Известия НАН Армении, Математика, том 54, н. 5, 2019, стр. 27 – 43

ULAM STABILITIES FOR NONLINEAR VOLTERRA DELAY INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. The present paper is devoted to the study of existence and uniqueness of a solution and Ulam type stabilities for Volterra delay integro-differential equations on a finite interval. Our analysis is based on the Pachpatte's inequality and Picard operator theory. Examples are provided to illustrate the stability results obtained in the case of a finite interval. Also, we give an example to illustrate that the Volterra delay integro-differential equations are not Ulam–Hyers stable on the infinite interval.

MSC2010 numbers: 45N05, 45M10, 34G20, 35A23.

Keywords: delay integro-differential equation; Ulam–Hyers stability; Ulam–Hyers–Rassias stability; integral inequality; Picard operator.

1. Introduction

The basic Ulam stability problem of functional equations, formulated by Ulam in 1940 (see [23]), has been studied and generalized by many researchers to various kinds of differential equations, integral equations, difference equations and fractional differential equations. The basic idea behind Ulam stability of any kind of equation is to deal with the existence of an exact solution near to every approximate solution. The concept of Ulam stability is applicable in various branches of mathematical analysis and is used in the cases where finding the exact solution is very difficult.

In recent years, many researchers have involved in the study on Ulam type stabilities of differential and integro-differential equations and obtained a number of remarkable results. At start, using the fixed point approach, implemented by Cadariu and Radu [1], S. M. Jung [9] has proved the Hyers–Ulam–Rassias stability of the Volterra integral equation $x(t) = \int_c^t f(s, x(s))ds$, where f is a continuous function and c is a fixed real number. Applying the fixed point arguments used in [9], Castro and Ramos [3] obtained Hyers–Ulam–Rassias stability and Hyers–Ulam stability for the following more general nonlinear Volterra integral equation:

$$x(t) = \int_a^t f(t, s, x(s))ds, -\infty < a \le t \le b < +\infty$$

both in finite and infinite intervals.

The concept of fixed point approach to study Ulam–Hyers stability has been extended by many authors. Here we mention few interesting contributions on Ulam type stabilities of different kinds of differential and integral equations. Tunc and Bicer [22] obtained results on the Hyers–Ulam–Rassias and Hyers–Ulam stability for the first order delay differential equation. Castro and Guerra [2] obtained weak conditions guaranteeing the Hyers–Ulam– Rassias stability of nonlinear Volterra integral equations with delay. Otrocol and Ilea [17] investigated Ulam stability for a delay differential equation. Using the idea of Cadariu, Radu and Jung, the Ulam–Hyers stability results for Volterra integral integro-differential equations was proved in [8] and [21]. Gachpazan and Baghani [5, 6] and Morales and Rojas [13] applied the successive approximation method to prove the Hyers–Ulam stability of a nonlinear integral equation. Using the method of successive approximation Huang and Li [7] established Ulam–Hyers stability of delay differential equations.
Recently, employing Pachpatte's inequality, Kucche and Shikhare [10] have discussed Ulam–Hyers stabilities of semilinear Volterra integro-differential equations in Banach spaces.

Motivated by the work of Rus [20] and Otrocol et al.[16, 17], in the present paper we obtain existence and uniqueness results and establish Ulam type stabilities (viz. Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability) for nonlinear Volterra delay integro-differential equation (VDIE) of the form:

$$x'(t) = f\left(t, x(t), x(g(t)), \int_0^t h(t,s,x(s),x(g(s)))ds\right),\ t \in I = [0,b],\ b > 0, \tag{1.1}$$

where $f \in C\left(I \times \mathbb{R}^3, \mathbb{R}\right)$, $h \in C\left(I \times I \times \mathbb{R}^2, \mathbb{R}\right)$, $g \in C\left(I, [-r,b]\right)$, $0 < r < \infty$ and $g(t) \le t$.

We apply Picard's operator theory, the abstract Gronwall lemma and the Pachpatte's inequality to achieve our results. The results obtained in this paper are more general than the known results and include the study of [3, 9, 16, 17, 20] – [22] as special cases of (1.1). For existence, uniqueness and other qualitative properties of various forms of nonlinear delay integro-differential equations we refer the papers by Ntouyas et al. [14, 15], Dauer and Balchandran [4], Kucche et al. [11, 12] and the references cited therein.

The rest of this paper is organized as follows. In Section 2, we define the Ulam type stability concepts for equation (1.1) and state theorems, needed to obtain our main results. In Section 3, we establish different Ulam type stability results for

VDIE (1.1) on a finite interval. Further, we give some applications of the obtained results, and discuss examples illustrating the results.

2. Preliminaries

In what follows we use the notation and definitions given in [20] to discuss the Ulam type stabilities of VDIE (1.1). Consider the following nonlinear Volterra delay integro-differential equations:

$$x'(t) = f\left(t, x(t), x(g(t)), \int_0^t h(t,s,x(s),x(g(s)))ds\right), \ t \in I, \tag{2.1}$$

$$x(t) = \phi(t), \ t \in [-r, 0], \tag{2.2}$$

where $\phi \in C([-r,0],\mathbb{R})$.

Definition 2.1. *A function $x \in C([-r,b],\mathbb{R}) \cap C^{'}([0,b],\mathbb{R})$ that verifies the equations (2.1) and (2.2) is called a solution of the initial value problem (2.1), (2.2).*

For a given $\epsilon > 0$ and a positive nondecreasing continuous function $\psi \in C([-r,b],\mathbb{R}_+)$, we consider the following inequalities:

$$\left| y'(t) - f\left(t, y(t), y(g(t)), \int_0^t h(t,s,y(s),y(g(s)))ds\right)\right| \le \epsilon, \ t \in I, \tag{2.3}$$

$$\left| y'(t) - f\left(t, y(t), y(g(t)), \int_0^t h(t,s,y(s),y(g(s)))ds\right)\right| \le \psi(t), \ t \in I, \tag{2.4}$$

$$\left| y'(t) - f\left(t, y(t), y(g(t)), \int_0^t h(t,s,y(s),y(g(s)))ds\right)\right| \le \epsilon\psi(t), \ t \in I. \tag{2.5}$$

