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LIMIT THEOREMS FOR TAPERED TOEPLITZ QUADRATIC FUNCTIONALS OF CONTINUOUS-TIME GAUSSIAN STATIONARY PROCESSES

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Abstract. Let $\{X(t),\, t \in \mathbb{R}\}$ be a centered real-valued stationary Gaussian process with spectral density f. The paper considers a question concerning asymptotic distribution of tapered Toeplitz type quadratic functional Q_T^h of the process $X(t)$, generated by an integrable even function g and a taper function h. Sufficient conditions in terms of functions f, g and h ensuring central limit theorems for standard normalized quadratic functionals Q_T^h are obtained, extending the results of Ginovyan and Sahakyan (Probab. Theory Relat. Fields 138 (2007), 551–579) to the tapered case and sharpening the results of Ginovyan and Sahakyan (Electronic Journal of Statistics 13 (2019), 255–283) for the Gaussian case.

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1. Introduction

1.1. **The problem.** Let $\{X(t),\, t \in \mathbb{R}\}$ be a centered real-valued stationary Gaussian process with spectral density $f(\lambda)$ and covariance function $r(t)$. The functions $r(t)$ and $f(\lambda)$ are connected by the Fourier integral:

$$r(t) = \int_{\mathbb{R}} e^{i\lambda t} f(\lambda)\, d\lambda. \tag{1.1}$$

We consider a question concerning asymptotic distribution (as $T \to \infty$) of the following tapered Toeplitz type quadratic functional of the process $X(t)$:

$$Q_T^h = \int_0^T \int_0^T \widehat{g}(t-s) h_T(t) h_T(s) X(t) X(s)\, dt\, ds, \tag{1.2}$$

where

$$\widehat{g}(t) = \int_{\mathbb{R}} e^{i\lambda t} g(\lambda) d\lambda, \quad t \in \mathbb{R}. \tag{1.3}$$

is the Fourier transform of some integrable even function $g(\lambda)$, $\lambda \in \mathbb{R}$, and $h_T(t) = h(t/T)$ with a taper function $h(\cdot)$ to be specified below.

We refer to $g(\lambda)$ and to its Fourier transform $\widehat{g}(t)$ as a *generating function* and *generating kernel* for the functional Q_T^h, respectively.

Throughout the paper we assume that the taper function $h(\cdot)$ satisfies the following assumption.

Assumption (T). The taper $h : \mathbb{R} \to \mathbb{R}$ is a continuous nonnegative function of bounded variation and of bounded support $[0, 1]$, such that $H_2 \neq 0$, where

$$H_k := \int_0^1 h^k(t)dt, \quad k \in \mathbb{N} := \{1, 2, \ldots\}. \tag{1.4}$$

Remark 1.1. *The case where $h(t) = \mathbb{I}_{[0,1]}(t)$, where $\mathbb{I}_{[0,1]}(\cdot)$ denotes the indicator of the segment $[0, 1]$, will be referred to as the non-tapered case, and the corresponding non-tapered quadratic functional will be denoted by Q_T.*

The limit distribution of the functional (1.2) is completely determined by the functions f, g and h, and depending on their properties it can be either Gaussian (that is, Q_T^h with an appropriate normalization obey central limit theorem), or non-Gaussian. We naturally arise the following two questions:

a) Under what conditions on f, g and h will the limits be Gaussian?

b) Describe the limit distributions, if they are non-Gaussian.

In this paper we discuss the question a), and obtain sufficient conditions in terms of functions f, g and h ensuring central limit theorems for a standard normalized tapered quadratic functional Q_T^h, extending the results of Ginovyan and Sahakyan [17] to the tapered case and sharpening the results of Ginovyan and Sahakyan [18] for the Gaussian case.

1.2. **Statistical motivation.** Quadratic functionals of the form (1.2) appear both in nonparametric and parametric estimation of the spectrum of the process $X(t)$ based on the tapered data:

$$\{h_T(t)X(t),\ 0 \leq t \leq T\}. \tag{1.5}$$

For instance, when we are interested in nonparametric estimation of a linear integral functional in $L^p(\mathbb{R})$, $p > 1$ of the form:

$$J = J(f) = \int_{\mathbb{R}} f(\lambda)g(\lambda)d\lambda, \tag{1.6}$$

where $g(\lambda) \in L^q(\mathbb{R}),\ 1/p + 1/q = 1$, then a natural statistical estimator for $J(f)$ is the linear integral functional of the empirical periodogram of the process $X(t)$. To define this estimator, we first introduce some notation.

Denote by $H_{k,T}(\lambda)$ the continuous-time tapered Dirichlet type kernel, defined by

$$H_{k,T}(\lambda) = \int_{\mathbb{R}} h_T^k(t)e^{-i\lambda t}dt = \int_0^T h_T^k(t)e^{-i\lambda t}dt. \tag{1.7}$$

Define the finite Fourier transform of the tapered data (1.5):

$$d_T^h(\lambda) = \int_0^T h_T(t)X(t)e^{-i\lambda t}dt, \tag{1.8}$$

and the tapered continuous periodogram $I_T^h(\lambda)$ of the process $X(t)$:

$$\begin{aligned} I_T^h(\lambda) : &= \frac{1}{C_T} d_T^h(\lambda) d_T^h(-\lambda) = \frac{1}{C_T}\left|\int_0^T h_T(t)X(t)e^{-i\lambda t}dt\right|^2 \\ &= \frac{1}{C_T}\int_0^T\int_0^T h_T(t)h_T(s)e^{-i\lambda(t-s)}X(t)X(s)dtds, \end{aligned} \tag{1.9}$$

where

$$C_T := 2\pi H_{2,T}(0) = 2\pi\int_0^T h_T^2(t)dt = 2\pi H_2\, T \neq 0. \tag{1.10}$$

Notice that for non-tapered case ($h(t) = \mathbb{I}_{[0,1]}(t)$), we have $C_T = 2\pi T$.

As an estimator J_T^h for functional $J(f)$, given by (1.6), based on the tapered data (1.5), we consider the averaged tapered periodogram (or a simple "plug-in"statistic), defined by

$$\begin{aligned} J_T^h &= J(I_T^h) := \int_{\mathbb{R}} I_T^h(\lambda)g(\lambda)d\lambda \\ &= \frac{1}{C_T}\int_0^T\int_0^T \widehat{g}(t-s)h_T(t)h_T(s)X(t)X(s)\,dt\,ds, \end{aligned} \tag{1.11}$$

where C_T is as in (1.10), and $\widehat{g}(t)$ is the Fourier transform of function $g(\lambda)$ given by (1.3).

