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A NOTE ON THE GENERALIZED CESÁRO MEANS OF TRIGONOMETRIC FOURIER SERIES

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Abstract. Different generalized Cesáro summation methods are compared with each other. Analogous of Hardy's theorem, concerning the order of the partial sums of trigonometric Fourier series, for generalized Cesáro means are obtained.

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1. Introduction

Let f be a 2π-periodic locally integrable function and

$$a_n = a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad b_n = b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

be its Fourier coefficients, and let

$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx) \tag{1.1}$$

be the partial sums of the Fourier series of a function f with respect to the trigonometric system.

Let (α_n) $(\alpha_n > -1)$ and (S_n), $n \in \mathbb{N}$, be sequences of real numbers, and let

$$\sigma_n^{\alpha_n} \equiv \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha_n-1} S_\nu / A_n^{\alpha_n}, \tag{1.2}$$

where

$$A_k^{\alpha_n} = (\alpha_n+1)(\alpha_n+2)\cdot \ldots \cdot(\alpha_n+k)/k!. \tag{1.3}$$

It is clear that $\sigma_n^0 = S_n$. If (α_n) is a constant sequence ($\alpha_n = \alpha,\ n \in \mathbb{N}$), then $\sigma_n^{\alpha_n}$ coincide with the usual Cesáro σ_n^α-means (see [18, Chapter III]). If in (1.2) instead of S_ν we substitute $S_\nu(f,x)$ (see (1.1)), then the corresponding means $\sigma_n^{\alpha_n}$ we will denote by $\sigma_n^{\alpha_n}(f,x)$.

These means, called generalized Cesáro (C, α_n)-means, were studied by Kaplan [6], where the author compared the (C, α_n) and (C, α) summability methods, and obtained necessary and sufficient conditions, in terms of the sequence (α_n), for the inclusion $(C, \alpha_n) \subset (C, \alpha)$, and sufficient conditions for the inclusion $(C, \alpha) \subset (C, \alpha_n)$. Later on Akhobadze [1] – [4] and Tetunashvili [11] – [16] have investigated problems concerning (C, α_n) summability of trigonometric Fourier series. In papers [1] – [4] the behavior of generalized Cesáro (C, α_n)-means $(\alpha_n \in (-1; d), d > 0)$ of trigonometric Fourier series of functions from various classes of continuous functions were studied, and the sharpness of the obtained results were shown. Lebesgue [8] proved that every function from $L[0; 2\pi]$ has a Fourier series the sequence of $(C, 1)$-means of which is a.e. convergent, and then M. Riesz [10] generalized this result for (C, α)-means $(\alpha > 0)$.

Observe that if $\alpha_n \to 0+$, then the behavior of (C, α_n)-means for Fourier series of integrable functions is different in the sense of pointwise convergence.

In [16], Tetunashvili proved the following theorem.

Theorem 1.1. *Let the sequence (α_n) be such that for some positive number m we have*

$$\alpha_n \leq \frac{c}{\ln n},$$

where $0 \leq c < \ln 2$ and $n > m$. Then for any series with partial sums S_n satisfying the condition:

$$\limsup_{n \to +\infty} |S_n| = +\infty,$$

the following is true:

$$\limsup_{n \to +\infty} |\sigma_n^{\alpha_n}| = +\infty.$$

Throughout the paper the letter c is used to denote positive constants depending only on the indicated parameters, the value of which can vary from line to line.

If the sequence (α_n) satisfies the condition of Theorem 1.1, then in view of Kolmogorov's well known result (see, e.g., [7], [5, Chapter V]), we can conclude that there exists an integrable function f_0 such that the sequence $\sigma_n^{\alpha_n}(f_0, x)$ diverges almost everywhere. On the other hand, in [13] Tetunashvili proved the following theorem.

Theorem 1.2. *For any function $f \in L(0; 2\pi)$ and a number $\varepsilon > 0$, there exist a sequence of numbers $\alpha_n \downarrow 0$ and a set $F \subset [0; 2\pi]$ with $|F| > 2\pi - \varepsilon$, such that*

$$\lim_{n \to +\infty} \sigma_n^{\alpha_n}(f, x) = f(x)$$

at every point $x \in F$, where $|F|$ denotes the Lebesgue measure of the set F.