Definition 2.2. *The equation (2.1) is said to be Ulam–Hyers stable if there exists a real number $C > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C^{'}([-r,b],\mathbb{R})$ of (2.3) there exists a solution $x \in C^{'}([-r,b],\mathbb{R})$ of (2.1) with $|y(t) - x(t)| \le C\,\epsilon$ for $t \in [-r,b]$.*

Definition 2.3. *The equation (2.1) is said to be generalized Ulam–Hyers stable if there exists $\theta_f \in C(\mathbb{R}_+,\mathbb{R}_+)$, $\theta_f(0) = 0$ such that for each solution $y \in C^{'}([-r,b],\mathbb{R})$ of (2.3) there exists a solution $x \in C^{'}([-r,b],\mathbb{R})$ of (2.1) with $|y(t) - x(t)| \le \theta_f(\epsilon)$ for $t \in [-r,b]$.*

Definition 2.4. *The equation (2.1) is said to be Ulam–Hyers–Rassias stable with respect to the positive nondecreasing continuous function $\psi : [-r,b] \to \mathbb{R}_+$ if there exists $C_\psi > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C^{'}([-r,b],\mathbb{R})$ of (2.5) there exists a solution $x \in C^{'}([-r,b],\mathbb{R})$ of (2.1) with $|y(t) - x(t)| \le C_\psi\epsilon\,\psi(t)$ for $t \in [-r,b]$.*

Definition 2.5. *The equation* (2.1) *is said to be generalized Ulam–Hyers–Rassias stable with respect to the positive nondecreasing continuous function* $\psi : [-r, b] \to \mathbb{R}_+$ *if there exists* $C_\psi > 0$ *such that for each solution* $y \in C^{'}([-r, b], \mathbb{R})$ *of* (2.4) *there exists a solution* $x \in C^{'}([-r, b], \mathbb{R})$ *of* (2.1) *with* $|y(t) - x(t)| \leq C_\psi\, \psi(t)$ *for* $t \in [-r, b]$.

Remark 2.1. Observe that a function $y \in C^{'}(I, \mathbb{R})$ is a solution of the inequality (2.3) if there exists a function $q_y \in C(I, \mathbb{R})$ (which depends on y) such that

(i) $|q_y(t)| \leq \epsilon,\ t \in I$;
(ii) $y^{'}(t) = f\left(t, y(t), y(g(t)), \int_0^t h(t, s, y(s), y(g(s)))ds\right) + q_y(t),\ t \in I.$

Similar arguments hold for the inequalities (2.4) and (2.5).

Remark 2.2. If $y \in C^{'}(I, \mathbb{R})$ satisfies the inequality (2.3), then y is a solution of the following integral inequality:

$$\left| y(t) - y(0) - \int_0^t f\left(s, y(s), y(g(s)), \int_0^s h(s, \tau, y(\tau), y(g(\tau)))d\tau\right) ds \right| \leq \epsilon t,\ t \in I. \tag{2.6}$$

Indeed, if $y \in C^{'}(I, \mathbb{R})$ satisfies the inequality (2.3), then by Remark 2.1, we have

$$y^{'}(t) = f\left(t, y(t), y(g(t)), \int_0^t h(t, s, y(s), y(g(s)))ds\right) + q_y(t),\ t \in I.$$

This gives

$$\begin{aligned}\left| y(t) - y(0) - \int_0^t f\left(s, y(s), y(g(s)), \int_0^s h(s, \tau, y(\tau), y(g(\tau)))d\tau\right) ds \right| &\leq \int_0^t |q_y(s)|ds \\ &\leq \epsilon t,\ t \in I.\end{aligned}$$

Similar estimates can also be obtained for the inequalities (2.4) and (2.5).

We use the following inequality to obtain our main results.

Theorem 2.1 (Pachpatte's inequality (see [18], p. 39)). *Let* $u(t)$, $f(t)$ *and* $q(t)$ *be nonnegative continuous functions defined on* $\mathbb{R}_+$, *and let* $n(t)$ *be a positive and nondecreasing continuous function defined on* $\mathbb{R}_+$ *for which the inequality*

$$u(t) \leq n(t) + \int_0^t f(s)\left[u(s) + \int_0^s q(\tau)u(\tau)d\tau\right] ds,$$

holds for $t \in \mathbb{R}_+$. *Then*

$$u(t) \leq n(t)\left[1 + \int_0^t f(s) \exp\left(\int_0^s [f(\tau) + q(\tau)]d\tau\right) ds\right],$$

for $t \in \mathbb{R}_+$.

Now we give the definition of the Picard operator and state the abstract Gronwall lemma (see Rus [19]), which are used in our subsequent analysis.

Definition 2.6 (Picard operator [19]). *Let (X, d) be a metric space. An operator $A : X \to X$ is said to be a Picard operator if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$, *where* $F_A = \{x \in X : A(x) = x\}$ *is the fixed point set of* A;

(ii) *the sequence* $(A^n(x_0))_{n\in\mathbb{N}}$ *converges to* x^* *for all* $x_0 \in X$.

Lemma 2.1 (Gronwall lemma [19]). *Let $(X, d, \leq)$ be an ordered metric space and let $A : X \to X$ be an increasing Picard operator ($F_A = x_A^*$). Then for $x \in X$, $x \leq A(x)$ implies $x \leq x_A^*$, while $x \geq A(x)$ implies $x \geq x_A^*$.*

3. Ulam type stabilities for VDIE on $I = [0, b]$

3.1. **The main results.** The following assumptions are needed to state and prove our main results.

(H1) (i) Let $f \in C\left([0, b] \times \mathbb{R}^3, \mathbb{R}\right)$, $h \in C\left([0, b] \times [0, b] \times \mathbb{R}^2, \mathbb{R}\right)$ and $g \in C\left([0, b], [-r, b]\right)$ be such that $g(t) \leq t$.

