small functions of e^{γ}. Suppose that z_0 is a zero of $\gamma_1 e^{\gamma} - 1$, and is not a zero of $\beta_1 e^{\beta} - 1$. If z_0 is not a zero of $e^{\gamma} - 1$, then by (4) it would be a pole of $\Delta_c f$. However, the equation (2) would imply that $\Delta_c f$ is analytic at z_0, yielding a contradiction. If z_0 is a zero of $e^{\gamma}-1$, then $\gamma_1(z_0)e^{\gamma(z_0)}-1=0$ and $e^{\gamma(z_0)}-1=0$ imply $\gamma_1(z_0)-1=0$. If $\gamma_1(z)-1\not\equiv 0$, then the second main theorem gives

$$\begin{aligned}T(r,e^{\gamma}) &\le \overline{N}(r,\frac{1}{\gamma_1 e^{\gamma}-1})+\overline{N}(r,\frac{1}{e^{\gamma}})+\overline{N}(r,e^{\gamma})+S(r,e^{\gamma})\\ &\le N(r,\frac{1}{\beta_1 e^{\beta}-1})+N(r,\frac{1}{\gamma_1-1})+S(r,e^{\gamma})=S(r,e^{\gamma}),\end{aligned}$$

which is impossible. Thus, $\gamma_1(z)=e^{\gamma(z+c)-\gamma(z)}=1$, which means that $\deg\gamma=1$. Noting that by assumption $\deg\beta<\deg\gamma$, we conclude that β is a constant.

Next, by (4), we get

$$\begin{aligned}\Delta_c f &= [P_1(z+c)-P_1(z)]+[P_2(z+c)-P_1(z+c)]\frac{\beta_1 e^{\beta(z)}-1}{\gamma_1 e^{\gamma(z)}-1}\\ &-[P_2(z)-P_1(z)]\frac{e^{\beta(z)}-1}{e^{\gamma(z)}-1}\\ &=[P_1(z+c)-P_1(z)]+[P_2(z+c)-P_1(z+c)-P_2(z)+P_1(z)]\frac{e^{\beta(z)}-1}{e^{\gamma(z)}-1}.\end{aligned}$$

On the other hand, by (2) we have

$$\begin{aligned}\Delta_c f &= P_2(z)+[P_2(z)-P_1(z)]\frac{1-e^{-\alpha(z)}}{e^{\gamma(z)}-1}\\ &=P_2(z)+[P_1(z)-P_2(z)]e^{-\beta(z)}+[P_1(z)-P_2(z)]\frac{e^{-\beta(z)}-1}{e^{\gamma(z)}-1},\end{aligned}$$

where $\gamma(z)=\beta(z)-\alpha(z)$. Here, by careful calculation, it can be shown that $degP_2(z)<degP_1(z)$, which is a contradiction.

Case 2. Let $\deg\beta>\deg\gamma$.

Then e^{γ} is a small function of e^{β}, and, as in the Case 1, we can conclude that β_1,γ_1 also are small functions of e^{β}. Assume that a_0 is a zero of $e^{\beta}-1$ and is not a zero of $e^{\gamma}-1$. Then, a_0 is a zero of $f-P_1$. Note that f and $\Delta_c f$ share P_1 CM. So a_0 is also a zero of $\Delta_c f-P_1$. Putting a_0 into the last form of $\Delta_c f$ in (4), we get

$$P_1(a_0)=[P_1(z+c)-P_1(z)]+[P_2(z+c)-P_1(z+c)]\frac{\beta_1(z)-1}{\gamma_1(z)e^{\gamma(z)}-1}|_{a_0}.$$

Next, we show that

$$P_1(z)=[P_1(z+c)-P_1(z)]+[P_2(z+c)-P_1(z+c)]\frac{\beta_1(z)-1}{\gamma_1(z)e^{\gamma(z)}-1}. \tag{5}$$

Indeed, otherwise, by the second main theorem, we would have

$$T(r,e^{\beta})\leq \overline{N}(r,\frac{1}{e^{\beta}-1})+\overline{N}(r,\frac{1}{e^{\beta}})+\overline{N}(r,e^{\beta})+S(r,e^{\beta})$$
$$\leq N(r,\frac{1}{e^{\gamma}-1})+N(r,\frac{1}{[P_1(z+c)-P_1(z)]+[P_2(z+c)-P_1(z+c)]\frac{\beta_1-1}{\gamma_1e^{\gamma}-1}-P_1(z)})$$
$$+S(r,e^{\beta})=S(r,e^{\beta}),$$

which is absurd.

