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A DIFFERENCE ANALOGUE OF CARTAN'S SECOND MAIN THEOREM FOR MEROMORPHIC MAPPINGS

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Abstract. In this paper, we prove a difference analogue of Cartan's second main theorem for a meromorphic mapping on $\mathbb{C}^m$ intersecting a finite set of fixed hyperplanes in general position on $\mathbb{P}^n(\mathbb{C})$. As an application, we prove a uniqueness theorem for a class of holomorphic curves by inverse images of $n+4$ hyperplanes. This result is so far the best result about the uniqueness problem for holomorphic curves by inverse images of hyperplanes.

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1. Introduction and main results

Recently, Nevanlinna theory have been studied for difference operators. In 2006, R. Halburd and R. Korhonen [6, 7] have built the second main theorem for a difference operator of meromorphic functions. Since then, many authors have studied applications of Nevanlinna theory for difference operators. In 2014, R. Halburd, R. Korhonen and K. Tohge [8] proved a difference analogue of Cartan's second main theorem for holomorphic curves. In 2016, T. B. Cao and R. Korhonen [1] gave a new version of the difference second main theorem for meromorphic mappings intersecting hyperplanes in several complex variables.

However, to the best of our knowledge, a little is known concerning uniqueness problem of holomorphic curves by applying difference second main theorems. When one applies inequalities of type second main theorem, it is often crucial to have an inequality with truncated counting functions. For instance, all the existing constructions of unique range sets depend on the second main theorem with truncated counting functions. The above quoted results motivate us to consider the difference second main theorem for holomorphic curves intersecting hyperplanes with the level of truncation. In order to reduce the number of hyperplanes in the uniqueness problem, we first establish a difference analogue of Cartan's second main theorem

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with truncated level 1. As an application of this result, we prove a uniqueness theorem for holomorphic curves by inverse images of $n+4$ hyperplanes.

To state our results, we first recall some notation and notions from Nevanlinna theory. We set

$$|z|^2=\sum_{j=1}^{m}|z_j|^2 \quad \text{for all} \quad z=(z_1,\ldots,z_m)\in\mathbb{C}^m,$$
$$S_m(r)=\{z\in\mathbb{C}^m:|z|=r\},\quad \overline{B}_m(r)=\{z\in\mathbb{C}^m:|z|\le r\},$$
$$d=\partial+\overline{\partial},\qquad d^c=\frac{1}{4\pi i}(\partial-\overline{\partial}),$$
$$\omega_m=dd^c\log|z|^2,\quad \sigma_m=d^c\log|z|^2\wedge {\omega_m}^{m-1}(z),\nu_m(z)=dd^c|z|^2.$$

Let ν be a divisor in $\mathbb{C}^m$. We set $\mathrm{supp}\nu=\overline{\{z:\nu(z)\neq 0\}}$, and define the counting function of ν by

$$N_\nu(r)=\int_1^r\frac{n(t)}{t^{2m-1}}dt,\quad 1<r<+\infty,$$

where $n(t)=\int_{\mathrm{supp}\nu\cap\overline{B}_m(t)}\nu_m^{m-1}$ for $m\ge 2$, and $n(t)=\sum_{|z|\le t}\nu(z)$ for $m=1$. Let M be a positive integer, we define ν^M by $\nu^M(z)=\min\{M,\nu(z)\}$ and the counting function of ν^M by

$$N_\nu^M(r)=\int_1^r\frac{n^M(t)}{t^{2m-1}}dt,\quad 1<r<+\infty,$$

where $n^M(t)=\int_{\mathrm{supp}\,\min\{M,\nu\}\cap\overline{B}_m(t)}\nu_m^{m-1}$ for $m\ge 2$, and $n^M(t)=\sum_{|z|\le t}\min\{M,\nu(z)\}$ for $m=1$. When $M=1$, we get the reduced counting function $\overline{N}_\nu(r)$.

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For a set $\alpha=(\alpha_1,\ldots,\alpha_m)$ of nonnegative integers, we set $|\alpha|=\alpha_1+\cdots+\alpha_m$ and $D^{|\alpha|}F=\dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1}z_1\ldots\partial^{\alpha_m}z_m}$. We define the zero divisor ν_F of F by

$$\nu_F=\max\{p:D^{|\alpha|}F(z)=0 \quad \text{for all} \quad \alpha:|\alpha|<p\}.$$

Let ϕ be a nonzero meromorphic function on $\mathbb{C}^m$. For each $z_0\in\mathbb{C}^m$, the zero divisor ν_ϕ of ϕ is defined as follows. We choose nonzero holomorphic functions F and G defined on a neighborhood U of z_0 such that $\phi=\dfrac{F}{G}$ on U and $\dim(F^{-1}(0)\cap G^{-1}(0))\le m-2$, then we put $\nu_\phi=\nu_{\phi=0}=\nu_F$, and $\nu_{\phi=\infty}=\nu_G$ is called the polar divisor of ϕ. For each $a\in\mathbb{P}^1(\mathbb{C})$ with $\phi^{-1}(a)\neq\mathbb{C}^m$, the counting function of an a-point of ϕ is defined as follows. We denote by $\nu_\phi(a)$ the a-divisor of ϕ. This means that if $\phi=(\phi_0:\phi_1)$ is an expression reducing ϕ, then the a-divisor $\nu_\phi(a)$ is the divisor associated with the holomorphic function $\phi_1-a\phi_0$. Thus, we have $\nu_\phi(a)=\sum_{z\in\mathbb{C}^m}\nu_{\phi_1-a\phi_0}(z)$. We define

$$n_\phi(r,a)=\int_{\mathrm{supp}\nu_\phi(a)\cap\overline{B}_m(r)}\nu_\phi(a)\nu_m^{m-1}$$

outside a set analysis of codimension 2, that is, $\dim((\phi_1 - a\phi_0)^{-1}(0) \cap \phi_0^{-1}(0)) \leq m-2$ for all $m \geq 1$ and $r > 0$, where $\mathrm{supp}\nu_\phi(a)$ denotes the closure of the set $\{z \in \mathbb{C}^m : \nu_\phi(a)(z) \neq 0\}$. The *counting function* of an a-point of ϕ is defined by

$$N_\phi(r,a) = \int_1^r \frac{n_\phi(t,a)}{t^{2m-1}} dt.$$

The *proximity function* of ϕ is defined by

$$m_\phi(r,a) = \begin{cases} \int_{S_m(r)} \log^+ \dfrac{1}{|\phi(z) - a|} \sigma_m(z), & a \neq \infty \\ \int_{S_m(r)} \log^+ |\phi(z)| \sigma_m(z), & a = \infty \end{cases}.$$