In view of (1.2) and (1.11) we have

$$J_T^h = C_T^{-1}Q_T^h, \tag{1.12}$$

and thus, to study the asymptotic properties of the estimator J_T^h, we have to study the asymptotic distribution (as $T \to \infty$) of the tapered Toeplitz type quadratic functional Q_T^h given by (1.2).

Some brief history. The question of describing the asymptotic distribution of non-tapered Toeplitz type quadratic forms and functionals of stationary processes has a long history, and goes back to the classical monograph by Grenander and Szegö [23], where the problem was considered as an application of authors' theory of the asymptotic behavior of the trace of products of truncated Toeplitz matrices and operators.

Later the problem have been studied by a number of authors. Here we mention only some significant contributions. For discrete-time short memory processes, the problem was studied by Ibragimov [26] and M. Rosenblatt [29], in connection with statistical estimation of the spectral and covariance functions, respectively. Since 1986, there has been a renewed interest in this problem, related to the statistical inferences for long memory (long-range dependent) and intermediate memory (anti-persistent) processes (see, e.g., Avram [1], Fox and Taqqu [12], Giraitis and Surgailis [20], Giraitis and Taqqu [21], Hhas'minskii and Ibragimov [25], Ginovian and Sahakian [16], Terrin and Taqqu [30], and references therein). In particular, Avram [1], Fox and Taqqu [12], Ginovian and Sahakian [16], Giraitis and Surgailis [20], Giraitis and Taqqu [21] have obtained sufficient conditions for non-tapered quadratic form Q_T to obey the central limit theorem (CLT).

For continuous-time stationary Gaussian processes the problem of describing the asymptotic distribution of non-tapered Toeplitz type quadratic functionals was studied in a number of papers. We cite merely the papers Avram et al. [2, 3], Bai et al. [4, 5], Bryc and Dembo [7], Ginovyan [13, 14, 15], Ginovyan and Sahakyan [17], Ibragimov [26], where can be found additional references.

In spectral analysis of stationary processes, however, the data are frequently tapered before calculating the statistics of interest. Instead of the original data $\{X(t),\ 0 \le t \le T\}$ the tapered data $\{h(t)X(t),\ 0 \le t \le T\}$ with the data taper $h(t)$ are used for all further calculations. Benefits of tapering the data have been widely reported in the literature. For example, data-tapers are introduced to reduce leakage effects, especially in the case when the spectrum of the model contains high peeks. Other application of data-tapers is in situations in which some of the data values are missing. Also, the use of tapers leads to the bias reduction, which is especially important when dealing with spatial data. In this case, the tapers can be used to fight the so-called "edge effects" (see Brillinger [6], R. Dahlhaus [8, 9], R. Dahlhaus and H. Künsch [10], Guyon [24], and references therein).

Central and non-central limit theorems for tapered quadratic forms of a discrete-time long memory Gaussian stationary fields have been proved in Doukhan et al. [11]. A central limit theorem for tapered quadratic functionals Q_T^h, in the case where the underlying model $X(t)$ is a Lévy-driven continuous-time stationary linear process has been proved in Ginovyan and Sahakyan [18] with time-domain conditions.

Remark 1.2. *Recall that a stationary process $X(t)$ with spectral density $f(\lambda)$ is said to have (a) short memory, (b) long memory or (c) intermediate memory if $f(\lambda)$ (a) is bounded away from zero and infinity at $\lambda = 0$, (b) has a pole at $\lambda = 0$, or (c) vanishes at $\lambda = 0$, respectively.*

1.3. **The approach.** To study the asymptotic distribution (as $T \to \infty$) of the functional $\widetilde{Q}_T^h$, given by (1.2), we use the method of cumulants, the frequency-domain approach, and the technique of truncated tapered Toeplitz operators.

By $W_T^h(\psi)$ we denote the truncated tapered Toeplitz operator generated by a function $\psi \in L^1(\mathbb{R})$ defined as follows (see [19], [23], [26] for non-tapered case):

$$[W_T^h(\psi)u](t) = \int_0^T \hat{\psi}(t-s)h_T(t)h_T(s)u(s)ds, \quad u(t) \in L^2[0,T], \tag{1.13}$$

where $\hat{\psi}(\cdot)$ is the Fourier transform of $\psi(\cdot)$.

Let $W_T^h(f)$ and $W_T^h(g)$ be the truncated tapered Toeplitz operators generated by the spectral density f, and the generating function g, respectively. Similar to the non-tapered case, we have the following results (cf. [19], [23], [26], see also the proof of Lemma 4.8 below).

1. The quadratic functional Q_T^h in (1.2) has the same distribution as the sum $\sum_{j=1}^{\infty} \lambda_{j,T}^2 \xi_j^2$, where $\{\xi_j, j \geq 1\}$ are independent $N(0,1)$ Gaussian random variables and $\{\lambda_{j,T}, j \geq 1\}$ are the eigenvalues of the operator $W_T^h(f)\, W_T^h(g)$.
2. The characteristic function $\varphi(t)$ of Q_T^h is given by formula:

$$\varphi(t) = \prod_{j=1}^{\infty} |1 - 2it\lambda_{j,T}|^{-1/2}. \tag{1.14}$$

3. The k–th order cumulant $\chi_k(Q_T^h)$ of Q_T^h is given by formula:

$$\chi_k(Q_T) = 2^{k-1}(k-1)! \sum_{j=1}^{\infty} \lambda_{j,T}^k = 2^{k-1}(k-1)!\, \mathrm{tr}\, [W_T^h(f)\, W_T^h(g)]^k, \tag{1.15}$$

where $\mathrm{tr}[A]$ stands for the trace of an operator A.

Thus, to describe the asymptotic distributions of the quadratic functional Q_T^h, we have to control the traces and eigenvalues of the products of truncated tapered Toeplitz operators.

Throughout the paper the letters C, c and M with or without indices are used to denote positive constants, the values of which can vary from line to line. Also, by $\mathbb{I}_A(\cdot)$ we denote the indicator of a set $A \subset \mathbb{R}$.

The remainder of the paper is structured as follows. In Section 2 we state the main results of the paper – Theorems 2.1 – 2.5. In Section 3 we apply the results of Section 2 to show that the avaraged tapered periodogram is an asymptotically normal estimator for the linear spectral functional. In Section 4 we prove preliminary results that are used in the proofs of main results, and also represent independent interest. Section 5 is devoted to the proofs of results stated in Section 2.