Now we rewrite (5) in the following form

$$[P_2(z+c)-P_1(z+c)]e^{\beta(z+c)-\beta(z)}-[P_2(z+c)-P_1(z+c)]$$
$$=[2P_1(z)-P_1(z+c)]e^{\gamma(z+c)}-[2P_1(z)-P_1(z+c)], \tag{6}$$

and show that γ is a constant. Suppose that $\deg\gamma\geq 1$. Then, combining (6) and the assumption $\deg\beta>\deg\gamma$, we get

$$\begin{aligned}&[P_2(z+c)-P_1(z+c)]e^{\beta(z+c)-\beta(z)}=[2P_1(z)-P_1(z+c)]e^{\gamma(z+c)},\\&P_2(z+c)-P_1(z+c)=2P_1(z)-P_1(z+c),\end{aligned} \tag{7}$$

implying that $\beta_1(z)=e^{\beta(z+c)-\beta(z)}=e^{\gamma(z+c)}$.

Next, rewriting (1.4) in the form

$$[P_2(z)-P_1(z+c)+P_1(z)](\gamma_1e^{\gamma}-1)(e^{\gamma}-1)e^{\beta}+[P_2(z)-P_1(z)](\gamma_1e^{\gamma}-1)(e^{\beta}-e^{\gamma})$$
$$=[P_2(z+c)-P_1(z+c)](\beta_1e^{\beta}-1)e^{\beta}(e^{\gamma}-1)-[P_2(z)-P_1(z)](e^{\beta}-1)(\gamma_1e^{\gamma}-1)e^{\beta},$$

after a routine computation, we get

$$a_0e^{2\beta}+a_1e^{\beta}+a_2=0,$$

where $a_0=[P_2(z+c)-P_1(z+c)](e^{\gamma(z)}-1)\beta_1(z)-[P_2(z)-P_1(z)](\gamma_1(z)e^{\gamma(z)}-1)$, and a_1,a_2 are small functions of e^{β}. The above equation shows that $a_0=0$, and hence, we have

$$[P_2(z+c)-P_1(z+c)](e^{\gamma(z)}-1)\beta_1(z)=[P_2(z)-P_1(z)](\gamma_1(z)e^{\gamma(z)}-1). \tag{8}$$

We put $\beta_1(z)=e^{\gamma(z+c)}$ into (7) to obtain

$$[P_2(z+c)-P_1(z+c)]e^{\gamma(z+c)+\gamma(z)}-[P_2(z+c)-P_1(z+c)-P_2(z)+P_1(z)]e^{\gamma(z+c)}$$
$$+[P_2(z)-P_1(z)]=0,$$

implying that γ is a constant, say $\gamma=A$. Thus, we have proved that γ is a constant. In addition, the form of f shows that f is an entire function. Then, by (4), we can get

$$\begin{aligned}\Delta_c f =&[P_1(z+c)-P_1(z)]+[P_2(z+c)-P_1(z+c)]\frac{\beta_1(z)e^{\beta(z)}-1}{\gamma_1(z)e^{\gamma(z)}-1}\\&-[P_2(z)-P_1(z)]\frac{e^{\beta(z)}-1}{e^{\gamma(z)}-1}\\&=\frac{e^{\beta(z)}}{e^A-1}\{[P_2(z+c)-P_1(z+c)]\beta_1(z)-[P_2(z)-P_1(z)]\}\\&+\frac{1}{e^A-1}[P_2(z)-P_1(z)-P_2(z+c)+P_1(z+c)]+[P_1(z+c)-P_1(z)].\end{aligned}$$