The *characteristic function* of ϕ is defined by $T_\phi(r) = m_\phi(r,\infty) + N_\phi(r,\infty)$. We also define $T_\phi(r,a) := m_\phi(r,a) + N_\phi(r,a), \quad a \neq \infty$. In some cases, we also use the notation: $T_\phi(r,a) = T(r, \dfrac{1}{\phi - a})$ and $m_\phi(r,a) = m(r, \dfrac{1}{\phi - a})$. The first main theorem states that $T_\phi(r,a) = T_\phi(r) + O(1)$. The difference operator of a meromorphic function ϕ is defined by

$$\Delta_{\mathbf{c}}(\phi) = \phi(z_1 + c_1, \ldots, z_m + c_m) - \phi(z_1, \ldots, z_m),$$

where $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{C}^m$. The hyper-order of ϕ is defined by

$$\varsigma(\phi) = \limsup_{r \to \infty} \frac{\log \log T_\phi(r)}{\log r}.$$

Let f be a meromorphic map of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. For arbitrary fixed homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$, we can choose holomorphic functions $f_0, f_1, \ldots, f_n$ defined on $\mathbb{C}^m$ such that $I_f = \{z \in \mathbb{C}^m : f_0(z) = \cdots = f_n(z) = 0\}$ is of dimension at most $m-2$ and $f = (f_0 : \cdots : f_n)$. Usually, the function $\tilde{f} = (f_0, \ldots, f_n) : \mathbb{C}^m \longrightarrow \mathbb{C}^{n+1}$ is called a reduced representation of f. Set $||\tilde{f}(z)|| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$. The characteristic function of f is defined by

$$T_f(r) = \int_{S_m(r)} \log ||\tilde{f}(z)|| \sigma_m(z),$$

where the above definition is independent (up to an additive constant) of the choice of the reduced representation of f. The order of f and the hyper-order f of are defined by

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log r} \quad \text{and} \quad \varsigma(f) = \limsup_{r \to \infty} \frac{\log \log T_f(r)}{\log r},$$

respectively.

Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$, and let

$$L(z_0, \ldots, z_n) = \sum_{j=0}^n a_j z_j$$

be a linear form defined on H, where $a_j \in \mathbb{C},\ j = 0, \dots, n$, are constants. Denote by $\mathbf{a} = (a_0, \dots, a_n)$ the non-zero vector associated with H, and define

$$L(\tilde{f}) = (H, f) = (\mathbf{a}, \tilde{f}) = \sum_{j=0}^{n} a_j f_j.$$

Under the assumption that $(\mathbf{a}, \tilde{f}) \not\equiv 0$ for $1 < r < +\infty$, the proximity function of f with respect to H is defined as follows:

$$m_f(r, H) = \int_{S_m(r)} \log \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}, \tilde{f})(z)|} \sigma_m(z),$$

where the above definition is independent (up to an additive constant) of the choice of the reduced representation of f. The counting function of f is defined to be $N_{\nu_{(H,f)}}(r)$, meaning that $N_{\nu(H,f)}(r) = N_{(H,f)}(r, 0)$. In some cases, we use the notation $N_f(r, H)$ instead of $N_{\nu(H,f)}(r)$.

The Casorati determinant of f is defined by

$$W_{\mathbf{c}}(f) = W_{\mathbf{c}}(f_0, \dots, f_n) = \begin{vmatrix} f_0(z) & f_1(z) & \cdot & f_n(z) \\ f_0(z+\mathbf{c}) & f_1(z+\mathbf{c}) & \cdot & f_n(z+\mathbf{c}) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(z+n\mathbf{c}) & f_1(z+n\mathbf{c}) & \cdot & f_n(z+n\mathbf{c}) \end{vmatrix},$$

where $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{C}^m \setminus \{0\}$.

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve, the Casorati type determinant of f is defined by

$$D_c(f) = D_c(f_0, \dots, f_n) = \begin{vmatrix} f_0'(z) & f_1'(z) & \cdot & f_n'(z) \\ f_0(z+c) & f_1(z+c) & \cdot & f_n(z+c) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(z+nc) & f_1(z+nc) & \cdot & f_n(z+nc) \end{vmatrix},$$

where $c \in \mathbb{C} \setminus \{0\}$.

Let F be a nonzero holomorphic function on $\mathbb{C}^m$ and $z_0 = (z_{0,1}, \dots, z_{0,m}) \in \mathbb{C}^m$ be such that $F(z_0) = 0$ with multiple $p \in \mathbb{N}^*$, then

$$F(z) = \sum_{|k|=p}^{\infty} b_k (z - z_0)^k$$

on a neighborhood of z_0, where $b_k \in \mathbb{C}$ and

$$(z - z_0)^k = (z_1 - z_{0,1})^{k_1} \dots (z_m - z_{0,m})^{k_m}, k_1 + \dots + k_m = |k|, k = (k_1, \dots, k_m) \in \mathbb{N}^m.$$

Observe that on a neighborhood of z_0, we also have

$$F(z + \mathbf{c}) = \sum_{|k|=q}^{\infty} c_k (z - z_0)^k, \quad q \geq 0,$$

where c_k are complex constants.

We denote by $\overset{\sim\mathbf{c}}{N}_F(r,0)$ the counting function at all zeros z_0 of $F(z)$, and observe that z_0 is also a zero of $F(z+\mathbf{c})$ in the following sense. If z_0 is a zero of $F(z)$ with multiplicity $p \geq 1$ and also is a zero of $F(z+\mathbf{c})$ with multiplicity $q \geq 1$, then z_0 is counted $p-q$ times in $\overset{\sim\mathbf{c}}{N}_F(r,0)$. If $q=0$, the point z_0 is counted p times in $\overset{\sim\mathbf{c}}{N}_F(r,0)$. If $F(z)=0$ implies $F(z+\mathbf{c})=0$, then we denote by $\widetilde{N}_{F(z+\mathbf{c})}(r,0)$ the counting function at the points $F(z+\mathbf{c})=0$ when $F(z)=0$ with counting multiplicity. This means that if z_0 is a zero of $F(z)$ with multiple $p \geq 1$ and z_0 also is a zero of $F(z+\mathbf{c})$ with multiple $q \geq 1$, then z_0 is counted q times in $\widetilde{N}_{F(z+\mathbf{c})}(r,0)$. We have $N_F(r,0) = \overset{\sim\mathbf{c}}{N}_F(r,0) + \widetilde{N}_{F(z+\mathbf{c})}(r,0)$. Note that $\overset{\sim\mathbf{c}}{N}_F(r,0)$ may be negative, positive or zero if $F(z) \equiv F(z+\mathbf{c})$.

The following definition was given in Korhonen et. al [9].

Definition 1.1. *Let $n \in \mathbb{N}^*$, $c \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{P}^1(\mathbb{C})$. An a-point z_0 of a meromorphic function $h(z)$ is said to be n-successive and c-separated if the n meromorphic functions $h(z+jc)$ $(j=1,\ldots,n)$ take the value a at $z=z_0$ with multiplicity not less than that of $h(z)$ at $z=z_0$. All the other a-points of $h(z)$ are called n-aperiodic of pace c. By $\tilde{N}{}_h^{[n,c]}(r,a)$ we denote the counting function of n-aperiodic zeros of the function $h-a$ of pace c.*

Therefore, we denote by $\tilde{N}{}^{[n,c]}_{(H,g)}(r,0)$ the counting function of the n-aperiodic zeros of function (H,g) for holomorphic curve $g: \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$. Also, we denote by $N_h^{[n,c]}(r,a)$ (resp. $\overline{N}_h^{[n,c]}(r,a)$) the counting with multiplicity (resp. without counting multiplicity) function of n-successive and c-separated a-points of a function h.