(ii) There exist constants L_f, $L_h > 0$ such that

$$|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \leq L_f\left(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\right);$$
$$|h(t, s, u_1, u_2) - h(t, s, v_1, v_2)| \leq L_h\left(|u_1 - v_1| + |u_2 - v_2|\right)$$

for all $t, s \in I$, $u_i, v_i \in \mathbb{R}$ $(i = 1, 2, 3)$.

(H2) The function $\psi : [-r, b] \to \mathbb{R}_+$ is positive, nondecreasing and continuous and there exists $\lambda > 0$ such that

$$\int_0^t \psi(s)ds \leq \lambda\psi(t),\ t \in [0, b].$$

Theorem 3.1. *Let the functions f and h in (2.1) satisfy (H1) and assume that (H2) holds. If $bL_f\left[2 + L_h b\right] < 1$, then the following assertions hold:*

(i) *the initial value problem* (2.1), (2.2) *has a unique solution* $x \in C\left([-r, b], \mathbb{R}\right) \cap C'\left([0, b], \mathbb{R}\right)$;

(ii) *the equation* (2.1) *is Ulam–Hyers–Rassias stable with respect to the function* ψ.

Proof. (i) Observe first that in view of assumption (H1)(i), the initial value problem (2.1), (2.2) is equivalent to the following integral equations:

$$x(t) = \phi(0) + \int_0^t f\left(s, x(s), x(g(s)), \int_0^s h(s, \tau, x(\tau), x(g(\tau)))d\tau\right) ds,\ t \in I,$$
$$x(t) = \phi(t),\ t \in [-r, 0].$$

Consider the Banach space $X = C([-r,b],\mathbb{R})$ with Chebyshev norm $\|\cdot\|_C$, and define the operator $B_f : X \to X$ by

$$B_f(x)(t) = \phi(0) + \int_0^t f\left(s, x(s), x(g(s)), \int_0^s h(s,\tau,x(\tau),x(g(\tau)))d\tau\right) ds,\ t \in I,$$
$$B_f(x)(t) = \phi(t),\ t \in [-r,0].$$

Now using the contraction principle we show that B_f has a fixed point. Note that

$$|B_f(x)(t) - B_f(y)(t)| = 0,\quad x,y \in C([-r,b],\mathbb{R}),\quad t \in [-r,0]. \tag{3.1}$$

Next, for any $t \in I$, we can write

$$\begin{aligned}
&|B_f(x)(t) - B_f(y)(t)| \\
&\le \int_0^t L_f \{|x(s)-y(s)| + |x(g(s)) - y(g(s))| \\
&\qquad + \int_0^s L_h [|x(\tau)-y(\tau)| + |x(g(\tau)) - y(g(\tau))|] d\tau\Big\} ds \\
&\le \int_0^t L_f \Big\{ \max_{0\le\sigma_1\le s} |x(\sigma_1) - y(\sigma_1)| + \max_{0\le\sigma_1\le s} |x(g(\sigma_1)) - y(g(\sigma_1))| \\
&\qquad + \int_0^s L_h \Big[\max_{0\le\sigma_2\le\tau} |x(\sigma_2)-y(\sigma_2)| + \max_{0\le\sigma_2\le\tau} |x(g(\sigma_2)) - y(g(\sigma_2))| \Big] d\tau \Big\} ds \\
&\le \int_0^t L_f \Big\{ \max_{-r\le\sigma_1\le b} |x(\sigma_1)-y(\sigma_1)| + \max_{-r\le\tau_1\le b} |x(\tau_1)-y(\tau_1)| \\
&\qquad + \int_0^s L_h \Big[\max_{-r\le\sigma_2\le b} |x(\sigma_2)-y(\sigma_2)| + \max_{-r\le\tau_2\le b} |x(\tau_2)-y(\tau_2)| \Big] d\tau \Big\} ds \\
&\le \int_0^t L_f \Big\{ 2\|x-y\|_C + 2\int_0^s L_h \|x-y\|_C\, d\tau \Big\} ds \\
&\le bL_f(2 + L_h b)\|x-y\|_C.
\end{aligned} \tag{3.2}$$

From (3.1) and (3.2), it follows that

$$\|B_f(x) - B_f(y)\|_C \le bL_f(2+L_h b)\|x-y\|_C,\ x,y \in C([-r,b],\mathbb{R}).$$

Since $bL_f(2+L_h b) < 1$, the operator B_f is a contraction on the complete space X. Hence by Banach contraction principle the operator B_f has a fixed point $x^* : [-r,b] \to \mathbb{R}$, which is a solution of the problem (2.1), (2.2).

(ii) Let $y \in C([-r,b],\mathbb{R}) \cap C'([0,b],\mathbb{R})$ be a solution of the inequality (2.5). Denote by $x \in C([-r,b],\mathbb{R}) \cap C'([0,b],\mathbb{R})$ the unique solution of the problem:

$$x'(t) = f\left(t, x(t), x(g(t)), \int_0^t h(t,s,x(s),x(g(s)))ds\right),\ t \in I,$$
$$x(t) = y(t),\ t \in [-r,0].$$

Then assumption (H1)(i) allows to write the following (equivalent to the above problem) integral equation:

$$x(t) = y(0) + \int_0^t f\left(s, x(s), x(g(s)), \int_0^s h(s,\tau,x(\tau),x(g(\tau)))d\tau\right) ds,\ t \in I, \tag{3.3}$$

$$x(t) = y(t),\ t \in [-r,0]. \tag{3.4}$$

If $y \in C([-r,b],\mathbb{R}) \cap C'([0,b],\mathbb{R})$ satisfies the inequality (2.5), then using assumption (H2) and Remarks 2.1 and 2.2, we obtain

$$\begin{aligned} &\left| y(t) - y(0) - \int_0^t f\left(s, y(s), y(g(s)), \int_0^s h(s,\tau,y(\tau),y(g(\tau)))d\tau\right) ds \right| \\ &\quad \le \int_0^t |q_y(s)|\, ds \le \int_0^t \epsilon\psi(s) ds \le \lambda\epsilon\psi(t),\ t \in I. \end{aligned} \tag{3.5}$$