Note that by (2)

$$\Delta_c f = P_2(z)+[P_2(z)-P_1(z)]\frac{1-e^{-\alpha(z)}}{e^{\gamma(z)}-1} = P_2(z)+[P_2(z)-P_1(z)]\frac{1-e^Ae^{-\beta(z)}}{e^A-1}.$$

Combining the above equations, we get

$$h_0e^{2\beta}+h_1e^{\beta}+h_2=0,$$

where h_i $(i=0,1,2)$ are small functions of e^{β} and $h_2=[P_2(z)-P_1(z)]\frac{-e^A}{e^A-1}$. Obviously, $h_2=0$, which shows that $P_1(z)=P_2(z)$, and we get a contradiction.

Thus, we have proved that $\deg\beta=\deg\gamma$. We can assume that

$$\deg\beta=\deg\gamma:=n\geq 1,$$

since f is a transcendental function.

Note that $\beta_1(z)=e^{\beta(z+c)-\beta(z)}$ and $\gamma_1(z)=e^{\gamma(z+c)-\gamma(z)}$ are two small functions of e^{β} and e^{γ}. Multiplying both sides of equation (4) by the factor $e^{\beta}(e^{\gamma}-1)(\gamma_1e^{\gamma}-1)$, we get

(9)

$$\begin{aligned}&[P_2(z)-P_1(z+c)+P_1(z)](\gamma_1e^{\gamma}-1)(e^{\gamma}-1)e^{\beta}+[P_2(z)-P_1(z)](\gamma_1e^{\gamma}-1)(e^{\beta}-e^{\gamma})\\&=[P_2(z+c)-P_1(z+c)](\beta_1e^{\beta}-1)e^{\beta}(e^{\gamma}-1)-[P_2(z)-P_1(z)](e^{\beta}-1)(\gamma_1e^{\gamma}-1)e^{\beta}.\end{aligned}$$

From (9) we obtain

$$b_0e^{2\gamma}+b_1e^{\beta+2\gamma}+b_2e^{\beta+\gamma}+b_3e^{2\beta}+b_4e^{2\beta+\gamma}+b_5e^{\beta}+b_6e^{\gamma}=0,$$

where

$$\begin{cases}b_0=[P_1(z)-P_2(z)]\gamma_1(z),\\ b_1=[P_2(z)+P_1(z)-P_1(z+c)]\gamma_1(z),\\ b_2=[P_1(z+c)-P_1(z)-P_2(z)]\gamma_1(z)+P_2(z+c)-P_2(z)-P_1(z),\\ b_3=[P_2(z+c)-P_1(z+c)]\beta_1(z)-P_2(z)+P_1(z),\\ b_4=[P_1(z+c)-P_2(z+c)]\beta_1(z)+[P_2(z)-P_1(z)]\gamma_1(z),\\ b_5=P_1(z)+P_2(z)-P_2(z+c),\\ b_6=P_2(z)-P_1(z).\end{cases}$$

Obviously, b_i $(i=0,1,\cdots,6)$ are small functions of e^{β} and e^{γ}. The equation (9) can be written as follows:

$$\sum_{i=0}^{6} b_i e^{g_i}=0, \tag{10}$$

where

$$\begin{cases} g_0=2\gamma, \qquad g_1=\beta+2\gamma, \qquad g_2=\beta+\gamma,\\ g_3=2\beta,\ g_4=2\beta+\gamma,\ \ g_5=\beta,\ \ g_6=\gamma. \end{cases}$$