Recall that the hyperplanes $H_1,\ldots,H_q,\ q>n$, in $\mathbb{P}^n(\mathbb{C})$ are said to be in general position if for any distinct $i_1,\ldots,i_{n+1} \in \{1,\ldots,q\}$, we have $\bigcap_{k=1}^{n+1} \operatorname{supp}(H_{i_k}) = \emptyset$, which is equivalent to the $H_{i_1},\ldots,H_{i_{n+1}}$ being linearly independent.

In this paper, we consider the following family of meromorphic maps:

$$\mathcal{F} = \Big\{ f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}) \text{ such that } T_{f_i}(r) \leq O(T_f(r)) \text{ for all } i=0,\ldots,n \Big\}.$$

Observe that $\mathcal{F} \neq \emptyset$, since $f=(f_0:f_1:\cdots:f_n) \in \mathcal{F}$, where $f_i=1$ for some $i \in \{0,\ldots,n\}$. Indeed, we have for all $j \neq i$,

$$\begin{aligned} T_{f_j}(r) &= \int_{S_m(r)} \log^+ |f_j(z)| \sigma_m(z) \leq \int_{S_m(r)} \log(1+\max_{t\in\{0,\ldots,n\}\setminus i}\{|f_t(z)|\})\sigma_m(z) \\ &\leq \int_{S_m(r)} \log ||\tilde{f}(z)|| \sigma_m(z) + O(1) = T_f(r) + O(1). \end{aligned}$$

Now we are in position to state the main results of this paper. The next theorem is a difference analogue of Cartan's second main theorem.

Theorem 1.2. *Let $f = (f_0 : f_1 : \cdots : f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve in $\mathcal{F}$ with $\varsigma(f) = \varsigma < 1$, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that the image of f is not contained in $H_j, j = 1, \ldots, q$. Suppose that $D_c(f) \not\equiv 0$. Then for any $1 < r < +\infty$, we have*

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} (\tilde{N}{}^{[n,c]}_{(H_j,f)}(r,0) + \overline{N}^{[n,c]}_{(H_j,f)}(r,0)) + S(r,f),$$

r lies outside of a possible exceptional set $E \subset [1, \infty)$ of finite logarithmic measure.

As an immediate consequence of Theorem 1.2, we have the following result.

Corollary 1.1. *Let $f = (f_0 : f_1 : \cdots : f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve in $\mathcal{F}$ with $\varsigma(f) = \varsigma < 1$, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that the image of f is not contained in $H_j, j = 1, \ldots, q$. Suppose that $D_c(f) \not\equiv 0$ and for any $1 < r < +\infty$,*

$$\sum_{j=1}^{q} \tilde{N}{}^{[n,c]}_{(H_j,f)}(r,0) = S(r,f).$$

Then we have

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} \overline{N}^{[n,c]}_{(H_j,f)}(r,0) + S(r,f) \leq \sum_{j=1}^{q} \overline{N}_{(H_j,f)}(r,0) + S(r,f)$$

for all r lying outside a of possible exceptional set $E \subset [1, \infty)$ of finite logarithmic measure.

Next, we consider the family $\mathcal{G} \subset \mathcal{F}$ of holomorphic curves with the following properties:
(i) $D_c(f) \not\equiv 0$ for all $f \in \mathcal{G}$;
(ii) Let $H_1, \ldots, H_q, q \geq n+4$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that the image of f is not contained in $H_j, j = 1, \ldots, q$, and $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ for all $i \neq j$, and $f \in \mathcal{G}$. We also assume that $\sum_{j=1}^{q} \tilde{N}{}^{[n,c]}_{(H_j,f)}(r,0) = S(r,f)$ for all $f \in \mathcal{G}$.
(iii) $\varsigma(f) = \varsigma < 1$ for all $f \in \mathcal{G}$.

As an application of Corollary 1.1, we have the following uniqueness theorem for holomorphic curves from $\mathcal{G}$.

Theorem 1.3. *Let f and g be two holomorphic curves in $\mathcal{G}$, and let $H_1, \ldots, H_q, q \geq n+4$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Suppose that $f(z) = g(z)$ on $\cup_{j=1}^{q}(f^{-1}(H_j) \cup g^{-1}(H_j))$. Then we have $f \equiv g$.*

Remark 1.1. In 2010, Z. Chen and Q. Yan [3] have proved a uniqueness theorem for holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ by inverse images of $2n+3$ hyperplanes.

Our Theorem 1.3 gives a uniqueness theorem for holomorphic curves by inverse images of $n+4$ hyperplanes.

Theorem 1.4. *Let $f = (f_0 : f_1 : \cdots : f_n) : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic non-degenerate linear map in $\mathcal{F}$ with $\varsigma(f) = \varsigma < 1$, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that $H_j(f(0)) \neq 0, j = 1, \ldots, q$. Then for any $1 < r < +\infty$, we have*

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q}(\tilde{N}^{\mathbf{c}}_{(H_j,f)}(r,0) + \tilde{N}^{n}_{(H_j,f(z+\mathbf{c}))}(r,0)) + S(r,f),$$

r lies outside of a possible exceptional set $E \subset [1,\infty)$ of finite logarithmic measure.

2. Some results from Nevanlinna theory

In this section we state some known results from Nevanlinna theory that will be used in the proofs of the theorems.

Lemma 2.1 ([1]). *Let f be a non-constant meromorphic function in $\mathbb{C}^m$ such that $f(0) \neq 0, \infty$, and let $\mathbf{c} \in \mathbb{C}^m$. If $\varsigma(f) = \varsigma < 1$, then*

$$m(r, \frac{f(z+\mathbf{c})}{f(z)}) = S(r,f),$$

for all $r > 0$ outside of a possible exceptional set $E \subset [1,+\infty)$ of finite logarithmic measure $\int_E dt/t < +\infty$.

Lemma 2.2 ([8, 10]). *Let $f : \mathbb{C}^m \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function, and let $\mathbf{c} \in \mathbb{C}^m$. If $\varsigma(f) = \varsigma < 1$, then $T_{f(z+\mathbf{c})}(r) \leq T_f(r) + o(T_f(r))$, where $r \to \infty$ outside of an exceptional set of finite logarithmic measure.*

Lemma 2.3 ([8]). *Let f be a non-constant meromorphic function, $\varepsilon > 0$ and $c \in \mathbb{C}$. If $\varsigma(f) < 1$ and $\varepsilon > 0$, then*

$$m(r, \frac{f(z+c)}{f(z)}) = o(\frac{T_f(r)}{r^{1-\varsigma-\varepsilon}})$$

for all r outside of a set of finite logarithmic measure.