Note that $|y(t) - x(t)| = 0$ for $t \in [-r,0]$. Next, using assumption (H1)(ii), the equation (3.3) and the estimate in (3.5), for any $t \in I$, we can write

$$\begin{aligned} |y(t) - x(t)| &= \left| y(t) - y(0) - \int_0^t f\left(s, x(s), x(g(s)), \int_0^s h(s,\tau,x(\tau),x(g(\tau)))d\tau\right) ds \right| \\ &\le \left| y(t) - y(0) - \int_0^t f\left(s, y(s), y(g(s)), \int_0^s h(s,\tau,y(\tau),y(g(\tau)))d\tau\right) ds \right| \\ &\quad + \int_0^t \left| f\left(s, y(s), y(g(s)), \int_0^s h(s,\tau,y(\tau),y(g(\tau)))d\tau\right) \right. \\ &\quad\quad \left. - f\left(s, x(s), x(g(s)), \int_0^s h(s,\tau,x(\tau),x(g(\tau)))d\tau\right) \right| ds \\ &\le \epsilon\lambda\psi(t) + \int_0^t L_f \Big\{ |y(s) - x(s)| + |y(g(s)) - x(g(s))| \\ &\quad\quad + \int_0^s L_h \left[|y(\tau) - x(\tau)| + |y(g(\tau)) - x(g(\tau))| \right] d\tau \Big\} ds. \end{aligned} \tag{3.6}$$

According to (3.6), we consider operator $A : C([-r,b],\mathbb{R}_+) \to C([-r,b],\mathbb{R}_+)$ defined by

$$A(u)(t) = 0,\quad t \in [-r,0],$$
$$A(u)(t) = \epsilon\lambda\psi(t) + L_f \int_0^t \left\{ u(s) + u(g(s)) + L_h \int_0^s [u(\tau) + u(g(\tau))]\, d\tau \right\} ds,\quad t \in [0,b].$$

Next, we prove that A is a Picard operator (see Definition 2.6). To this end, observe first that for any $u, v \in C([-r,b],\mathbb{R}_+)$ we have $|A(u)(t) - A(v)(t)| = 0$, $t \in [-r,0]$.

Using hypothesis (H1)(ii), for all $t \in I$, we can write

$$
\begin{aligned}
&|A(u)(t) - A(v)(t)| \\
&\leq L_f \int_0^t \left\{ |u(s) - v(s)| + |u(g(s)) - v(g(s))| + L_h \int_0^s \left[|u(\tau) - v(\tau)| + |u(g(\tau)) - v(g(\tau))| \right] d\tau \right\} ds \\
&\leq \int_0^t L_f \left\{ \max_{0 \leq \sigma_1 \leq s} |u(\sigma_1) - v(\sigma_1)| + \max_{0 \leq \sigma_1 \leq s} |u(g(\sigma_1)) - v(g(\sigma_1))| \right. \\
&\qquad \left. + \int_0^s L_h \left[\max_{0 \leq \sigma_2 \leq \tau} |u(\sigma_2) - v(\sigma_2)| + \max_{0 \leq \sigma_2 \leq \tau} |u(g(\sigma_2)) - v(g(\sigma_2))| \right] d\tau \right\} ds \\
&\leq \int_0^t L_f \left\{ \max_{-r \leq \sigma_1 \leq b} |u(\sigma_1) - v(\sigma_1)| + \max_{-r \leq \tau_1 \leq b} |u(\tau_1) - v(\tau_1)| \right. \\
&\qquad \left. + \int_0^s L_h \left[\max_{-r \leq \sigma_2 \leq b} |u(\sigma_2) - v(\sigma_2)| + \max_{-r \leq \tau_2 \leq b} |u(\tau_2) - v(\tau_2)| \right] d\tau \right\} ds \\
&\leq \int_0^t L_f \left\{ 2 \|u - v\|_C + 2 \int_0^s L_h \|u - v\|_C \, d\tau \right\} ds \leq b L_f \left(2 + L_h b\right) \|u - v\|_C .
\end{aligned}
$$

Therefore,

$$
\|A(u) - A(v)\|_C \leq b L_f \left(2 + L_h b\right) \|u - v\|_C , \text{ for all } u, v \in C\left([-r, b], \mathbb{R}_+\right).
$$

Since $bL_f \left(2 + L_h b\right) < 1$, A is a contraction on $C\left([-r, b], \mathbb{R}_+\right)$, using Banach contraction principle, we conclude that A is a Picard operator and $F_A = \{u^*\}$. Then, for $t \in I$, we have

$$
u^*(t) = \epsilon \lambda \psi(t) + L_f \int_0^t \left\{ u^*(s) + u^*(g(s)) + L_h \int_0^s \left[u^*(\tau) + u^*(g(\tau)\right] d\tau \right\} ds.
$$

Note that u^* is increasing and $(u^*)' \geq 0$ on I. Therefore $u^*(g(t)) \leq u^*(t)$ for $g(t) \leq t, \ t \in I$, and hence

$$
u^*(t) \leq \epsilon \lambda \psi(t) + \int_0^t 2L_f \left(u^*(s) + \int_0^s L_h u^*(\tau) d\tau \right) ds.
$$

Next, applying Pachpatte's inequality given in Theorem 2.1, we obtain

$$
\begin{aligned}
u^* &\leq \epsilon \lambda \psi(t) \left[1 + \int_0^t 2L_f \exp\left(\int_0^s \left[2L_f + L_h\right] d\tau \right) ds \right] \\
&\leq \epsilon \lambda \psi(t) \left\{ 1 + 2L_f \left(\frac{\exp(2L_f + L_h)b - 1}{2L_f + L_h} \right) \right\}. \qquad (3.7)
\end{aligned}
$$