2. Central limit theorems for tapered quadratic functional Q_T^h

Below we assume that $f, g \in L^1(\mathbb{R})$, and with no loss of generality, that $g \geq 0$. We use the following notation: By $\widetilde{Q}_T^h$ we denote the standard normalized quadratic functional:

$$\widetilde{Q}_T^h = T^{-1/2}\left(Q_T^h - \mathbb{E}[Q_T^h]\right). \tag{2.1}$$

Then by (1.15) we have

$$\chi_k(\widetilde{Q}_T^h) = \begin{cases} 0, & \text{for } k=1 \\ T^{-k/2}2^{k-1}(k-1)!\,\mathrm{tr}\,[W_T^h(f)W_T^h(g)]^k, & \text{for } k\ge 2. \end{cases} \tag{2.2}$$

We set

$$\sigma_h^2 := 16\pi^3 H_4 \int_{\mathbb{R}} f^2(\lambda)g^2(\lambda)\,d\lambda, \tag{2.3}$$

where H_4 is as in (1.4). The notation

$$\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0,\sigma_h^2) \quad \text{as} \quad T\to\infty \tag{2.4}$$

means that the distribution of the random variable $\widetilde{Q}_T^h$ tends (as $T\to\infty$) to the centered normal distribution with variance σ_h^2.

The main results of the paper are the following theorems.

Theorem 2.1. *Assume that $f\cdot g \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, the taper function h satisfies assumption (T), and for $T\to\infty$*

$$\chi_2(\widetilde{Q}_T^h) = \frac{2}{T}tr\big[W_T^h(f)W_T^h(g)\big]^2 \longrightarrow \sigma_h^2, \tag{2.5}$$

where σ_h^2 is as in (2.3). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0,\sigma_h^2)$ as $T\to\infty$.

Theorem 2.2. *Assume that the function*

$$\varphi(x_1,x_2,x_3) = \int_{\mathbb{R}} f(u)g(u-x_1)f(u-x_2)g(u-x_3)\,du \tag{2.6}$$

belongs to $L^2(\mathbb{R}^3)$ and is continuous at $(0,0,0)$, and the taper function h satisfies assumption (T). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0,\sigma_h^2)$ as $T\to\infty$.

Theorem 2.3. *Assume that $f(\lambda)\in L^p(\mathbb{R})$ $(p\ge 1)$ and $g(\lambda)\in L^q(\mathbb{R})$ $(q\ge 1)$ with $1/p+1/q\le 1/2$, and the taper function h satisfies assumption (T). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0,\sigma_h^2)$ as $T\to\infty$.*

Theorem 2.4. *Let $f\in L^2(\mathbb{R}),\ g\in L^2(\mathbb{R}),\ fg\in L^2(\mathbb{R})$,*

$$\int_{\mathbb{R}} f^2(\lambda)g^2(\lambda-\mu)\,d\lambda \longrightarrow \int_{\mathbb{R}} f^2(\lambda)g^2(\lambda)\,d\lambda \quad \text{as} \quad \mu\to 0, \tag{2.7}$$

and let the taper function h satisfy assumption (T). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0,\sigma_h^2)$ as $T\to\infty$.

To state the next theorem, we need to introduce a class of slowly varying at zero functions. Recall that a function $u(\lambda)$, $\lambda \in \mathbb{R}$, is called slowly varying at zero if it is nonnegative and for any $t > 0$

$$\lim_{\lambda \to 0} \frac{u(t\lambda)}{u(\lambda)} \to 1.$$

Denote by $SV_0(\mathbb{R})$ the class of slowly varying at zero functions $u(\lambda)$, $\lambda \in \mathbb{R}$, satisfying the following conditions: for some $a > 0$, $u(\lambda)$ is bounded on $[-a, a]$, $\lim_{\lambda \to 0} u(\lambda) = 0$, $u(\lambda) = u(-\lambda)$ and $0 < u(\lambda) < u(\mu)$ for $0 < \lambda < \mu < a$. An example of a function belonging to $SV_0(\mathbb{R})$ is $u(\lambda) = |\ln |\lambda||^{-\gamma}$ with $\gamma > 0$ and $a = 1$.

Theorem 2.5. *Assume that the functions f and g are integrable on $\mathbb{R}$ and bounded outside any neighborhood of the origin, and satisfy for some $a > 0$*

$$f(\lambda) \le |\lambda|^{-\alpha} L_1(\lambda), \quad |g(\lambda)| \le |\lambda|^{-\beta} L_2(\lambda), \quad \lambda \in [-a, a], \tag{2.8}$$

for some $\alpha < 1$, $\beta < 1$ with $\alpha + \beta \le 1/2$, where $L_1(x)$ and $L_2(x)$ are slowly varying at zero functions satisfying

$$L_i \in SV_0(\mathbb{R}), \quad \lambda^{-(\alpha+\beta)} L_i(\lambda) \in L^2[-a, a], \ i = 1, 2. \tag{2.9}$$

Also, let the taper function h satisfy assumption (T). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0, \sigma_h^2)$ as $T \to \infty$.

Remark 2.1. The conditions $\alpha < 1$ and $\beta < 1$ in Theorem 2.5 ensure that the Fourier transforms of f and g are well defined. Observe that when $\alpha > 0$ the process $X(t)$ may exhibit long-range dependence. We also allow here $\alpha + \beta$ to assume the critical value $1/2$. The assumptions $f \cdot g \in L^1(\mathbb{R})$, $f, g \in L^\infty(\mathbb{R} \setminus [-a, a])$ and (2.9) imply that $f \cdot g \in L^2(\mathbb{R})$, so that the variance σ_h^2 in (2.3) is finite.

Remark 2.2. In Theorem 2.5, the assumption that $L_1(x)$ and $L_2(x)$ belong to $SV_0(\mathbb{R})$ instead of merely being slowly varying at zero is done in order to deal with the critical case $\alpha + \beta = 1/2$. Suppose that we are away from this critical case, namely, $f(x) = |x|^{-\alpha} l_1(x)$ and $g(x) = |x|^{-\beta} l_2(x)$, where $\alpha + \beta < 1/2$, and $l_1(x)$ and $l_2(x)$ are slowly varying at zero functions. Assume also that $f(x)$ and $g(x)$ are integrable and bounded on $(-\infty, -a) \cup (a, +\infty)$ for any $a > 0$. We claim that Theorem 2.5 applies. Indeed, choose $\alpha' > \alpha$, $\beta' > \beta$ with $\alpha' + \beta' < 1/2$. Write $f(x) = |x|^{-\alpha'} |x|^{\delta} l_1(x)$, where $\delta = \alpha' - \alpha > 0$. Since $l_1(x)$ is slowly varying, when $|x|$ is small enough, for some $\epsilon \in (0, \delta)$ we have $|x|^{\delta} l_1(x) \le |x|^{\delta - \epsilon}$. Then one can bound $|x|^{\delta-\epsilon}$ by $c\,|\ln |x||^{-1} \in SV_0(\mathbb{R})$ for small $|x| < 1$. Hence one has when $|x| < 1$ is small enough, $f(x) \le |x|^{-\alpha'} \left(c\,|\ln |x||^{-1} \right)$. Similarly, when $|x| < 1$ is small enough, one has $g(x) \le |x|^{-\beta'}\left(c\,|\ln|x||^{-1}\right)$. All the assumptions in Theorem 2.5 are now readily checked with α, β replaced by α' and β', respectively.