3. Proof of Theorems

We first prove a number of lemmas.

Lemma 3.1. *Let $f = (f_0 : f_1 : \cdots : f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve in $\mathcal{F}$ with hyper-order $\varsigma(f) < 1$, and let $H_1, \ldots, H_q$ be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$ such that the image of f is not contained in $H_j, j = 1, \ldots, q$. Let $\mathbf{a}_j$ by the non-zero* *vector associated with* $H_j, j = 1, \ldots, q$. *Suppose that* $D_c(f) \not\equiv 0$. *Then the following inequality*

$$\int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|\tilde{f}(re^{i\theta})\|}{|(\mathbf{a}_l, \tilde{f})(re^{i\theta})|} \frac{d\theta}{2\pi} \leqslant (n+1)T_f(r) - N_{D_c(f)}(r,0) + S(r,f)$$

holds for all r *outside of an exceptional set of finite logarithmic measure. Here the maximum is taken over all subsets* K *of* $\{1, \ldots, q\}$ *such that* $\mathbf{a}_l$, $l \in K$ *are linearly independent.*

Proof. Let $K \subset \{1, \ldots, q\}$ be a set such that $\mathbf{a}_l$ $(l \in K)$ are linearly independent. Without loss of generality, we may assume that $q \geqslant n+1$ and $\#K = n+1$. Let $\mathcal{T}$ be the set of all injective maps $\mu : \{0, 1, \ldots, n\} \to \{1, \ldots, q\}$. Then we can write

$$\begin{aligned}
&\int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|\tilde{f}(re^{i\theta})\|}{|(\mathbf{a}_j, \tilde{f})(re^{i\theta})|} \frac{d\theta}{2\pi} = \int_0^{2\pi} \max_{\mu \in \mathcal{T}} \sum_{l=0}^{n} \log \frac{\|\tilde{f}(re^{i\theta})\|}{|(\mathbf{a}_{\mu(l)}, \tilde{f})(re^{i\theta})|} \frac{d\theta}{2\pi} \\
&\quad = \int_0^{2\pi} \max_{\mu \in \mathcal{T}} \log \left\{ \frac{\|\tilde{f}(re^{i\theta})\|^{n+1}}{\prod_{l=0}^{n} |(\mathbf{a}_{\mu(l)}, \tilde{f})(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} \\
&\quad \leq \int_0^{2\pi} \max_{\mu \in \mathcal{T}} \log \left\{ \frac{\|\tilde{f}(re^{i\theta})\|^{n+1}}{|D_c((\mathbf{a}_{\mu(0)}, \tilde{f}), \ldots, (\mathbf{a}_{\mu(n)}, \tilde{f}))(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} \\
&\quad + \int_0^{2\pi} \max_{\mu \in \mathcal{T}} \log \left\{ \frac{|D_c((\mathbf{a}_{\mu(0)}, \tilde{f}), \ldots, (\mathbf{a}_{\mu(n)}, \tilde{f}))(re^{i\theta})|}{\prod_{l=0}^{n} |(\mathbf{a}_{\mu(l)}, \tilde{f})(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} + O(1) \\
&\quad = \int_0^{2\pi} \log \max_{\mu \in \mathcal{T}} \left\{ \frac{\|\tilde{f}(re^{i\theta})\|^{n+1}}{|D_c((\mathbf{a}_{\mu(0)}, \tilde{f}), \ldots, (\mathbf{a}_{\mu(n)}, \tilde{f}))(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} \\
&\quad + \int_0^{2\pi} \log \max_{\mu \in \mathcal{T}} \left\{ \frac{|D_c((\mathbf{a}_{\mu(0)}, \tilde{f}), \ldots, (\mathbf{a}_{\mu(n)}, \tilde{f}))(re^{i\theta})|}{\prod_{l=0}^{n} |(\mathbf{a}_{\mu(l)}, \tilde{f})(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} + O(1) \\
&\quad \leq \int_0^{2\pi} \log \sum_{\mu \in \mathcal{T}} \frac{|D_c((\mathbf{a}_{\mu(0)}, \tilde{f}), \ldots, (\mathbf{a}_{\mu(n)}, \tilde{f}))(re^{i\theta})|}{\prod_{l=0}^{n} |(\mathbf{a}_{\mu(l)}, \tilde{f})(re^{i\theta})|} \frac{d\theta}{2\pi} \\
&\quad + \int_0^{2\pi} \log \sum_{\mu \in \mathcal{T}} \frac{\|\tilde{f}(re^{i\theta})\|^{n+1}}{|D_c((\mathbf{a}_{\mu(0)}, \tilde{f}), \ldots, (\mathbf{a}_{\mu(n)}, \tilde{f}))(re^{i\theta})|} \frac{d\theta}{2\pi} + O(1).
\end{aligned}$$

By the property of Casorati-type determinant, we see that $|D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\dots,(\mathbf{a}_{\mu(n)},\tilde{f}))| = C_{1,\mu}|D_c(f_0,\dots,f_n)|$, where $C_{1,\mu}>0$ is a constant. So, we obtain

$$\begin{aligned}\int_0^{2\pi}\max_K\sum_{l\in K}\log\frac{\|\tilde{f}(re^{i\theta})\|}{|(\mathbf{a}_l,\tilde{f})(re^{i\theta})|}\frac{d\theta}{2\pi} &\leqslant \int_0^{2\pi}\log\sum_{\mu\in T}\frac{|D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\dots,(\mathbf{a}_{\mu(n)},\tilde{f}))(re^{i\theta})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde{f})(re^{i\theta})|}\frac{d\theta}{2\pi}\\ &\quad+\int_0^{2\pi}\log\frac{\|\tilde{f}(re^{i\theta})\|^{n+1}}{|D_c(f_0,\dots,f_n)(re^{i\theta})|}\frac{d\theta}{2\pi}+O(1).\end{aligned}\tag{3.1}$$

We have

$$\frac{D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\dots,(\mathbf{a}_{\mu(n)},\tilde{f}))(z)}{\prod\limits_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde{f})(z)} = \begin{vmatrix} \frac{(a_{\mu(0)},\tilde{f})'(z)}{(\mathbf{a}_{\mu(0)},\tilde{f})(z)} & \frac{(\mathbf{a}_{\mu(1)},\tilde{f})'(z)}{(\mathbf{a}_{\mu(1)},\tilde{f})(z)} & . & \frac{(\mathbf{a}_{\mu(n)},\tilde{f})'(z)}{(\mathbf{a}_{\mu(n)},\tilde{f})(z)}\\ \frac{(\mathbf{a}_{\mu(0)},\tilde{f})(z+c)}{(\mathbf{a}_{\mu(0)},\tilde{f})(z)} & \frac{(\mathbf{a}_{\mu(1)},\tilde{f})(z+c)}{(\mathbf{a}_{\mu(1)},\tilde{f})(z)} & . & \frac{(\mathbf{a}_{\mu(n)},\tilde{f})(z+c)}{(\mathbf{a}_{\mu(n)},\tilde{f})(z)}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{(\mathbf{a}_{\mu(0)},\tilde{f})(z+nc)}{(\mathbf{a}_{\mu(0)},\tilde{f})(z)} & \frac{(\mathbf{a}_{\mu(1)},\tilde{f})(z+nc)}{(\mathbf{a}_{\mu(1)},\tilde{f})(z)} & . & \frac{(\mathbf{a}_{\mu(n)},\tilde{f})(z+nc)}{(\mathbf{a}_{\mu(n)},\tilde{f})(z)} \end{vmatrix}.$$