Taking $C_\psi = \lambda \left\{ 1 + 2L_f \left(\frac{\exp(2L_f + L_h)b - 1}{2L_f + L_h} \right) \right\}$, from inequality (3.7) we get

$$
u^*(t) \leq C_\psi \, \epsilon \, \psi(t), \quad t \in [-r, b].
$$

For $u(t) = |y(t) - x(t)|$ the inequality (3.6) gives that $u(t) \leq A(u)(t)$. So, we have proved that $A : C\left([-r, b], \mathbb{R}_+\right) \to C\left([-r, b], \mathbb{R}_+\right)$ is an increasing Picard operator such that for $u \in C\left([-r, b], \mathbb{R}_+\right)$, $u(t) \leq Au(t)$ and $F_A = \{u^*\}$. Hence, applying the abstract Gronwall lemma (Lemma 2.1), we obtain $u(t) \leq u^*(t),\ t \in [-r,b]$, implying that

$$|y(t) - x(t)| \leq C_\psi\, \epsilon\, \psi(t), \quad \forall\, t \in [-r,b]. \tag{3.8}$$

Thus, the equation (2.1) is Ulam–Hyers–Rassias stable with respect to the function ψ. Theorem 3.1 is proved. □

Corollary 3.1. *Let the functions f and h in* (2.1) *satisfy (H1) and assume that (H2) holds. If $bL_f\,[2 + L_h b] < 1$, then the problem* (2.1), (2.2) *has a unique solution and the equation* (2.1) *is generalized Ulam–Hyers–Rassias stable with respect to the function ψ.*

Proof. By taking $\epsilon = 1$ in the proof of Theorem 3.1, we obtain (cf. (3.8)):

$$|y(t) - x(t)| \leq C_\psi\, \psi(t), \quad \forall\, t \in [-r,b],$$

showing that the equation (2.1) is generalized Ulam–Hyers–Rassias stable with respect to the function ψ. □

Using arguments similar to those applied in the proof of Theorem 3.1, one can prove Ulam–Hyers stability of equation (2.1).

Observing that for $\psi(t) = 1,\ \forall\, t \in [-r,b]$ the assumption (H2) holds, we can state the following corollary of Theorem 3.1.

Corollary 3.2. *Let the functions f and h in* (2.1) *satisfy the hypothesis (H1). If $bL_f\,[2 + L_h b] < 1$, then the problem* (2.1), (2.2) *has a unique solution and the equation* (2.1) *is Ulam–Hyers stable.*

Proof. By taking $\psi(t) = 1,\ \forall\, t \in [-r,b]$ in the proof of Theorem 3.1, we obtain (cf. (3.8)):

$$|y(t) - x(t)| \leq C\, \epsilon, \quad \forall\, t \in [-r,b],$$

and the result follows. □

Corollary 3.3. *Let the functions f and h in* (2.1) *satisfy the hypothesis (H1). If $bL_f\,[2 + L_h b] < 1$, then the problem* (2.1), (2.2) *has a unique solution and the equation* (2.1) *is generalized Ulam–Hyers stable.*

Proof. The result follows from Corollary 3.2, by taking $\theta_f(\epsilon) = C\,\epsilon$. □

3.2. **Applications.** In this section we consider some important special cases of the problem (2.1), (2.2).

Fix any $r > 0$, and define $g_1(t) = t - r,\ t \in [0, b]$. Then we get the following special case of the problem (2.1), (2.2):

$$x^{'}(t) = f_1\left(t, x(t), x(t-r), \int_0^t h_1(t, s, x(s), x(s-r))ds\right),\ t \in [0, b], \tag{3.9}$$

$$x(t) = \phi(t),\ t \in [-r, 0], \tag{3.10}$$

which is an initial value problem for a nonlinear Volterra integro-differential difference equation. Consider the following inequality:

$$\left|y^{'}(t) - f_1\left(t, y(t), y(t-r), \int_0^t h_1(t, s, y(s), y(s-r))ds\right)\right| \leq \epsilon\psi(t),\ t \in [0, b],$$

where ϵ, ψ and ϕ are as specified in Section 2 (Preliminaries).

As an application of Theorem 3.1, we have the following theorem for the problem (3.9), (3.10).

Theorem 3.2. *Suppose that the following assumptions are fulfilled:*

(A1) (i) $f_1 \in C\left([0, b] \times \mathbb{R}^3, \mathbb{R}\right),\ h_1 \in C\left([0, b] \times [0, b] \times \mathbb{R}^2, \mathbb{R}\right)$ *and* $g_1 \in C\left([0, b], [-r, b]\right)$ *be such that* $g_1(t) \leq t$;

(ii) *there exist constants* $L_{f_1},\ L_{h_1} > 0$ *such that*

$$|f_1(t, u_1, u_2, u_3) - f_1(t, v_1, v_2, v_3)| \leq L_{f_1}\left(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\right);$$

$$|h_1(t, s, u_1, u_2) - h_1(t, s, v_1, v_2)| \leq L_{h_1}\left(|u_1 - v_1| + |u_2 - v_2|\right);$$

for all $t, s \in [0, b],\ u_i, v_i \in \mathbb{R}\ (i = 1, 2, 3)$;

(A2) *the function* $\psi : [-r, b] \to \mathbb{R}_+$ *is positive, nondecreasing and continuous, and there exists* $\lambda > 0$ *such that* $\int_0^t \psi(s)ds \leq \lambda\psi(t),\ t \in [0, b]$;

(A3) $bL_{f_1}\left[2 + L_{h_2}b\right] < 1$.

Then the problem (3.9), (3.10) *has a unique solution* $x \in C\left([-r, b], \mathbb{R}\right) \cap C^{'}\left([0, b], \mathbb{R}\right)$, *and the equation* (3.9) *is Ulam–Hyers–Rassias stable with respect to the function* ψ.

Another special case of the problem (2.1), (2.2) we obtain by taking the delay $g_2(t) = t^2,\ t \in I = [0, 1]$. Then we have

$$x^{'}(t) = f_2\left(t, x(t), x(t^2), \int_0^t h_2(t, s, x(s), x(s^2))ds\right),\ t \in I = [0, 1], \tag{3.11}$$

$$x(t) = \phi(t), \quad t \in [-r, 0], \tag{3.12}$$

which is an initial value problem for a nonlinear Volterra integro-differential equation. Consider the following inequality:

$$\left|y^{'}(t) - f_2\left(t, y(t), y(s^2), \int_0^t h_2(t, s, y(s), y(s^2))ds\right)\right| \leq \epsilon\psi(t),\ t \in [0, 1].$$

where ϵ, ψ and ϕ are as specified in Section 2 (Preliminaries).