We claim that $\deg(\gamma-\beta)=n$. On the contrary, suppose that $\deg(\gamma-\beta)<n$. Then $e^{\gamma-\beta}$ is a small function of e^{β} and e^{γ}. We denote by $N_E(r)$ the counting function of the common zeros of $e^{\beta}-1$ and $e^{\gamma}-1$. Assume that c_0 is a common zero of $e^{\beta}-1$ and $e^{\gamma}-1$. Then c_0 is a zero of $e^{\gamma-\beta}-1$. Notice $e^{\beta}\neq e^{\gamma}$, then $e^{\gamma-\beta}-1\neq 0$. Therefore

$$N_E(r)\leq N(r,\frac{1}{e^{\gamma-\beta}-1})=S(r,e^{\gamma}).$$

Since e^{γ} is of finite order, we have $S(r+|c|,e^{\gamma})=S(r,e^{\gamma})$. Assume that d_0 is a zero of $\gamma_1e^{\gamma}-1$, and is not a zero of $\beta_1e^{\beta}-1$. Similarly as above, we can conclude that d_0 is also a zero of $e^{\gamma}-1$. Furthermore, d_0 is a zero of γ_1-1. If $\gamma_1-1\neq 0$, then, it follows from the second main theorem that

$$\begin{aligned} T(r,e^{\gamma}) &\leq \overline{N}(r,\frac{1}{\gamma_1e^{\gamma}-1})+\overline{N}(r,\frac{1}{e^{\gamma}})+\overline{N}(r,e^{\gamma})+S(r,e^{\gamma})\\ &\leq N_E(r+|c|)+N(r,\frac{1}{\gamma_1-1})+S(r,e^{\gamma})\\ &\leq T(r,\frac{1}{\gamma_1-1})+S(r+|c|,e^{\gamma})+S(r,e^{\gamma})=S(r,e^{\gamma}), \end{aligned}$$

which is a contradiction. Thus, $\gamma_1(z)=e^{\gamma(z+c)-\gamma(z)}=1$, which implies that $e^{\gamma(z+c)}=e^{\gamma(z)}$ and $\deg\gamma=1$. As a consequence, noting that $\deg(\beta-\gamma)<1$, we see that $\beta-\gamma$ is a constant, say A_1. Recall $e^{\gamma(z+c)}=e^{\gamma(z)}$. One has $e^{\beta(z+c)-\beta(z)}=e^{\beta(z+c)-\gamma(z+c)-(\beta(z)-\gamma(z))}=e^{A_1-A_1}=1$. So $e^{\beta(z+c)}=e^{\beta(z)}$. By (4), we can get

$$\Delta_c f=[P_1(z+c)-P_1(z)]+[P_2(z+c)-P_1(z+c)-P_2(z)+P_1(z)]\frac{e^{\beta(z)}-1}{e^{\gamma(z)}-1},$$

where $\deg\beta\geq 1$. But in view of (2), we have $\deg(-\alpha)=\deg(\gamma-\beta)<1$, yielding a contradiction. Thus, we have shown that $\deg(\gamma-\beta)=n$.

Furthermore, one has $\deg(g_2-g_j)=n$, for $j=0,1,3,4,5,6$, because

$$\begin{cases} g_2-g_0=\beta-\gamma,\ g_2-g_1=-\gamma,\ g_2-g_3=\gamma-\beta,\\ g_2-g_4=-\beta, \qquad\quad g_2-g_5=\gamma, \qquad\quad g_2-g_6=\beta. \end{cases}$$

We assume that $b_2=[P_1(z+c)-P_1(z)-P_2(z)]\gamma_1(z)+P_2(z+c)-P_2(z)-P_1(z)\not\equiv 0$.

Then, we consider $\psi_j=b_je^{g_j}(j=0,\cdots,6)$. From (10) we deduce that there exist a set $I\subset\{0,1,3,4,5,6\}$ and complex numbers $\lambda_j\neq 0 (j\in I)$ such that

$\psi_2 = \sum_{j\in I}\lambda_j\psi_j$, and ψ_j $(j \in I)$ are linearly independent. Rewriting this in the form:

$$\sum_{j\in I}\lambda_j\frac{b_j}{b_2}e^{g_j-g_2}=1,$$

we can apply Nevanlinna's lemma to the functions

$$\varphi_j=\lambda_j\frac{b_j}{b_2}e^{g_j-g_2}, j\in I,$$

which are linearly independent and satisfy $\sum_{j\in I}\varphi_j=1$.