By Lemma 2.3, we obtain

$$m\Big(r,\frac{(\mathbf{a}_{\mu(l)},\tilde{f})(z+jc)}{(\mathbf{a}_{\mu(l)},\tilde{f})(z)}\Big)=o(T_{(\mathbf{a}_{\mu(l)},\tilde{f})(z)}(r)),\tag{3.2}$$

for all $r>0$ outside of a possible exceptional set $E\subset[1,+\infty)$ of finite logarithmic measure $\int_E dt/t<+\infty$, for all $l=0,\dots,n$ and for all $j=1,\dots,n$. We have

$$T_{(\mathbf{a}_{\mu(l)},\tilde{f})(z)}(r)\le\sum_{j=0}^{n}T_{f_j}(r)+O(1)\le O(T_f(r))$$

for all $l=0,\dots,n$. Thus, (3.2) implies

$$\|\quad m\Big(r,\frac{(\mathbf{a}_{\mu(l)},\tilde{f})(z+jc)}{(\mathbf{a}_{\mu(l)},\tilde{f})(z)}\Big)=o(T_f(r)),\tag{3.3}$$

for all $l=0,\dots,n$ and for all $j=1,\dots,n$.

From (3.3) and the lemma on the logarithmic derivative, for any $\mu\in\mathcal{T}$, we have

$$\int_0^{2\pi}\log^+\frac{|D_c((\mathbf{a}_{\mu(0)},\tilde{f}),\dots,(\mathbf{a}_{\mu(n)},\tilde{f}))(re^{i\theta})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde{f})(re^{i\theta})|}\frac{d\theta}{2\pi}\leqslant S(r,f).$$

This implies that

$$\int_0^{2\pi} \log \sum_{\mu\in\mathcal{T}} \frac{|D_c((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(re^{i\theta})|}{\prod\limits_{l=0}^{n} |(\mathbf{a}_{\mu(l)},\tilde f)(re^{i\theta})|}\frac{d\theta}{2\pi}$$
$$\leqslant \int_0^{2\pi} \log^+ \sum_{\mu\in\mathcal{T}} \frac{|D_c((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(re^{i\theta})|}{\prod\limits_{l=0}^{n} |(\mathbf{a}_{\mu(l)},\tilde f)(re^{i\theta})|}\frac{d\theta}{2\pi}$$
$$\leqslant \sum_{\mu\in\mathcal{T}} \int_0^{2\pi} \log^+ \frac{|D_c((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(re^{i\theta})|}{\prod\limits_{l=0}^{n} |(\mathbf{a}_{\mu(l)},\tilde f)(re^{i\theta})|}\frac{d\theta}{2\pi} + O(1) \leqslant S(r,f). \tag{3.4}$$

Now the statement of the lemma follows from (3.1), (3.4) and Jensen's formula. Lemma 3.1 is proved. □

Lemma 3.2. *(see [5]) Let $f_0, f_1, \dots, f_n$ be linearly independent meromorphic functions in $\mathbb{C}^m$, and let $f = (f_0, f_1, \dots, f_n)$. Then there are multi-indices $\nu_i \in \mathbb{Z}_+^m$, $i = 1,\dots,n$ such that $0 < |\nu_i| \le i$ and $f, \partial^{\nu_1} f, \dots, \partial^{\nu_n} f$ are linearly independent over $\mathbb{C}^m$.*

Fix multi-indices $\nu_i \in \mathbb{Z}_+^m$ with $\nu_0 = 0$ and $|\nu_i| > 0$ $(i = 1,\dots,n)$, and set $l = |\nu_1| + \dots + |\nu_n|$. For meromorphic functions $f_0,\dots,f_n$ in $\mathbb{C}^m$, the Wronskian determinant is defined by

$$W(f_0,\dots,f_n) = W_{\nu_1\dots\nu_n}(f_0,\dots,f_n) = \begin{vmatrix} f_0 & f_1 & \cdot & f_n \\ \partial^{\nu_1} f_0 & \partial^{\nu_1} f_1 & \cdot & \partial^{\nu_1} f_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{\nu_n} f_0 & \partial^{\nu_n} f_1 & \cdot & \partial^{\nu_n} f_n \end{vmatrix}.$$

Observe that if $f_0, f_1, \dots, f_n$ are linearly independent meromorphic functions in $\mathbb{C}^m$, then $W(f_0,\dots,f_n) \not\equiv 0$.

Lemma 3.3. *Let $f = (f_0 : \dots : f_n) : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a non-degenerate meromorphic map in $\mathcal{F}$ with $\varsigma(f) < 1$, and let $H_1,\dots,H_q$ be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$ such that $H_j(f(0)) \neq 0, j = 1,\dots,q$. Let $\mathbf{a}_j$ be the non-zero vector associated with $H_j, j = 1,\dots,q$. Then the following inequality*

$$\int_{S_m(r)} \max_K \sum_{l\in K} \log \frac{\|\tilde f(z)\|}{|(\mathbf{a}_l,\tilde f)(z)|}\sigma_m(z) \leqslant (n+1)T_f(r) - N_{W(f(z+\mathbf{c}))}(r,0) + S(r,f)$$

holds for all r outside of an exceptional set of finite logarithmic measure. Here the maximum is taken over all subsets K of $\{1,\dots,q\}$ such that $\mathbf{a}_l$ $(l \in K)$ are linearly independent.

Proof. By Lemma 3.2, there are multi-indices $\nu_i \in \mathbb{Z}_+^m (i = 1,\dots,n)$ such that $0 < |\nu_i| \le i$ and $\tilde f(z+\mathbf{c}), \partial^{\nu_1}\tilde f(z+\mathbf{c}),\dots,\partial^{\nu_n}\tilde f(z+\mathbf{c})$ are linearly independent over