Remark 2.3. The analogs of Theorems 2.1 - 2.5 for non-tapered case ($h(t) = \mathbb{I}_{[0,1]}(t)$) were proved in Ginovyan and Sahakyan [17].

Remark 2.4. In Ginovyan and Sahakyan [18] was proved a central limit theorem for tapered functional Q_T^h for more general case where $X(t)$ is a Lévy-driven stationary linear process. Specifically, in [18] was proved the following result (see [18], Theorem 5.1). Let $\{X(t),\ t \in \mathbb{R}\}$ be a stationary linear process defined by

$$X(t) = \int_{\mathbb{R}} a(t-s)\xi(ds),$$

where $a(\cdot)$ is a filter from $L^2(\mathbb{R})$, and $\xi(t)$ is a Lévy process satisfying the conditions: $\mathbb{E}\xi(t) = 0$, $\mathbb{E}\xi^2(1) = 1$ and $\mathbb{E}\xi^4(1) < \infty$. Assume that the filter $a(\cdot)$ and the generating kernel $\hat{g}(\cdot)$ are such that

$$a(\cdot) \in L^p(\mathbb{R}) \cap L^2(\mathbb{R}), \quad \hat{g}(\cdot) \in L^q(\mathbb{R}), \quad 1 \le p, q \le 2, \quad \frac{2}{p} + \frac{1}{q} \ge \frac{5}{2}, \tag{2.10}$$

and the taper h satisfies assumption (T). Then $\widetilde{Q}_T^h \xrightarrow{d} \eta \sim N(0, \sigma^2_{L,h})$ as $T \to \infty$, where

$$\sigma^2_{L,h} = 16\pi^3 H_4 \int_{\mathbb{R}} f^2(\lambda) g^2(\lambda) d\lambda + \kappa_4 4\pi^2 H_4 \left[\int_{\mathbb{R}} f(\lambda) g(\lambda) d\lambda\right]^2. \tag{2.11}$$

Notice that if the underlying process $X(t)$ is Gaussian, then in formula (2.11) we have only the first term and so $\sigma^2_{L,h} = \sigma^2_h$, because in this case $\kappa_4 = 0$. On the other hand, the condition (2.10) is more restrictive than the conditions in Theorems 2.1 - 2.5. Thus, for Gaussian processes Theorems 2.1 - 2.5 improve the above stated result.

3. An application

In this section we apply the results of Section 2 to prove that the statistic J_T^h given by (1.11) is an asymptotically normal estimator for the linear functional $J(f)$ given by (1.6). To state the corresponding result we introduce the L^p-Hölder class and set up an assumption.

Given numbers $p \ge 1$, $0 < \alpha < 1$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we set $\beta = \alpha + r$ and denote by $H_p(\beta)$ the L^p-Hölder class, that is, the class of those functions $\psi(\lambda) \in L^p(\mathbb{R})$, which have r-th derivatives in $L^p(\mathbb{R})$ and with some positive constant C satisfy

$$||\psi^{(r)}(\cdot + h) - \psi^{(r)}(\cdot)||_p \le C|h|^\alpha.$$

Assumption (A). Let the spectral density $f(\lambda) \in H_p(\beta_1)$, $\beta_1 > 0$, $p \geq 1$ and let the generating function $g(\lambda) \in H_q(\beta_2)$, $\beta_2 > 0$, $q \geq 1$ with $1/p + 1/q = 1$. Assume that one of the conditions a)–d) is fulfilled:

a) $\beta_1 > 1/p$, $\beta_2 > 1/q$

b) $\beta_1 \leq 1/p$, $\beta_2 \leq 1/q$ and $\beta_1 + \beta_2 > 1/2$

c) $\beta_1 > 1/p$, $1/q - 1/2 < \beta_2 \leq 1/q$

d) $\beta_2 > 1/q$, $1/p - 1/2 < \beta_1 \leq 1/p$.

Theorem 3.1. *Let the functionals $J = J(f)$ and $J_T^h = J(I_T^h)$ be defined by (1.6) and (1.11), respectively. Then under the conditions (A) and (T) the statistic J_T^h is an asymptotically normal estimator for functional J. More precisely, we have*

$$T^{1/2}\left[J_T^h - J\right] \xrightarrow{d} \eta \quad \text{as} \quad T \to \infty, \tag{3.1}$$

where η is a normally distributed random variable with mean zero and variance $\sigma_h^2(J)$ given by

$$\sigma_h^2(J) = 4\pi e(h) \int_{\mathbb{R}} f^2(\lambda) g^2(\lambda) d\lambda, \quad e(h) := H_4 H_2^{-2}, \tag{3.2}$$

and H_k is as in (1.4).

Remark 3.1. In Theorem 2.3 of Ginovyan and Sahakyan [18] was proved the asymptotic normality of the estimator J_T^h for more general case where $X(t)$ is a Lévy-driven stationary linear process, but under the additional restrictive condition (2.10). Thus, for Gaussian processes Theorem 3.1 improve Theorem 2.3 of Ginovyan and Sahakyan [18].

4. Preliminaries

For a number k $(k = 2, 3, \ldots)$ and a taper function h satisfying assumption (T) consider the following "tapered" Fejér type kernel function:

$$\Phi_{k,T}^h(\mathbf{u}) = \frac{G_{k,T}(\mathbf{u})}{(2\pi)^{k-1} H_{k,T}(0)}, \quad \mathbf{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{R}^{k-1}, \tag{4.1}$$

where

$$G_{k,T}(\mathbf{u}) = H_{1,T}(u_1) \cdots H_{1,T}(u_{k-1}) H_{1,T}\left(-\sum_{j=1}^{k-1} u_j\right), \tag{4.2}$$

and the function $H_{k,T}$ is defined by (1.7) with $H_{k,T}(0) = T \cdot H_k \neq 0$ (see (1.4)).

Remark 4.1. Observe that by a change of variables $u_1 = x_1 - x_2$, $u_2 = x_2 - x_3$, …, $u_{k-1} = x_{k-1} - x_k$, the function $G_{k,T}(\mathbf{u})$ in (4.2) can be written in the following "symmetric" form:

$$G_{k,T}(\mathbf{x}) = H_{1,T}(x_1 - x_2) H_{1,T}(x_2 - x_3) \cdots H_{1,T}(x_{k-1} - x_k) H_{1,T}(x_k - x_1), \tag{4.3}$$

where $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$.