We use the fact that the zeros and poles of φ_j and their Wronskians can come only from the zeros and poles of functions b_j whose Nevanlinna characteristic is

$$T(r,b_j)=O(r^{n-1})=S(r,\varphi_j),$$

since $\deg(g_2-g_j)=n$ for $j\in I$. So, by Nevanlinna's lemma we obtain that

$$T(r,\varphi_j)\le S(r),$$

for all $j\in I$ with $S(r)$ as above. This is a contradiction. Thus, $b_2\equiv 0$. Now, we consider the case

$$b_2=[P_1(z+c)-P_1(z)-P_2(z)]\gamma_1(z)+P_2(z+c)-P_1(z)-P_2(z)=0.$$

If $P_1(z+c)-P_1(z)-P_2(z)=0$, then $P_2(z+c)-P_2(z)-P_1(z)=0$. We can obtain $P_1(z+c)=P_2(z+c)$, which is a contradiction. Thus, $P_1(z+c)-P_1(z)-P_2(z)\not\equiv 0$, and we can get

$$\gamma_1(z)=\frac{P_2(z+c)-P_2(z)-P_1(z)}{P_1(z+c)-P_1(z)-P_2(z)}. \tag{11}$$

Note that γ_1 is not a constant function. This contradicts the fact that $\gamma_1(z)$ is an entire function. Thus, γ_1 is a constant, which implies that $\deg\gamma=n=1$. So, we have $\deg\beta=n=1$ and γ_1 is a constant. Suppose that

$$\deg(\gamma+\beta)=n=1,\quad \deg(\gamma-2\beta)=n=1.$$

Then, one has $\deg(g_6-g_j)=n$, for $j=0,1,3,4,5$, because

$$g_6-g_0=-\gamma, g_6-g_1=-\gamma-\beta, g_6-g_2=\gamma-2\beta, g_6-g_4=-2\beta, g_6-g_5=\gamma-\beta.$$

Again applying Nevanlinna's Lemma and replacing g_2 by g_6 in the above discussion, we get a contradiction.

Now, we assume that either $\gamma+\beta$ or $\gamma-2\beta$ is constant. If $\gamma+\beta$ is constant, then for the functions g_j with some constants c_j, we have

$$g_0=-2\beta+c_0,\ g_1=-\beta+c_1,\ g_3=2\beta+c_3,\ g_4=\beta+c_4,\ g_5=\beta+c_5,$$

and b_j are polynomials (since β_1 and γ_1 are constants). So, the identity (10) gives

$$b_0^*e^{-2\beta}+b_1^*e^{-\beta}+b_3^*e^{2\beta}+b_4^*e^{\beta}=0,$$

with certain polynomials b_j^*. This identity obviously implies that all b_j^* are 0, where $b_3^* = \{[P_2(z+c) - P_1(z+c)]\beta_1 + P_1(z) - P_2(z)\}e^{c_3}$.

If $b_3^* \equiv 0$, we can get $[P_2(z+c) - P_1(z+c)]\beta_1(z) + P_1(z) - P_2(z) = 0$. Say $P_3(z) = P_2(z) - P_1(z)$, so $[P_3(z+c) - P_3(z)]\beta_1(z) = P_3(z)(1 - \beta_1(z))$. We show that $P_3(z)$ is a constant. Indeed, assume the opposite that $degP_3(z) \geq 1$. Then, we can get $\beta_1(z) = 1$ and $P_3(z+c) - P_3(z) = 0$, implying that $P_3(z)$ is a constant, which is a contradiction. Thus, $P_3(z)$ is a constant, say $c(\neq 0)$. This implies that $P_1(z) = P_2(z) + c$. By (11) we can get that $\gamma_1(z) = 1 + \frac{c}{P_2(z+c) - 2P_2(z)}$, showing that $\gamma_1(z)$ has a pole. Taking into account that $\gamma_1(z)$ is an entire function, we get a contradiction. Thus, we have $b_3^* \not\equiv 0$. This rules out the case where $\gamma + \beta$ is constant. The case where $\gamma - 2\beta$ is constant can be treated in the same way. This completes the proof of the theorem. □

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