$\mathbb{C}^m$. Therefore, we have $W(f_0(z+\mathbf{c}),\dots,f_n(z+\mathbf{c}))\not\equiv 0$. Let $K\subset\{1,\dots,q\}$ be a set such that $\mathbf{a}_l$ $(l\in K)$ are linearly independent. Without loss of generality, we may assume that $q\geqslant n+1$ and $\#K=n+1$. Let $\mathcal{T}$ be the set of all injective maps $\mu:\{0,1,\dots,n\}\to\{1,\dots,q\}$. Then, we can write

$$\begin{aligned}
&\int_{S_m(r)}\max_K\sum_{l\in K}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_j,\tilde f)(z)|}\sigma_m(z)=\int_{S_m(r)}\max_{\mu\in\mathcal{T}}\sum_{l=0}^{n}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\sigma_m(z)\\
&\quad=\int_{S_m(r)}\max_{\mu\in\mathcal{T}}\log\left\{\frac{\|\tilde f(z)\|^{n+1}}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\right\}\sigma_m(z)\\
&\quad\le\int_{S_m(r)}\max_{\mu\in\mathcal{T}}\log\left\{\frac{\|\tilde f(z)\|^{n+1}}{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}\right\}\sigma_m(z)\\
&\quad+\int_{S_m(r)}\max_{\mu\in\mathcal{T}}\log\left\{\frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\right\}\sigma_m(z)+O(1)\\
&\quad=\int_{S_m(r)}\log\max_{\mu\in\mathcal{T}}\left\{\frac{\|\tilde f(z)\|^{n+1}}{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}\right\}\sigma_m(z)\\
&\quad+\int_{S_m(r)}\log\max_{\mu\in\mathcal{T}}\left\{\frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\right\}\sigma_m(z)+O(1)\\
&\quad\le\int_{S_m(r)}\log\sum_{\mu\in\mathcal{T}}\frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\sigma_m(z)\\
&\qquad+\int_{S_m(r)}\log\sum_{\mu\in\mathcal{T}}\frac{\|\tilde f(z)\|^{n+1}}{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}\sigma_m(z)+O(1).
\end{aligned}$$

By the property of Wronskian determinant, we get $|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|=C_{2,\mu}|W(f_0,\dots,f_n)(z+\mathbf{c})|$, where $C_{2,\mu}>0$ is a constant. So, we obtain

$$\begin{aligned}
(3.5)\qquad&\int_{S_m(r)}\max_K\sum_{l\in K}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_l,\tilde f)(z+\mathbf{c})|}\sigma_m(z)\\
&\qquad\leqslant\int_{S_m(r)}\log\sum_{\mu\in T}\frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\sigma_m(z)\\
&\qquad\quad+\int_{S_m(r)}\log\frac{\|\tilde f(z)\|^{n+1}}{|W(f_0,\dots,f_n)(z+\mathbf{c})|}\sigma_m(z)+O(1).
\end{aligned}$$

Next, we have

$$\frac{W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})}{\prod_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z)}$$

$$= \frac{W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})}{\prod_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}\cdot\frac{\prod_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{\prod_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z)}$$

$$= \begin{vmatrix} 1 & 1 & \cdot & 1 \\ \frac{\partial^{\nu_1}(\mathbf{a}_{\mu(0)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(0)},\tilde f)(z+\mathbf{c})} & \frac{\partial^{\nu_1}(\mathbf{a}_{\mu(1)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(1)},\tilde f)(z+\mathbf{c})} & \cdot & \frac{\partial^{\nu_1}(\mathbf{a}_{\mu(n)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(n)},\tilde f)(z+\mathbf{c})} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{\nu_n}(\mathbf{a}_{\mu(0)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(0)},\tilde f)(z+\mathbf{c})} & \frac{\partial^{\nu_n}(\mathbf{a}_{\mu(1)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(1)},\tilde f)(z+\mathbf{c})} & \cdot & \frac{\partial^{\nu_n}(\mathbf{a}_{\mu(n)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(n)},\tilde f)(z+\mathbf{c})} \end{vmatrix}$$

$$\cdot\frac{\prod_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{\prod_{l=0}^{n}(\mathbf{a}_{\mu(l)},\tilde f)(z)}.$$

By Lemma 2.1, we obtain

$$m\Big(r,\frac{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(l)},\tilde f)(z)}\Big)=o(T_{(\mathbf{a}_{\mu(l)},\tilde f)(z)}(r)), \tag{3.6}$$

for all $r>0$ outside of a possible exceptional set $E\subset[1,+\infty)$ of finite logarithmic measure $\int_E dt/t<+\infty$, for all $l=0,\dots,n$.

By Lemma 2.2, we have

$$T_{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}(r)\le T_{(\mathbf{a}_{\mu(l)},\tilde f)(z)}(r)+S(r,f)\le\sum_{j=0}^{n}T_{f_j}(r)+S(r,f)\le O(T_f(r))$$

for all $l=0,\dots,n$. Thus, (3.6) implies that

$$\| \quad m\Big(r,\frac{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(l)},\tilde f)(z)}\Big)=o(T_f(r)), \tag{3.7}$$

for all $l=0,\dots,n$.

Hence, by the lemma on the logarithmic derivative of several variables, for any $\mu\in\mathcal{T}$, we have

$$\int_{S_m(r)}\log^+\frac{\partial^{\nu_i}(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}\sigma_m(z)=o(T_{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}(r))=S(r,f),$$

for all $l=0,\dots,n$ and $i=1,\dots,n$. Therefore

$$\int_{S_m(r)}\log^+\frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})|}\sigma_m(z)\leqslant S(r,f). \tag{3.8}$$

Next, in view of (3.7) and (3.8), we have

$$\begin{aligned}
&\int_{S_m(r)} \log \sum_{\mu\in\mathcal{T}} \frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z)|}\sigma_m(z)\\
&\qquad\leqslant \int_{S_m(r)} \log^+ \sum_{\mu\in\mathcal{T}} \frac{|W((\mathbf{a}_{\mu(0)},\tilde f),\dots,(\mathbf{a}_{\mu(n)},\tilde f))(z+\mathbf{c})|}{\prod\limits_{l=0}^{n}|(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})|}\sigma_m(z)\\
&\qquad+\sum_{l=0}^{n}\int_{S_m(r)}\log^+\Big|\frac{(\mathbf{a}_{\mu(l)},\tilde f)(z+\mathbf{c})}{(\mathbf{a}_{\mu(l)},\tilde f)(z)}\Big|\sigma_m(z)\leqslant S(r,f). \qquad (3.9)
\end{aligned}$$

The statement of the lemma follows from (3.5), (3.9) and Jensen's formula. □

Lemma 3.4. *Let $f=(f_0:\dots:f_n):\mathbb{C}^m\longrightarrow\mathbb{P}^n(\mathbb{C})$ be a meromorphic map, and let $H_1,\dots,H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position such that the image of f is not contained in $H_j, j=1,\dots,q$. Let $\mathbf{a}_j$ be the vector associated with H_j for $j=1,\dots,q$. Then*

$$\sum_{j=1}^{q} m_f(r,H_j)\leqslant \int_{S_m(r)}\max_K\sum_{l\in K}\log\frac{\|\tilde f(z)\|}{|(\mathbf{a}_l,\tilde f)(z)|}\sigma_m(z)+O(1),$$

where the maximum is taken over all subsets K of $\{1,\dots,q\}$ such that $\#K=n+1$.