In Lemma 3.4 of Ginovyan and Sahakyan [18], it was proved that, similar to the classical Fejér kernel, the "tapered"kernel $\Phi^h_{k,T}(\mathbf{u})$ is an approximation identity. In particular, it was shown that the kernel $\Phi^h_{k,T}$ possesses the following property.

Lemma 4.1. *If a function $\psi(\mathbf{u}) \in L^1(\mathbb{R}^{k-1}) \bigcap L^{k-2}(\mathbb{R}^{k-1})$ is continuous at $\mathbf{v} = (v_1, \dots, v_{k-1})$ $(k = 2, 3, \dots)$, then*

$$\lim_{T\to\infty} \int_{\mathbb{R}^{k-1}} \psi(\mathbf{u}+\mathbf{v}) \Phi^h_{k,T}(\mathbf{u}) d\mathbf{u} = \Psi(\mathbf{v}), \tag{4.4}$$

where $\mathbf{u} = (u_1, \dots, u_{k-1})$ and $\Phi^h_{k,T}(\mathbf{u})$ is defined by (4.1) and (4.2).

The next lemma contains a formula for trace of product of truncated tapered Toeplitz operators.

Lemma 4.2. *Let $W^h_T(f)$ and $W^h_T(g)$ be the truncated tapered Toeplitz operators generated by functions $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, respectively. Then*

$$tr \left[W^h_T(f) W^h_T(g)\right]^2 = \int_{\mathbb{R}^4} G_T(\mathbf{x}) f(x_1) g(x_2) f(x_3) g(x_4)\, d\mathbf{x}, \tag{4.5}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $G_T(\mathbf{x}) := G_{4,T}(\mathbf{x})$, that is,

$$G_T(\mathbf{x}) := H_{1,T}(x_1 - x_2) H_{1,T}(x_2 - x_3) H_{1,T}(x_3 - x_4) H_{1,T}(x_4 - x_1), \tag{4.6}$$

and $H_{1,T}(\cdot)$ is as in (1.7) with $k = 1$.

Proof. It is easy to check that the result follows from (1.1), (1.3), (1.7), (1.13), and the formula for traces of integral operators (see [22], §3.10). Lemma 4.2 is proved.

Denote

$$\mu_T(A) = \frac{1}{T} \int_A G_T(\mathbf{x})\, d\mathbf{x}, \tag{4.7}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $G_T(\mathbf{x}))$ is as in (4.6), and let $C_{loc}(\mathbb{R}^n)$ be the space of continuous functions on $\mathbb{R}^n$ with bounded support.

Lemma 4.3. *If $\phi \in C_{loc}(\mathbb{R}^4)$, then*

$$\lim_{T\to\infty} \int_{\mathbb{R}^4} \phi(\mathbf{x})\, d\mu_T = 8\pi^3 H_4 \int_{\mathbb{R}} \phi(u,u,u,u) du, \tag{4.8}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mu_T(A)$ is as in (4.7) and H_4 is as in (1.4).

Proof. Making a change of variables

$$x_1 = u, \quad x_1 - x_2 = u_1, \quad x_2 - x_3 = u_2, \quad x_3 - x_4 = u_3,$$

in view of (4.1), (4.2) and (4.7), we can write

$$\begin{aligned}\int_{\mathbb{R}^4} \phi(\mathbf{x})\, d\mu_T &= \frac{1}{T}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \phi(u, u-u_1, u-u_1-u_2, u-u_1-u_2-u_3)du \\ &\times \; H_{1,T}(u_1)H_{1,T}(u_2)H_{1,T}(u_3)H_{1,T}(-u_1-u_2-u_3)\, du_1\, du_2\, du_3 \\ &= 8\pi^3 H_4 \int_{\mathbb{R}^3} \Psi(\mathbf{u})\, \Phi_T^h(\mathbf{u})\, d\mathbf{u}, \qquad (4.9)\end{aligned}$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\Phi_T^h(\mathbf{u}) := \Phi_{4,T}^h(\mathbf{u})$ and

$$\Psi(\mathbf{u}) = \int_{\mathbb{R}} \phi(u, u-u_1, u-u_1-u_2, u-u_1-u_2-u_3)\, du.$$

It is not difficult to check that the function Ψ satisfies conditions of Lemma 4.1 and

$$\lim_{\mathbf{u}\to(0,0,0)} \Psi(\mathbf{u}) = \int_{\mathbb{R}} \phi(u,u,u,u)\, du. \qquad (4.10)$$

Hence applying Lemma 4.1 from (4.9) and (4.10) we get (4.8). Lemma 4.3 is proved.

Lemma 4.4. *Let* $\phi(u_1,u_2,u_3,u_4) = \prod_{i=1}^{4} \phi_i(u_i)$, *where* $\phi_i \in L^1(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$, $i = 1,2,3,4$. *Then the asymptotic relation (4.8) holds.*

Proof. Suppose $\|\phi_i\|_\infty \le M < \infty$, $i = 1,2,3,4$. Using Lusin's theorem for any $\varepsilon > 0$ we can find functions φ_i, ψ_i, $i = 1,2,3,4$, satisfying

$$\phi_i = \varphi_i + \psi_i, \quad \varphi_i \in C_{loc}(\mathbb{R}), \quad \|\psi_i\|_{L^1(\mathbb{R})} \le \varepsilon, \quad \|\varphi_i\|_C \le M. \qquad (4.11)$$

Therefore

$$\int_{\mathbb{R}^4} \phi d\mu_T = \int_{\mathbb{R}^4} \prod_{i=1}^{4} (\varphi_i + \psi_i) d\mu_T = \int_{\mathbb{R}^4} \prod_{i=1}^{4} \varphi_i d\mu_T + I_T, \qquad (4.12)$$

where by (4.11) and Lemma 4.5 below

$$\begin{aligned}|I_T| &\le \sum_{j=1}^{4} \int_{\mathbb{R}^4} |\psi_j| \prod_{i=1, i\neq j}^{4} (|\varphi_i| + |\psi_i|) d|\mu_T| \\ &\le C_M \sum_{j=1}^{4} \int_{\mathbb{R}^4} |\psi_j| d|\mu_T| \le C_M \sum_{j=1}^{4} \|\psi_j\|_{L^1(\mathbb{R})} \le C_M \varepsilon. \qquad (4.13)\end{aligned}$$

By Lemma 4.3 we have

$$\begin{aligned}\lim_{T\to\infty} \int_{\mathbb{R}^4} \prod_{i=1}^{4} \varphi_i(u_i) d\mu_T &= \int_{\mathbb{R}} \prod_{i=1}^{4} \varphi_i(u) du \\ &= \int_{\mathbb{R}} \prod_{i=1}^{4} [\phi_i(u) - \psi_i(u)]\, du = \int_{\mathbb{R}} \phi(u,u,u,u) du + J, \qquad (4.14)\end{aligned}$$

where

$$|J| \le \sum_{j=1}^{4} \int_{\mathbb{R}} |\psi_j(u)| \prod_{i=1, i\neq j}^{4} (|\phi_i(u)| + |\varphi_i(u)|) du \le C_M \varepsilon. \qquad (4.15)$$

From (4.12) – (4.15) we get (4.8). Lemma 4.4 is proved.