As an application of Theorem 3.1, we have the following theorem for the problem (3.11), (3.12).

Theorem 3.3. *Suppose that the following assumptions are fulfilled:*

(B1) (i) $f_2 \in C\left([0,1] \times \mathbb{R}^3, \mathbb{R}\right)$, $h_2 \in C\left([0,1] \times [0,1] \times \mathbb{R}^2, \mathbb{R}\right)$ *and* $g_2 \in C\left([0,1], [-r,1]\right)$ *be such that* $g_2(t) \leq t$*;*

(ii) *there exist constants* $L_{f_2},\ L_{h_2} > 0$ *such that*

$$|f_2(t,u_1,u_2,u_3) - f_2(t,v_1,v_2,v_3)| \leq L_{f_2}\left(|u_1-v_1| + |u_2-v_2| + |u_3-v_3|\right);$$
$$|h_2(t,s,u_1,u_2) - h_2(t,s,v_1,v_2)| \leq L_{h_2}\left(|u_1-v_1| + |u_2-v_2|\right);$$

for all $t,s \in [0,1],\ u_i, v_i \in \mathbb{R}\ (i=1,2,3)$*;*

(B2) *the function* $\psi : [-r,1] \to \mathbb{R}_+$ *is positive, nondecreasing and continuous, and there exists* $\lambda > 0$ *such that* $\int_0^t \psi(s)ds \leq \lambda\psi(t)$, $t \in [0,1]$;

(B3) $L_{f_2}\left[2 + L_{h_2}\right] < 1$.

Then the problem (3.11), (3.12) *has a unique solution* $x \in C\left([-r,1],\mathbb{R}\right) \cap C^{'}\left([0,1],\mathbb{R}\right)$, *and the equation* (3.11) *is Ulam–Hyers–Rassias stable with respect to the function* ψ.

Other Ulam type stability results for equations (3.9) and (3.11) can be obtained by using the corresponding results from Section 3.1.

3.3. **Examples.** In this section, we present concrete examples to illustrate our main results obtained in Section 3.1.

Example 1. *Consider the following nonlinear delay Volterra integro-differential equations:*

(3.13)
$$x^{'}(t) = 1 + \frac{t\cos(x(t))}{140} - \frac{3x(t)}{140} + \frac{t\cos(x(g(t)))}{70} + \frac{1}{20}\int_0^t \frac{t}{70}\left\{\sin(x(s)) - \sin(x(g(s)))\right\}ds,\ \ t \in [0,5],$$

(3.14)
$$x(t) = 0,\ \ t \in [-1,0],$$

where $g(t) = \frac{t}{2},\ t \in [0,5]$. Clearly we have $g(t) \leq t,\ t \in [0,5]$.

(i) Define $h : [0,5] \times [0,5] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$h\left(t,s,x(s),x(g(s))\right) = \frac{t}{70}\left[\sin(x(s)) - \sin(x(g(s)))\right],\quad t,s \in [0,5].$$

Then, for any $t,s \in [0,5]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$\begin{aligned}|h(t,s,x_1,x_2) - h(t,s,y_1,y_2)| &\leq \frac{t}{70}\left\{|\sin x_1 - \sin y_1| + |\sin x_2 - \sin y_2|\right\}\\ &\leq \frac{5}{70}\left\{|x_1 - y_1| + |x_2 - y_2|\right\}.\end{aligned}$$

(ii) Define $f : [0,5] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f\left(t, x(t), x(g(t)), \int_0^t h(t,s,x(s),x(g(s)))ds\right)$$
$$= 1 + \frac{t\cos(x(t))}{140} - \frac{3x(t)}{140} + \frac{t\cos(x(g(t)))}{70} + \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(x(s)) - \sin(x(g(s)))\right] ds, \ t \in [0,5]$$
$$= 1 + \frac{t\cos(x(t))}{140} - \frac{3x(t)}{140} + \frac{t\cos(x(g(t)))}{70} + \frac{1}{20}\int_0^t h\left(t,s,x(s),x(g(s))\right) ds.$$

Then, for any $t \in [0,5]$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$, we have

$$|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)|$$
$$\leq \left\{\frac{t}{140}|\cos x_1 - \cos y_1| + \frac{3}{140}|x_1 - y_1|\right\} + \frac{t}{70}|\cos x_2 - \cos y_2| + \frac{1}{20}|x_3 - y_3|.$$

Next, for any $x, y \in \mathbb{R}$ with $x < y$, by mean value theorem, there exists $p,\ x < p < y$ such that $\frac{\cos x - \cos y}{x-y} = -\sin p \Rightarrow |\cos x - \cos y| \leq |x-y|$. Therefore, we have

$$|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)| \leq \left\{\frac{5}{140}|x_1 - y_1| + \frac{3}{140}|x_1 - y_1|\right\} + \frac{5}{70}|x_2 - y_2| + \frac{1}{20}|x_3 - y_3|$$
$$\leq \frac{5}{70}\left\{|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|\right\}.$$

Hence the above defined functions f and h verify the assumptions (H1) and (H2) with $L_f = \frac{5}{70},\ L_h = \frac{5}{70},\ b = 5$. Further, we see that $bL_f(2+bL_h) = 5\frac{5}{70}\left[2 + \frac{5}{70}5\right] = 0.84183673 < 1$. Therefore, by Corollary 3.2, the problem (3.13), (3.14) has a unique solution on $[-1,5]$ and the equation (3.13) is Ulam–Hyers stable on $[0,5]$. Other stability results for the equation (3.13) can be discussed similarly.