Observe that for the function constructed by Kolmogorov the conclusion of Theorem 1.2 is true. It is clear that in this case the sequence (α_n) does not satisfy the condition of Theorem 1.1.

The theorems that follow give important information on the pointwise convergence of (C,α_n)-means of trigonometric Fourier series in the case where $\alpha_n \to 0+$.

Theorem 1.3. *Let $0 \le \alpha_n \le \beta_n$. Then the (C,α_n) summability of a number sequence (S_n) to S implies the (C,β_n) summability of (S_n) to S.*

Theorem 1.4. *Let $f \in L(0;2\pi)$ and $\alpha_n \to 0+$ as $n \to +\infty$. Then for almost every $x \in (0;2\pi)$ we have*

$$\lim_{n\to+\infty} \alpha_n \sigma_n^{\alpha_n}(f,x) = 0.$$

2. Proof of theorems 1.3 and 1.4

Proof of Theorem 1.3. To prove it we use the scheme proposed by Kaplan [6]. Let

$$\sigma_n^{\alpha_n} = \frac{1}{A_n^{\alpha_n}} S_n^{\alpha_n} = \frac{1}{A_n^{\alpha_n}} \sum_{k=0}^{n} A_{n-k}^{\alpha_n - 1} S_k.$$

We shall express $\sigma_n^{\beta_n}$ by numbers $\sigma_n^{\alpha_n}$ as their regular mean. For all $k \in \{0,1,...,n\}$ we examine the sums (see [18, Chapter III, (1.10)]):

$$S_k^{\alpha_k} = S_k^{\alpha_k - \beta_n + \beta_n} = \sum_{j=0}^{k} A_{k-j}^{\alpha_k - \beta_n - 1} S_j^{\beta_n}.$$

Let us consider these expressions as a system of linear equations with respect to the variables $S_k^{\beta_n},\ k \in \{0,1,...,n\}$. Taking into account that $A_0^{\alpha_k - \beta_n - 1} = 1$, we can easily obtain that the determinant of this system is equal to 1. By Cramer's rule we have

$$S_n^{\beta_n} = \begin{vmatrix} 1 & 0 & \dots & 0 & S_0^{\alpha_0} \\ A_1^{\alpha_1 - \beta_n - 1} & 1 & \dots & 0 & S_1^{\alpha_1} \\ \dots & \dots & \dots & \dots & \dots \\ A_{n-1}^{\alpha_{n-1} - \beta_n - 1} & A_{n-2}^{\alpha_{n-1} - \beta_n - 1} & \dots & 1 & S_{n-1}^{\alpha_{n-1}} \\ A_n^{\alpha_n - \beta_n - 1} & A_{n-1}^{\alpha_n - \beta_n - 1} & \dots & A_1^{\alpha_n - \beta_n - 1} & S_n^{\alpha_n} \end{vmatrix}.$$

Expanding this determinant by the last n-th column, we get

$$S_n^{\beta_n} = \sum_{k=0}^{n} A_{k,n} S_k^{\alpha_k},$$

where $A_{k,n}$ is the cofactor of the element $S_k^{\alpha_k}$, $k \in \{0, 1, ..., n\}$. It is easy to see that

$$\sigma_n^{\beta_n} = \sum_{k=0}^{n} \frac{A_{k,n} \cdot A_k^{\alpha_k}}{A_n^{\beta_n}} \sigma_k^{\alpha_k}.$$

Therefore, denoting

$$a_{nk} := A_{k,n} \cdot A_k^{\alpha_k} / A_n^{\beta_n},$$

we get

$$\sigma_n^{\beta_n} = \sum_{k=0}^{n} a_{nk} \sigma_k^{\alpha_k}. \tag{2.1}$$

Now we prove that matrix (a_{nk}) is regular.