Lemma 4.5. *If $f \in L^1(\mathbb{R})$, then the following inequalities hold:*

$$\text{(4.16)} \qquad 1) \int_{\mathbb{R}^4} |f(x_i)| d|\mu_T| \leq C_1 \|f\|_{L^1(\mathbb{R})}, \quad i = 1,2,3,4,$$

$$\text{(4.17)} \qquad 2) \int_{\mathbb{R}^4} |f(x_i) f(x_j)| d|\mu_T| \leq C_2 \|f\|^2_{L^2(\mathbb{R})}, \quad i,j = 1,2,3,4, \ i \neq j.$$

where C_1 and C_2 are absolute constants, and μ_T is as in (4.7).

Proof. Since h is a function of bounded variation with support on $[0,1]$, in view of (1.7), for $T > 0$ we have

$$\text{(4.18)} \qquad |H_{1,T}(x)| \leq C_h T \psi_T(x), \quad \text{where} \quad \psi_T(x) = \frac{1}{1 + T|x|}, \quad x \in \mathbb{R}.$$

We use the following inequality for function $\psi_T(x)$, which was proved in Ginovyan and Sahakyan [17] (see proof of Lemma 5):

$$\text{(4.19)} \qquad T \int_{\mathbb{R}} \psi_T(x+u) \psi_T(x+v) dx \leq C_\delta \psi_T^{1-\delta}(u - v), \quad \delta > 0, \quad u, v \in \mathbb{R}.$$

To prove (4.16) for $i = 1$ (say), we use (4.6), (4.7) and the inequality (4.19) with $\delta = 1/4$ to obtain

$$\begin{aligned} \int_{\mathbb{R}^4} |f(x_1)| d|\mu_T| \qquad &\leq CT^3 \int_{\mathbb{R}^4} |f(x_1)| \psi_T(x_1 - x_3) \psi_T(x_3 - x_2) \\ &\times \psi_T(x_4 - x_1) \psi_T(x_2 - x_4) dx_1 dx_2 dx_3 dx_4 \\ &\leq CT \int_{\mathbb{R}} |f(x_1)| \int_{\mathbb{R}} \psi_T^{3/2}(x_1 - x_2) dx_2 dx_1 \leq C_1 \|f\|_{L^1(\mathbb{R})}. \end{aligned}$$

This proves (4.16). To prove (4.17) for $i = 1, \ j = 2$ (say), we use (4.6), (4.7), the inequality (4.19) with $\delta = 1/4$, and Cauchy inequality to obtain

$$\begin{aligned} &\int_{\mathbb{R}^4} |f(x_1) f(x_2)| d|\mu_T| \leq CT^3 \int_{\mathbb{R}^4} |f(x_1) f(x_2)| \psi_T(x_1 - x_3) \psi_T(x_3 - x_2) \psi_T(x_4 - x_1) \\ &\leq CT \int_{\mathbb{R}^2} |f(x_1) f(x_2)| \psi_T^{3/2}(x_1 - x_2) dx_1 dx_2 \\ &\leq C \left\{ T \int_{\mathbb{R}^2} f^2(x_1) \psi_T^{3/2}(x_1 - x_2) dx_1 dx_2 \right\}^{1/2} \\ &+ \left\{ T \int_{\mathbb{R}^2} f^2(x_2) \psi_T^{3/2}(x_1 - x_2) dx_1 dx_2 \right\}^{1/2} \leq C_2 \int_{\mathbb{R}} f^2(x) dx. \end{aligned}$$

Lemma 4.5 is proved.

Lemma 4.6. *Let $\psi(u) \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, with $1 < p < \infty$, and let the taper function h satisfy assumption (T). Then*

$$\text{(4.20)} \qquad \lambda_T := ||W_T^h(\psi)||_\infty = o\big(T^{1/p}\big) \quad \textit{as} \quad T \to \infty.$$

Proof. Let N_T be a positive function of T, which we will specify later. We set

$$\text{(4.21)} \qquad M_T = \{\lambda \in \mathbb{R}; \ |\psi(\lambda)| > N_T\}.$$

We have

$$\lambda_T = ||W_T^h(\psi)||_\infty = \sup_{||u||_2=1} |(W_T^h(\psi)u, u)| =$$

$$\sup_{||u||_2=1} \left| \int_0^T \int_0^T \widehat{\psi}(t-s) u(t) u(s) h(t) h(s) \, dt \, ds \right| =$$

$$\sup_{||u||_2=1} \left| \int_0^T \int_0^T \left[\int_{\mathbb{R}} e^{i\lambda(t-s)} \psi(\lambda) \, d\lambda \right] u(t) u(s) h(t) h(s) \, dt \, ds \right|. \tag{4.22}$$

A square integrable function $u(t)$ is also integrable on $[0,T]$. Hence, switching the order of integration in (4.22), we get

$$\lambda_T = \sup_{||u||_2=1} \left| \int_{\mathbb{R}} \psi(\lambda) \left[\int_0^T u(t) h(t) e^{it\lambda} \, dt \int_0^T u(s) h(s) e^{-is\lambda} \, ds \right] d\lambda \right|$$

$$\leq \sup_{||u||_2=1} \int_{\mathbb{R}} |\psi(\lambda)| \left| \int_0^T u(t) h(t) \, e^{i\lambda t} dt \right|^2 d\lambda. \tag{4.23}$$

Since for $u(t) \in L^2[0,T]$ with $||u||_2 = 1$ and h is bounded, we have $\left|\int_0^T u(t)h(t)\, e^{i\lambda t} dt\right|^2 \leq C_h T$, and by Plancherel's theorem from (4.23) we obtain

$$\lambda_T \leq C_h \left(N_T + T \int_{M_T} |\psi(\lambda)| \, d\lambda \right), \tag{4.24}$$

where M_T is as in (4.21). We show that for every p $(1 < p < \infty)$

$$\int_{M_T} |\psi(\lambda)| \, d\lambda \leq \gamma_T^p \, N_T^{(1-p)}, \tag{4.25}$$

where

$$\gamma_T = \left(\int_{M_T} |\psi(\lambda)|^p \, d\lambda \right)^{1/p}. \tag{4.26}$$