In fact, we see that the function

$$x(t) = \begin{cases} t & \text{if } t \in [0,5], \\ 0 & \text{if } t \in [-1,0] \end{cases} \tag{3.15}$$

is the unique solution of the problem (3.13), (3.14). The verification is given below. For $x(t) = t,\ t \in [0,5]$ and $g(t) = \frac{t}{2},\ t \in [0,5]$, we have

$$1 + \frac{t\cos(x(t))}{140} - \frac{3x(t)}{140} + \frac{t\cos(x(g(t)))}{70} + \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(x(s)) - \sin(x(g(s)))\right] ds$$
$$= 1 + \frac{t\cos(t)}{140} - \frac{3t}{140} + \frac{t\cos(\frac{t}{2})}{70} + \frac{1}{140}\int_0^t t\left[\sin(s) - \sin\left(\frac{s}{2}\right)\right] ds = 1 = x'(t).$$

Next, we discuss the Ulam–Hyers stability of the equation (3.13) with fixed delay $g(t) = \frac{t}{2},\ t \in [0,5]$ by finding the exact solution $x(t)$ of equation (3.13) corresponding to given values of ϵ and given solutions $y(t)$ of the inequalities.

(i) Take $\epsilon = 0.7$ and $y_1(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0,5], \\ 0 & \text{if } t \in [-1,0]. \end{cases}$ Then for $t \in [0,5]$, we have

$$\left| y_1'(t) - \left(1 + \frac{t\cos(y_1(t))}{140} - \frac{3y_1(t)}{140} + \frac{t\cos(y_1(g(t)))}{70} + \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y_1(s)) - \sin(y_1(g(s)))\right] ds\right)\right|$$
$$= \left| y_1'(t) - 1 - \frac{t\cos(y_1(t))}{140} + \frac{3y_1(t)}{140} - \frac{t\cos(y_1(g(t)))}{70} - \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y_1(s)) - \sin(y_1(g(s)))\right] ds\right|$$
$$\leq \left| \frac{1}{2} - 1 - \frac{t\cos(\frac{t}{2})}{140} + \frac{3\left(\frac{t}{2}\right)}{140} - \frac{t\cos(\frac{t}{4})}{70} - \frac{1}{140}\int_0^t t\left[\sin\left(\frac{s}{2}\right) - \sin\left(\frac{s}{4}\right)\right] ds\right| \leq 0.667499 < \epsilon.$$

For the solution $x(t)$ of the problem (3.13), (3.14) given in (3.15) and the constant $C = 4$, we have $|y_1(t) - x(t)| = \left|\frac{t}{2} - t\right| \leq 2.5 < C\epsilon,\ t \in [0,5]$, and $|y_1(t) - x(t)| = 0,\ t \in [-1,0]$. Therefore

$$|y_1(t) - x(t)| < C\epsilon,\ t \in [-1,5].$$

(ii) Let $y_2(t) = 0, t \in [-1,5]$ and $\epsilon = 1.2$. Then, for $t \in [0,5]$, we have

$$\left| y_2'(t) - \left(1 + \frac{t\cos(y_2(t))}{140} - \frac{3y_2(t)}{140} + \frac{t\cos(y_2(g(t)))}{70} + \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y_2(s)) - \sin(y_2(g(s)))\right] ds\right)\right|$$
$$= \left| y_2'(t) - 1 - \frac{t\cos(y_2(t))}{140} + \frac{3y_2(t)}{140} - \frac{t\cos(y_2(g(t)))}{70} - \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y_2(s)) - \sin(y_2(g(s)))\right] ds\right|$$
$$= \left| -1 - \frac{t}{140} - \frac{t}{70}\right| \leq \frac{155}{140} < 1.2 = \epsilon.$$

For the solution $x(t)$ of the problem (3.13), (3.14) given in (3.15) and the constant $C = 6$, we have

$$|y_2(t) - x(t)| = |0 - t| \leq 5 < C\epsilon,\ t \in [0,5].$$

Further, $|y_2(t) - x(t)| = 0 < C\epsilon,\ t \in [-1,0]$. Therefore corresponding to $y_2(t) = 0, t \in [-1,5]$ and $\epsilon = 1.2$ we have the solution $x(t)$ given in (3.15) and the constant $C = 6$ that satisfy

$$|y_2(t) - x(t)| < C\epsilon,\ t \in [-1,5].$$

(iii) For $\epsilon = 1.5$ and $y_3(t) = \begin{cases} \frac{t}{10} & \text{if } t \in [0,5], \\ 0 & \text{if } t \in [-1,0], \end{cases}$ we have

$$\left| y_3'(t) - \left(1 + \frac{t\cos(y_3(t))}{140} - \frac{3y_3(t)}{140} + \frac{t\cos(y_3(g(t)))}{70} + \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y_3(s)) - \sin(y_3(g(s)))\right] ds\right)\right|$$
$$= \left| y_3'(t) - 1 - \frac{t\cos(y_3(t))}{140} + \frac{3y_3(t)}{140} - \frac{t\cos(y_3(g(t)))}{70} - \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y_3(s)) - \sin(y_3(g(s)))\right] ds\right|$$
$$\leq 1.0557 < \epsilon.$$

The solution $x(t)$ of the problem (3.13), (3.14) given in (3.15) and the constant $C = 3$ verify

$$|y_3(t) - x(t)| \leq 4.5 = C\epsilon,\ t \in [-1,5].$$

(iv) Take $\epsilon = 10$ and $y_4(t) = \begin{cases} t^2 & \text{if } t \in [0,5], \\ 0 & \text{if } t \in [-1,0]. \end{cases}$ Then, for $t \in [0,5]$, we have

$$\left| y_4^{'}(t) - \left(1 + \frac{t\cos(y_4(t))}{140} - \frac{3y_4(t)}{140} + \frac{t\cos(y_4(g(t)))}{70} + \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y_4(s)) - \sin(y_4(g(s)))\right] ds \right) \right|$$
$$= \left| y_4^{'}(t) - 1 - \frac{t\cos(y_4(t))}{140} + \frac{3y_4(t)}{140} - \frac{t\cos(y_4(g(t)))}{70} - \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y_4(s)) - \sin(y_4(g(s)))\right] ds \right| < \epsilon.$$