The following equalities are well known (see, e.g., [18, Chapter III, (1.10)]):

$$A_k^{\alpha_k} = \sum_{j=0}^{k} A_{k-j}^{\alpha_k - \beta_n - 1} A_j^{\beta_n}, \quad k \in \{0, 1, ..., n\}.$$

Observe that these equalities can be considered as a system of linear equations with respect to variables $A_j^{\beta_n}$. Then arguing analogously as above, we get

$$A_n^{\beta_n} = \begin{vmatrix} 1 & 0 & \dots & 0 & A_0^{\alpha_0} \\ A_1^{\alpha_1 - \beta_n - 1} & 1 & \dots & 0 & A_1^{\alpha_1} \\ \dots & \dots & \dots & \dots & \dots \\ A_{n-1}^{\alpha_{n-1} - \beta_n - 1} & A_{n-2}^{\alpha_{n-1} - \beta_n - 1} & \dots & 1 & A_{n-1}^{\alpha_{n-1}} \\ A_n^{\alpha_n - \beta_n - 1} & A_{n-1}^{\alpha_n - \beta_n - 1} & \dots & A_1^{\alpha_n - \beta_n - 1} & A_n^{\alpha_n} \end{vmatrix}.$$

Expanding this determinant by the last n-th column, we obtain

$$A_n^{\beta_n} = \sum_{k=0}^{n} A_{k,n} \cdot A_k^{\alpha_k}.$$

Let $A_n = \sum_{k=0}^{n} a_{nk}$. Then by the previous equality we have

$$A_n = \sum_{k=0}^{n} \frac{A_{k,n} \cdot A_k^{\alpha_k}}{A_n^{\beta_n}} = 1. \tag{2.2}$$

Thus, the first condition of regularity is fulfilled.

Next, we consider the cofactors $A_{k,n}$ $(k \in \{0, 1, ..., n\})$. It is clear that $A_{n,n} = 1$. Now we estimate the cofactor $A_{n-1,n}$ in the determinant $A_n^{\beta_n}$. To this end, in the determinant $A_n^{\beta_n}$, we rewrite the last n-th column by the $(n-1)$-th column, and observe that the obtained determinant is equal to zero. Then expanding it by the last column, we get $0 = A_1^{\alpha_n - \beta_n - 1} \cdot A_{n,n} + A_{n-1,n}$. Since $\alpha_n \leq \beta_n$ we have $A_1^{\alpha_n - \beta_n - 1} \leq 0$, and hence, taking into account that $A_{n,n} = 1$, from the last equality we obtain $A_{n-1,n} \geq 0$. Analogously, we can show that the cofactor $A_{n-2,n}$ is nonnegative. In particular, in the given matrix the last n-th column we can rewrite by the $(n-2)$-th column. Then the value of the corresponding determinant will be 0. If we expand the last determinant by the last column, we obtain

$$0 = A_2^{\alpha_n - \beta_n - 1} \cdot A_{n,n} + A_1^{\alpha_{n-1} - \beta_n - 1} \cdot A_{n-1,n} + A_{n-2,n}.$$

Therefore, $A_{n-2,n} \geq 0$. Repeating the above reasonings for each cofactor $A_{k,n}$ and taking into account that $A_{k+1,n}, ..., A_{n,n} \geq 0$, we get $A_{k,n} \geq 0$, $k \in \{0, 1, ..., n\}$. Thus, we have

$$N_n = \sum_{k=0}^{n} |a_{nk}| = A_n = 1, \qquad n \in \{0, 1, ...\}.$$

It is clear that

$$0 \leq a_{nk} \leq 1, \;\; k \in \{0, 1, ..., n\}. \tag{2.3}$$

Finally, we show that $\lim_{n\to\infty} a_{nk} = 0$. Let $\alpha'_n = \alpha_n - 1/2$ and $\beta'_n = \beta_n - 1/2$. It is clear that $\alpha'_k - \beta'_n = \alpha_k - \beta_n$, $k \in \{0, 1, ..., n\}$. Therefore, the above considered cofactors depend only on the difference $\alpha_k - \beta_n$, and hence, using the above arguments applied to these new sequences, we get

$$0 \leq a'_{nk} = A_{k,n} \cdot A_k^{\alpha'_k} / A_n^{\beta'_n} \leq 1,$$

implying that $A_{k,n} \leq A_n^{\beta'_n} / A_k^{\alpha'_k}$. Using this estimation for the inequalities $0 \leq a_{nk} \leq 1$, $k \in \{0, 1, ..., n\}$, and taking into account that for fixed k, the numbers $A_k^{\alpha_k}$ and $A_k^{\alpha'_k}$ are fixed, we get

$$0 \leq a_{nk} \leq A_k^{\alpha_k} \cdot A_n^{\beta'_n} / (A_n^{\beta_n} \cdot A_k^{\alpha'_k}) = O(n^{\beta'_n} / n^{\beta_n}) = O(1/\sqrt{n}) \to 0, \;\; n \to \infty.$$