Indeed, by Hölder inequality

$$\int_{M_T} |\psi(\lambda)| \, d\lambda \leq \gamma_T \left[m(M_T) \right]^{(p-1)/p}, \tag{4.27}$$

where $m(M_T)$ is the Lebesgue measure of the set M_T. Next, by Chebyshev inequality

$$m(M_T) \leq \gamma_T^p \, N_T^{-p}. \tag{4.28}$$

A combination of (4.27) and (4.28) yields (4.25). Now from (4.24) and (4.25) we have

$$\lambda_T \leq C_h \left(N_T + T \gamma_T^p \, N_T^{(1-p)} \right). \tag{4.29}$$

If $\psi \in L^\infty(\mathbb{R})$, then putting $N_T = \|\psi\|_\infty$ for all $T > 0$, we will have $\gamma_T = 0$ and (4.29) implies $\lambda_T = O(1)$.

Now suppose $\psi \notin L^\infty(\mathbb{R})$ and for fixed $T > 0$ consider the function

$$F(N) = N - T^{1/p} \left(\int_{\{\lambda : |\psi(\lambda)| > N\}} |\psi(\lambda)|^p \, d\lambda \right)^{1/p}, \qquad N \in [0, \infty).$$

Since $F(0) < 0$ and $\lim_{N\to\infty} F(N) = +\infty$ there exists a positive number $N = N_T$ with $F(N_T) = 0$, that is,

$$N_T = T^{1/p}\left(\int_{\{\lambda: |\psi(\lambda)| > N_T\}} |\psi(\lambda)|^p\right) = T^{1/p}\gamma_T. \tag{4.30}$$

It is easy to see that for $\psi \notin L^\infty(\mathbb{R})$ the equality (4.30) implies $\lim_{T\to\infty} N_T = \infty$. Hence $\gamma_T = o(1)$ and from (4.29) and (4.30) we obtain $\lambda_T < C_h T^{1/p}\gamma_T = o(T^{1/p})$ as $T \to \infty$. Lemma 4.6 is proved.

Lemma 4.7. *Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and let $W_T^h(\psi)$ be the tapered truncated Toeplitz operator defined by (1.13) with taper function h satisfying assumption (T). Then*

$$\frac{1}{T}||W_T^h(\psi)||_2^2 \longrightarrow 2\pi H_4 ||\psi||_2^2 \quad \text{as} \quad T \to \infty, \tag{4.31}$$

where H_4 is as in (1.4).

Proof. Using the formula for Hilbert–Schmidt norm of integral operators (see [22]), by (1.13) we have

$$||W_T^h(\psi)||_2^2 = \int_0^T \int_0^T |\widehat{\psi}(t-s)|^2 |h_T(t)h_T(s)|^2 \, dt\, ds. \tag{4.32}$$

Using the change of variables $t - s = u$ and taking into account that by assumption (T) the taper function h is supported on $[0, 1]$, from (4.32) we get

$$||W_T^h(\psi)||_2^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{\psi}(u)|^2 |h_T(s+u)h_T(s)|^2 \, du\, ds. \tag{4.33}$$

Next, taking into account that $h_T(t) = h(t/T)$ and using the change of variables $s/T = v$, from (4.33) we can write

$$\begin{aligned} \frac{1}{T}||W_T^h(\psi)||_2^2 &= \frac{1}{T}\int_{\mathbb{R}} |\widehat{\psi}(u)|^2| \left[\int_0^T |h(s/T + u/T)h(s/T)|^2 \, ds\right] du \\ &= \int_{\mathbb{R}} |\widehat{\psi}(u)|^2| \left[\int_0^1 |h(v + u/T)h(v)|^2 \, dv\right] du. \end{aligned} \tag{4.34}$$

For the inside integral on the right-hand side of (4.34), in view of (1.4), we have

$$\lim_{T\to\infty} \int_0^1 |h(v + u/T)h(v)|^2 \, dv = \int_0^1 h^4(v)\, dv = H_4. \tag{4.35}$$

Finally, using Parseval-Plancherel's equality, from (4.34) and (4.35), we obtain (4.31). Lemma 4.7 is proved.

Lemma 4.8. *Let $Y(t)$, $t \in \mathbb{R}$, be a real-valued, centered, separable stationary Gaussian process with the spectral density $f_Y(\lambda) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and let $h_T(t) = h(t/T)$ with a taper function h satisfying assumption (T). Define*

$$L_T^h := \int_0^T [h_T(t)Y(t)]^2 \, dt. \tag{4.36}$$

Then the distribution of the normalized quadratic functional

$$\widetilde{L}_T^h := T^{-1/2}\left(L_T^h - \mathbb{E}L_T^h\right) \tag{4.37}$$

tends (as $T \to \infty$) to the normal distribution $N(0, \sigma_Y^2)$ with variance

$$\sigma_Y^2 = 4\pi H_4 \int_{\mathbb{R}} f_Y^2(\lambda)\, d\lambda, \tag{4.38}$$

where H_4 is as in (1.4)*.*

Proof. Let $R(t)$ be the covariance function of $Y(t)$. For $T > 0$ denote by $\lambda_j = \lambda_j(T)$, $j \in \mathbb{N}$, the eigenvalues of the operator $W_T^h(f_Y)$ (see (1.13)), and let $e_j(t) = e_j(t, T) \in L_2[0.T]$, $j \in \mathbb{N}$, be the corresponding orthonormal eigenfunctions, that is,

$$\int_0^T K(t-s)e_j(s)\, ds = \lambda_j e_j(t), \quad j \in \mathbb{N}, \tag{4.39}$$

where $K(t-s) := R(t-s)h_T(t)h_T(s)$. Since by Mercer's theorem (see, e.g., [22], §3.10)

$$K(t-s) = \sum_{j=1}^{\infty} \lambda_j e_j(t)e_j(s) \tag{4.40}$$

with positive and summable eigenvalues $\{\lambda_j\}$, we have the Karhunen–Loéve expansion:

$$h_T(t)Y(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\xi_j e_j(t), \tag{4.41}$$

where $\{\xi_j\}$ are independent $N(0,1)$ random variables. Therefore by (4.37) and (4.41)

$$\widetilde{L}_T^h = T^{-1/2}\sum_{j=1}^{\infty} \lambda_j(\xi_j^2 - 1). \tag{4.42}$$