Proof. Let $\mathbf{a}_j=(a_{j,0},\dots,a_{j,n})$ be the associated vector of H_j, $1\leqslant j\leqslant q$, and let $\mathcal{T}$ be the set of all injective maps $\mu:\{0,1,\dots,n\}\longrightarrow\{1,\dots,q\}$. Since by hypothesis $H_1,\dots,H_q$ are in general position, for any $\mu\in\mathcal{T}$, the vectors $\mathbf{a}_{\mu(0)},\dots,\mathbf{a}_{\mu(n)}$ are linearly independent.

Let $\mu\in\mathcal{T}$, we have

$$(\tilde f,\mathbf{a}_{\mu(t)})=a_{\mu(t),0}f_0+\dots+a_{\mu(t),n}f_n,\quad t=0,1,\dots,n. \qquad (3.10)$$

Solve the system of linear equations (3.10), to get

$$f_t=b_{\mu(t),0}(\mathbf{a}_{\mu(0)},\tilde f)+\dots+b_{\mu(t),n}(\mathbf{a}_{\mu(n)},\tilde f),\quad t=0,1,\dots,n,$$

where $\Big(b_{\mu(t),j}\Big)_{t,j=0}^{n}$ is the inverse of the matrix $\Big(a_{\mu(t),j}\Big)_{t,j=0}^{n}$. So, there is a constant C_μ to satisfy

$$\|\tilde f(z)\|\leqslant C_\mu\max_{0\leqslant t\leqslant n}|(\mathbf{a}_{\mu(t)},\tilde f)(z)|.$$

Set $C=\max_{\mu\in\mathcal{T}}C_\mu$. Then for any $\mu\in\mathcal{T}$, we have

$$\|\tilde f(z)\|\leqslant C\max_{0\leqslant t\leqslant n}|(\mathbf{a}_{\mu(t)},\tilde f)(z)|.$$

For any $z\in\mathbb{C}^m\setminus\{\cup_{j=1}^{q}(H_j(\tilde f))^{-1}(0)\cup I_f\}$, there exists a mapping $\mu\in\mathcal{T}$ such that for $j\notin\{\mu(0),\dots,\mu(n)\}$,

$$0<|(\mathbf{a}_{\mu(0)},\tilde f)(z)|\leqslant|(\mathbf{a}_{\mu(1)},\tilde f)(z)|\leqslant\dots\leqslant|(\mathbf{a}_{\mu(n)},\tilde f)(z)|\leqslant|(\mathbf{a}_j,\tilde f)(z)|.$$

Therefore, we have

$$\prod_{j=1}^{q} \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}_j, \tilde{f})(z)|} \leqslant C^{q-n-1} \max_{\mu \in \mathcal{T}} \prod_{t=0}^{n} \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}_{\mu(t)}, \tilde{f})(z)|}.$$

Next, we have

$$\sum_{j=1}^{q} m_f(r, H_j) = \sum_{j=1}^{q} \int_{S_m(r)} \log \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}_j, \tilde{f})(z)|} \sigma_m(z) = \int_{S_m(r)} \log \prod_{j=1}^{q} \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}_j, \tilde{f})(z)|} \sigma_m(z)$$

$$\leqslant \int_{S_m(r)} \log \max_{\mu \in \mathcal{T}} \prod_{t=0}^{n} \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}_{\mu(t)}, \tilde{f})(z)|} \sigma_m(z) + O(1) = \int\limits_{S_m(r)} \max_{\mu \in T} \log \prod_{t=0}^{n} \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}_{\mu(t)}, \tilde{f})(z)|} \sigma_m(z)$$

$$+ O(1) = \int_{S_m(r)} \max_{\mu \in T} \sum_{t=0}^{n} \log \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}_{\mu(t)}, \tilde{f})(z)|} \sigma_m(z) + O(1).$$

Finally, we obtain

$$\sum_{j=1}^{q} m_f(r, H_j) \leqslant \int_{S_m(r)} \max_{K} \sum_{j \in K} \log \frac{\|\tilde{f}(z)\|}{|(\mathbf{a}_j, \tilde{f})(z)|} \sigma_m(z) + O(1).$$

This completes the proof of lemma 3.4. □

Now we are in position to prove the main results of this paper.

Proof of Theorem 1.2. By Lemmas 3.1 and 3.4, we obtain

$$\begin{aligned} \sum_{j=1}^{q} m_f(r, H_j) &\leqslant \int_0^{2\pi} \max_{K} \sum_{l \in K} \log \frac{\|\tilde{f}(re^{i\theta})\|}{|(\mathbf{a}_l, \tilde{f})(re^{i\theta})|} \frac{d\theta}{2\pi} \\ &\leqslant (n+1)T_f(r) - N_{D_c(f)}(r, 0) + S(r, f). \end{aligned} \tag{3.11}$$

By the first main theorem, we get $T_f(r) = N_{(H_j, f)}(r, 0) + m_f(r, H_j) + O(1)$ for any $j \in \{1, \ldots, q\}$. So, from (3.11), we have

$$(q - n - 1)T_f(r) \leqslant \sum_{j=1}^{q} N_{(H_j, f)}(r, 0) - N_{D_c(f)}(r, 0) + S(r, f).$$

For $z_0 \in \mathbb{C}$, we may assume that z_0 is a zero of $(\mathbf{a}_j, \tilde{f})$ for $1 \leqslant j \leq q_1 \leq n$, and $(\mathbf{a}_j, \tilde{f})$ does not vanish at z_0 for $j > q_1$. Without loss of generality, we may assume that $z_0 \in \mathbb{C}$ is an n-successive and c-separated zero of $(\mathbf{a}_j, \tilde{f})$ for $1 \leqslant j \leqslant p_1 \leq q_1 \leq n$. Hence, there exist integers k_j $(j = 1, \ldots, q)$ and nowhere vanishing holomorphic functions g_j $(j = 1, \ldots, q)$, defined on a neighborhood U of z_0, such that

$$(\mathbf{a}_j, \tilde{f})(z) = (z - z_0)^{k_j} g_j(z), \text{ for } j = 1, \ldots, q,$$

where $k_j = 0$ for $q_1 < j \leqslant q$. Also, we can assume that $k_j \geqslant 2$ for $1 \leqslant j \leq p_0$, and $k_j = 1$ for $p_0 < j \leqslant p_1$. From the definition of n-successive and c-separated 0-point, we have

$$(\mathbf{a}_j, \tilde{f})(z + kc) = (z - z_0)^{l_j} h_j^k(z), \text{ for } j = 1, \ldots, p_1,$$

for all $k = 1, \ldots, n$, where h_j^k $(j = 1, \ldots, p_1)$ are nowhere vanishing holomorphic functions, defined on a neighborhood U of z_0, and $l_j \geq k_j, 1 \leq j \leq p_1$. Let $\mathcal{T}$ be the set of all injective maps $\mu : \{0, 1, \ldots, n\} \to \{1, \ldots, q\}$. By a property of Wronskian, there exists a constant $\mathcal{C}_\mu \neq 0$ such that