Thus, we have proved that the matrix (a_{nk}) is regular. $\square$

Proof of Theorem 1.4. We have (see [18, Chap. III, (5.4)])

$$\sigma_n^{\alpha_n}(f, x) = \frac{1}{2\pi} \int_0^{\pi} \chi_x(t) K_n^{\alpha_n}(t) dt + f(x),$$

where

$$\chi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

In what follows we will need the following estimates for the kernel $K_n^{\alpha_n}(t)$ (see [18, Chapter III] and [2, Lemmas 1 and 2]):

$$|K_n^{\alpha_n}(t)| \leq n + 1, \quad |K_n^{\alpha_n}(t)| \leq \frac{c}{n^{\alpha_n} t^{1+\alpha_n}}.$$

Since for almost all Lebesgue point x the value of $|f(x)|$ is finite, we have $\alpha_n f(x) \to 0$ as $n \to +\infty$. On the other hand, for such point x, we can write

$$\frac{1}{2\pi}\int_0^\pi \chi_x(t)K_n^{\alpha_n}(t)dt = \frac{1}{2\pi}\int_0^{1/n} \chi_x(t)K_n^{\alpha_n}(t)dt + \frac{1}{2\pi}\int_{1/n}^\pi \chi_x(t)K_n^{\alpha_n}(t)dt =$$

$$:= A_1(n,x) + A_2(n,x).$$

Besides, for any $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that for all δ $(0 < \delta < \delta(\varepsilon))$

$$\frac{1}{\delta}\int_0^\delta |\chi_x(t)|dt < \varepsilon.$$

Let n_δ be a natural number for which $1/n_\delta < \delta < \delta(\varepsilon)$. Then, for $n > n_\delta$ we have

$$|A_1(n,x)| \le \frac{n+1}{2\pi}\int_0^{1/n} |\chi_x(t)|dt < \varepsilon.$$

On the other hand, using integration by parts, we get

$$|A_2(n,x)| \le \frac{c}{n^{\alpha_n}}\int_{1/n}^\pi |\chi_x(t)|t^{-1-\alpha_n}dt =$$

$$= \frac{c}{n^{\alpha_n}}\, t^{-1-\alpha_n}\int_0^t |\chi_x(u)|du\Big|_{1/n}^\pi + \frac{c(1+\alpha_n)}{n^{\alpha_n}}\int_{1/n}^\pi t^{-2-\alpha_n}\int_0^t |\chi_x(u)|dudt =$$

$$=: B_1(n,x) + B_2(n,x).$$

It is easy to see that

$$B_1(n,x) = O_x(1).$$

For $B_2(n,x)$ we have the estimate

$$B_2(n,x) \le \frac{c}{n^{\alpha_n}}\left(\int_{1/n}^\delta + \int_\delta^\pi\right) t^{-2-\alpha_n}\int_0^t |\chi_x(u)|dudt =: F_1(n,x) + F_2(n,x).$$

Next, the functions $F_1(n,x)$ and $F_2(n,x)$ can be estimated as follows:

$$F_1(n,x) = \frac{c}{n^{\alpha_n}}\int_{1/n}^\delta t^{-1-\alpha_n}\frac{1}{t}\int_0^t |\chi_x(u)|dudt \le \frac{c\varepsilon}{n^{\alpha_n}}\int_{1/n}^\delta t^{-1-\alpha_n}dt =$$

$$= \frac{c\varepsilon}{n^{\alpha_n}}\cdot\frac{1}{\alpha_n}\cdot\left(n^{\alpha_n} - \frac{1}{\delta^{\alpha_n}}\right) < \frac{c\varepsilon}{\alpha_n}$$

and

$$F_2(n,x) = \frac{c}{n^{\alpha_n}}\int_\delta^\pi t^{-2-\alpha_n}\int_0^t |\chi_x(u)|dudt \le$$

$$\le \frac{c}{n^{\alpha_n}\delta^{2+\alpha_n}}\int_\delta^\pi\int_0^t |\chi_x(u)|dudt = O_{x,\delta}\left(\frac{1}{n^{\alpha_n}}\right).$$