Further, for the solution $x(t)$ of the problem (3.13), (3.14) given in (3.15) and the constant $C = 2$, we have

$$|y_4(t) - x(t)| \leq 20 = C\epsilon, \ t \in [-1,5].$$

(v) Finally, we take $\epsilon = 77$ and $y_5(t) = \begin{cases} t^3 & \text{if } t \in [0,5], \\ 0 & \text{if } t \in [-1,0], \end{cases}$ to obtain

$$\left| y_5^{'}(t) - \left(1 + \frac{t\cos(y_5(t))}{140} - \frac{3y_5(t)}{140} + \frac{t\cos(y_5(g(t)))}{70} + \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y_5(s)) - \sin(y_5(g(s)))\right] ds \right) \right|$$
$$= \left| y_5^{'}(t) - 1 - \frac{t\cos(y_5(t))}{140} + \frac{3y_5(t)}{140} - \frac{t\cos(y_5(g(t)))}{70} - \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y_5(s)) - \sin(y_5(g(s)))\right] ds \right| < \epsilon.$$

For the solution $x(t)$ of the problem (3.13), (3.14) given in (3.15) and the constant $C = 2$, we have

$$|y_5(t) - x(t)| \leq 120 < C\epsilon, \ t \in [-1,5].$$

Remark 3.1. If $y(t)$ is a solution of the inequality

$$\left| y^{'}(t) - \left(1 + \frac{t\cos(y(t))}{140} - \frac{3y(t)}{140} + \frac{t\cos(y(g(t)))}{70} + \frac{1}{20}\int_0^t \frac{t}{70}\left[\sin(y(s)) - \sin(y(g(s)))\right] ds \right) \right| < \epsilon,$$

and $x(t)$ is the exact solution of the problem (3.13), (3.14), then from the inequality $|y(t) - x(t)| \leq C\epsilon, \ t \in [-1,5]$, it follows that $y(t) \to x(t)$ as $\epsilon \to 0$.

The same fact can be observed from the example given above and Figure 1 below.

Рис. 1

Remark 3.2. The equation (2.1) is not Ulam–Hyers stable on the infinite interval $I = [0, \infty)$.

The next example supports the assertion of Remark 3.2.

Example 2. Consider the following Volterra delay integro-differential equations:

(3.16)
$$x^{'}(t) = \frac{17}{30} + \frac{1}{60}\sin(x(t)) - \frac{1}{15}\cos(x(g(t))) - \frac{1}{12}\int_0^t \frac{1}{10}\left[\cos(x(s)) + \sin(x(g(s)))\right] ds, t \in [0, \infty),$$

(3.17)
$$x(t) = 0, \;\; t \in [-1, 0],$$

where $g(t) = \frac{t}{4} \leq t, \;\; t \in [0, \infty)$.

(i) Define the function $h : [0, \infty) \times [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$h\left(t, s, x(s), x(g(s))\right) = \frac{1}{10}\left[\cos(x(s)) + \sin(x(g(s)))\right], \;\; t, s \in [0, \infty), \;\; t \geq s.$$

Then, for any $t, s \in [0, \infty)$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$|h(t, s, x_1, x_2) - h(t, s, y_1, y_2)| \leq \frac{1}{10}\left\{|\cos x_1 - \cos y_1| + |\sin x_2 - \sin y_2|\right\}$$
$$\leq \frac{1}{10}\left\{|x_1 - y_1| + |x_2 - y_2|\right\}.$$

(ii) Define $f : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f\left(t, x(t), x(g(t)), \int_0^t h(t, s, x(s), x(g(s)))ds\right)$$
$$= \frac{17}{30} + \frac{1}{60}\sin(x(t)) - \frac{1}{15}\cos(x(g(t))) - \frac{1}{12}\int_0^t \left[\cos(x(s)) + \sin(x(g(s)))\right] ds$$
$$= \frac{17}{30} + \frac{1}{60}\sin(x(t)) - \frac{1}{15}\cos(x(g(t))) - \frac{1}{12}\int_0^t h\left(t, s, x(s), x(g(s))\right) ds.$$

Then, for any $t \in [0, \infty)$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$, we have

$$|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| \leq \frac{1}{60}|\sin x_1 - \sin y_1| + \frac{1}{15}|\cos x_2 - \cos y_2| + \frac{1}{12}|x_3 - y_3|$$
$$\leq \frac{1}{12}\left\{|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|\right\}.$$

The above defined functions f and h verify the assumptions (H1) and (H2) with $L_f = \frac{1}{12}$ and $L_h = \frac{1}{10}$. Further, one can easily verify that the function

$$x(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0, \infty), \\ 0 & \text{if } t \in [-1, 0] \end{cases}$$

is the solution of the initial value problem (3.16), (3.17). Now, choose any $\epsilon > \frac{1}{2}$ and let

$$y(t) = \begin{cases} \frac{t}{3} & \text{if } t \in [0, \infty), \\ 0 & \text{if } t \in [-1, 0]. \end{cases}$$

Then, for any $t \in [0, \infty)$, we have

$$\left|y^{'}(t) - \left(\frac{17}{30} + \frac{1}{60}\sin(y(t)) - \frac{1}{15}\cos(y(g(t))) - \frac{1}{12}\int_0^t \frac{1}{10}\left[\cos(y(s)) + \sin(y(g(s)))\right] ds\right)\right|$$
$$= \left|\frac{1}{3} - \frac{17}{30} - \frac{1}{60}\sin\left(\frac{t}{3}\right) + \frac{1}{15}\cos\left(\frac{t}{12}\right) + \frac{1}{120}\int_0^t \left[\cos\left(\frac{s}{3}\right) + \sin\left(\frac{s}{12}\right)\right] ds\right| \le \frac{19}{120} < \epsilon.$$

But for any solution $x(t)$ of equation (3.16) we have

$$|x(t) - y(t)| = \left|x(t) - \frac{t}{3}\right| \le |x(t)| + \frac{t}{3} \to \infty \quad as \quad t \to \infty.$$

Therefore, the equation (3.16) is not Ulam–Hyers stable on the infinite interval $I = [0, \infty)$.

Acknowledgments. The authors would like to thank an associate editor and an anonymous referee for valuable comments and useful suggestions.

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Поступила 15 июня 2017

После доработки 30 октября 2017

Принята к публикации 12 января 2018