$$D_c(f) = \mathcal{C}_\mu . \begin{vmatrix} (\mathbf{a}_{\mu(0)}, \tilde{f})' & (\mathbf{a}_{\mu(1)}, \tilde{f})' & \cdot & (\mathbf{a}_{\mu(n)}, \tilde{f})' \\ (\mathbf{a}_{\mu(0)}, \tilde{f})(z+c) & (\mathbf{a}_{\mu(1)}, \tilde{f})(z+c) & \cdot & (\mathbf{a}_{\mu(n)}, \tilde{f})(z+c) \\ \vdots & \vdots & \ddots & \vdots \\ (\mathbf{a}_{\mu(0)}, \tilde{f})(z+nc) & (\mathbf{a}_{\mu(1)}, \tilde{f})(z+nc) & \cdot & (\mathbf{a}_{\mu(n)}, \tilde{f})(z+nc) \end{vmatrix}$$
$$= \prod_{j=1}^{p_0} (z - z_0)^{k_j - 1} h(z),$$

where $h(z)$ is a holomorphic function on U. Then $D_c(f)$ vanishes at z_0 with order at least $\sum_{j=1}^{p_0} (k_j - 1)$. By the definitions of $N^{[n,c]}_{(H_j,f)}(r, 0)$ and $N_{D_c(f)}(r, 0)$, we have

$$\sum_{j=1}^{q} N^{[n,c]}_{(H_j,f)}(r, 0) - N_{D_c(f)}(r, 0) \leqslant \sum_{j=1}^{q} \overline{N}^{[n,c]}_{(H_j,f)}(r, 0).$$

Therefore, we get

$$(q - n - 1)T_f(r) \leqslant \sum_{j=1}^{q} (\widetilde{N}^{[n,c]}_{(H_j,f)}(r, 0) + \overline{N}^{[n,c]}_{(H_j,f)}(r, 0)) + S(r, f),$$

r lies outside of a exceptional set $E \subset [1, \infty)$ of finite logarithmic measure. $\square$

Proof of Theorem 1.3. We denote $f = (f_0 : \cdots : f_n)$ and $g = (g_0 : \cdots : g_n)$, and assume that $f \not\equiv g$. Then there are two numbers $\alpha, \beta \in \{0, \ldots, n\}$, $\alpha \neq \beta$ such that $f_\alpha g_\beta \not\equiv f_\beta g_\alpha$. Assume that $z_0 \in \mathbb{C}$ is a zero of (H_j, f) for some $j = 1, \ldots, q$, then z_0 is a zero of at most n entire functions $(H_t, f), t \in \{1, \ldots, q\}$. Since $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$ for all $i \neq j$, then z_0 is a zero of one entire function (H_j, f) for some $j \in \{1, \ldots, q\}$. From condition $f(z) = g(z)$, when $z \in \cup_{j=1}^{q} (f^{-1}(H_j) \cup g^{-1}(H_j))$, we get $f(z_0) = g(z_0)$. This implies that z_0 is a zero of $\dfrac{f_\alpha}{f_\beta} - \dfrac{g_\alpha}{g_\beta}$. Therefore, we have

$$\sum_{j=1}^{q} \overline{N}_{(H_j,f)}(r, 0) \leq N_{\frac{f_\alpha}{f_\beta} - \frac{g_\alpha}{g_\beta}}(r, 0) \leq T_f(r) + T_g(r) + O(1).$$

Applying Corollary 1.1, we obtain

$$\|(q - n - 1)T_f(r) \leq T_f(r) + T_g(r) + o(T_f(r)). \tag{3.12}$$

Similarly, we get

$$\|(q - n - 1)T_g(r) \leq T_f(r) + T_g(r) + o(T_g(r)). \tag{3.13}$$

Finally, combining (3.12) and (3.13), we obtain $\|(q - n - 3)(T_f(r) + T_g(r)) \leq o(T_f(r)) + o(T_g(r))$, which contradicts the condition $q \geq n + 4$. Hence $f \equiv g$. $\square$

Proof of Theorem 1.4. By Lemmas 3.3 and 3.4, we have

$$\begin{aligned}\sum_{j=1}^{q} m_f(r, H_j) &\leqslant \int_{S_m(r)} \max_K \sum_{l\in K} \log \frac{\|\tilde f(z)\|}{|(\mathbf{a}_l, \tilde f)(z)|}\sigma_m(z)\\ &\leqslant (n+1)T_f(r) - N_{W(f(z+\mathbf{c}))}(r,0) + S(r,f). \end{aligned} \tag{3.14}$$

Next, by the first main theorem, we get

$$T_f(r) = N_{(H_j,f)}(r,0) + m_f(r,H_j) + O(1)$$

for any $j \in \{1,\dots,q\}$. So, in view of (3.14), we can write

$$\begin{aligned}(q-n-1)T_f(r) &\leqslant \sum_{j=1}^{q} N_{(H_j,f)}(r,0) - N_{W(f(z+\mathbf{c}))}(r,0) + S(r,f)\\ &= \sum_{j=1}^{q}[\widetilde{N}^{\mathbf{c}}_{(H_j,f)}(r,0) + \widetilde{N}_{(H_j,f(z+\mathbf{c}))}(r,0)] \quad - N_{W(f(z+\mathbf{c}))}(r,0) + S(r,f)\\ &= \sum_{j=1}^{q}\widetilde{N}^{\mathbf{c}}_{(H_j,f)}(r,0) + \sum_{j=1}^{q}\widetilde{N}_{(H_j,f(z+\mathbf{c}))}(r,0) - N_{W(f(z+\mathbf{c}))}(r,0) + S(r,f).\end{aligned}$$

We assume that z_0 is a zero of (H_j,f) with multiple $k_j > 0, 1 \le j \le q_1 \le n$, and $k_j > n$ when $1 \le j \le q_0$, $k_j < n$ when $q_0 < j \le q_1$ and $k_j = 0$ when $q_1 < j \le q$. Hence, we may assume that z_0 is also a zero of $(H_j, f(z+\mathbf{c}))$ with multiple l_j, $l_j \ge 0, 1 \le j \le q_1$, and $l_j > n$ when $1 \le j \le p_0$, $1 \le l_j \le n$ when $p_0 < j \le p_1$ and $l_j = 0$ when $p_1 < j \le q_1$.

Therefore, it is easy to see that z_0 is counted in $N_{W(f(z+\mathbf{c}))}(r,0)$ with order at least $\sum_{j=1}^{p_0}(l_j - n)$. Then, we have

$$\sum_{j=1}^{q}\widetilde{N}_{(H_j,f(z+\mathbf{c})}(r,0) - N_{W(f(z+\mathbf{c}))}(r,0) \le \sum_{j=1}^{q}\widetilde{N}^{n}_{(H_j,f(z+\mathbf{c}))}(r,0).$$

Finally, we get

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q}(\widetilde{N}^{\mathbf{c}}_{(H_j,f)}(r,0) + \widetilde{N}^{n}_{(H_j,f(z+\mathbf{c}))}(r,0)) + S(r,f).$$

This completes the proof of theorem 1.3. □

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