Therefore, $A_2(n,x) = o_x(1/\alpha_n),\ n \to +\infty.$ $\square$

3. Appendix. The case of continuous functions

Let $C([0,2\pi])$ denote the space of 2π-periodic continuous functions with norm $||f||_{C([0,2\pi])} = \max_{x\in[0,2\pi]} |f(x)|$. If $f \in C([0,2\pi])$, then

$$\omega(\delta, f) = \max\{|f(x_1) - f(x_2)| : |x_1 - x_2| \le \delta,\ x_1, x_2 \in [0,2\pi]\}$$

is called the modulus of continuity of the function f. For a given modulus of continuity ω, by H^ω we denote the class of functions $f \in C([0,2\pi])$ for which (see [9]):

$$\omega(\delta, f) \le \omega(\delta),\ \delta \in [0, 2\pi).$$

If the sequence (α_n) satisfies the condition of Theorem 1.1, then there exists a continuous function f_0 such that $\sigma_n^{\alpha_n}(f_0, x)$ diverges at a point. On the other hand, Tetunashvili [12] showed that for any continuous function there exists a sequence of numbers $\alpha_n \downarrow 0$, $n \to +\infty$, such that the (C, α_n)-means of partial sums of trigonometric Fourier series of this function converge at every point. Then, Akhobadze [2] improved this result by proving the following theorem.

Theorem 3.1. *If $f \in H^\omega$ and $\alpha_n \in (0,1]$, $n = 3, 4, \ldots$, then*

$$||\sigma_n^{\alpha_n}(\cdot, f) - f(\cdot)||_C \le c \cdot \max\left\{ \frac{n^{\alpha_n} - 1}{\alpha_n \cdot n^{\alpha_n}}\omega(1/n), \frac{\alpha_n}{n}\int\limits_{\pi/n}^{\pi} \frac{\omega(t)}{t^2}dt \right\}, \tag{3.1}$$

where c is an absolute constant.

From the last statement we can easily conclude that for any modulus of continuity ω there exists a positive sequence $\alpha_n = o(1)$ as $n \to +\infty$, such that for any function $f \in H^\omega$ the generalized Cesáro means $\sigma_n^{\alpha_n}(f, x)$ converge uniformly. Indeed, every continuous function $f \in H^\omega$, where instead of ω can be considered the modulus of continuity of f. If α_n tends to zero sufficiently "slowly", then it can easily be proved that

$$\frac{n^{\alpha_n} - 1}{\alpha_n n^{\alpha_n}}\omega\left(\frac{1}{n}\right) \to 0, \quad n \to +\infty.$$

On the other hand, we have (see [17, p. 91, (2;8.82)]):

$$\frac{\alpha_n}{n}\int_{\pi/n}^{\pi} \frac{\omega(t)}{t^2} \le c_\omega \cdot \alpha_n \cdot \omega\left(\frac{1}{n}\right) \ln\frac{1}{\omega(1/n)} \to 0, \quad n \to +\infty.$$

The last reasoning can be completed as follows. It is well known (see [18, Chapter VIII, Theorem (2.1)]) that the condition $\omega(1/n) = O(1/\ln n)$ does not imply convergence of $S_n(f, x)$ for all continuous functions from H^ω, but for the generalized Cesáro means we have different result.

Theorem 3.2. *Let $\omega(1/n) = O(1/\ln n)$ and $\alpha_n \to 0+$ as $n \to +\infty$, and let*

$$\lim_{n\to+\infty} \alpha_n \cdot \ln n = +\infty,$$

then $\sigma_n^{\alpha_n}(f,x)$ uniformly converge to f for every function $f \in H^{\omega}$.

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