Denote by $\chi_k(\widetilde{L}_T^h)$ the k-th order cumulant of $\widetilde{L}_T^h$. By (4.42) (cf. (2.2)) we have

$$\chi_k(\widetilde{L}_T^h) = \begin{cases} 0, & \text{for } k = 1, \\ (k-1)!\, 2^{k-1}T^{-k/2}\mathrm{tr}[W_T^h(f_Y)]^k, & \text{for } k \geq 2. \end{cases} \tag{4.43}$$

By (4.43) and Lemma 4.7 we have

$$\chi_2(\widetilde{L}_T^h) = \frac{2}{T}||W_T^h(f_Y)||_2^2 \longrightarrow 4\pi H_4 \int_{\mathbb{R}} f_Y^2(\lambda)\, d\lambda \quad \text{as} \quad T \to \infty. \tag{4.44}$$

Next, by (4.43) for $k \geq 3$, we have

$$|\chi_k(\widetilde{L}_T^h)| \leq C\,\frac{1}{T}||W_T^h(f_Y)||_2^2\, T^{1-k/2}\lambda_T^{k-2}, \tag{4.45}$$

where $\lambda_T = ||W_T^h(f_Y)||_\infty$. By Lemmas 4.6 and 4.7 the right hand side of (4.45) tends to zero as $T \to \infty$. Lemma 4.8 is proved.

The next lemma, which is the well-known Hardy-Littlewood type embedding theorem for the Hölder classes $H_p(\beta)$ (see Nikol'skii [28]), will be used in the proof of Theorem 3.1.

Lemma 4.9. *Let $\psi(\lambda) \in H_p(\beta)$ with $\beta > 0$ and $p \geq 1$. The following assertions hold:*

a) if $\beta \leq 1/p$ and $p < p_1 < p/(1-\beta p)$, then

$$\psi(\lambda) \in H_{p_1}(\beta - \frac{1}{p} + \frac{1}{p_1})$$

b) if $\beta > 1/p$, then $\psi(\lambda)$ is continuous and $||\psi||_\infty < \infty$.

5. Proofs

Since the proofs of Theorems 2.3 and 2.4 are almost the same (with some minor changes) as in the non-tapered case given in Ginovyan and Sahakyan [17], here we prove only Theorems 2.1, 2.2 and 2.5.

Proof of Theorem 2.1. By Theorem 16.7.2 from [27] the underlying process $X(t)$ admits the moving average representation

$$X(t) = \int_{\mathbb{R}} \widehat{a}(t-s)\, d\xi(s), \tag{5.1}$$

where $\widehat{a}(\cdot)$ is a function from $L^2(\mathbb{R})$, and $\xi(s)$ is a process with orthogonal increments such that $\mathbb{E}[d\xi(s)] = 0$ and $\mathbb{E}|d\xi(s)|^2 = ds$. Moreover the spectral density $f(\lambda)$ can be represented as

$$f(\lambda) = 2\pi\, |a(\lambda)|^2, \tag{5.2}$$

where $a(\lambda)$ is the inverse Fourier transform of $\widehat{a}(t)$. We set

$$a_1(\lambda) = (2\pi)^{1/2}\, a(\lambda) \cdot [g(\lambda)]^{1/2}, \tag{5.3}$$

where $g(\lambda)$ is the generating function, and consider a process $Y(t)$ ($t \in \mathbb{R}$) defined by

$$Y(t) = \int_{\mathbb{R}} \widehat{a}_1(t-s)\, d\xi(s), \tag{5.4}$$

where $\widehat{a}_1(t)$ is the Fourier transform of $a_1(\lambda)$ and $\xi(s)$ is as in (5.1). Since $fg \in L^1(\mathbb{R})$ by Parseval-Plancherel's identity we have

$$\int_{\mathbb{R}} |\widehat{a}_1(t)|^2\, dt = 2\pi \int_{\mathbb{R}} |a_1(\lambda)|^2\, d\lambda = 4\pi^2 \int_{\mathbb{R}} f(\lambda)\, g(\lambda)\, d\lambda < \infty. \tag{5.5}$$

So, $Y(t)$ is well-defined stationary process with spectral density

$$f_Y(\lambda) := |a_1(\lambda)|^2 = 2\pi\, f(\lambda)\, g(\lambda). \tag{5.6}$$

Since by assumption $f(\lambda)g(\lambda) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the process $Y(t)$ defined by (5.4) satisfies the conditions of Lemma 4.8. Hence Lemma 4.8 and Lemma 5.1 that follows imply Theorem 2.1.

Lemma 5.1. *Under assumptions of Theorem 2.1*

$$\mathrm{Var}(Q_T^h - L_T^h) = o(T) \quad \text{as} \quad T \to \infty, \tag{5.7}$$

where Q_T^h and L_T^h are as in (1.2) and (4.36) respectively.

Proof. By (1.2) and (5.1) we have

$$Q_T^h = \int_{\mathbb{R}^2} \left[\int_0^T \int_0^T \widehat{g}(t-s)\widehat{a}(t-u_1)\widehat{a}(s-u_2)h_T(t)h_T(s)\,dt\,ds \right] d\xi(u_1)\,d\xi(u_2). \tag{5.8}$$

Similarly, by (4.36) and (5.4)

$$L_T^h = \int_{\mathbb{R}^2} \left[\int_0^T \widehat{a}_1(t-u_1)\widehat{a}_1(t-u_2)h_T^2(t)\,dt \right] d\xi(u_1)\,d\xi(u_2). \tag{5.9}$$

Setting

$$d_{1T}(u_1,u_2) := \int_0^T \int_0^T \widehat{g}(t-s)\widehat{a}(t-u_1)\widehat{a}(s-u_2)h_T(t)h_T(s)\,dt\,ds \tag{5.10}$$

and

$$\begin{aligned} d_{2T}(u_1,u_2) :=& \int_0^T \int_0^T \widehat{a}_1(t-u_1)\widehat{a}_1(s-u_2)h_T^2(t)\,dt\,ds \\ =& \int_0^T \widehat{a}_1(t-u_1)\widehat{a}_1(t-u_2)h_T^2(t)\,dt, \end{aligned} \tag{5.11}$$

from (5.8)–(5.11) we get

$$Q_T^h - L_T^h = \int_{\mathbb{R}^2} \left[d_{1T}(u_1,u_2) - d_{2T}(u_1,u_2) \right] d\xi(u_1)\,d\xi(u_2). \tag{5.12}$$

Since the underlying process $X(t)$ is Gaussian, we obtain

$$\mathrm{Var}(Q_T^h - L_T^h) = 2\int_{\mathbb{R}^2} \left[d_{1T}(u_1,u_2) - d_{2T}(u_1,u_2) \right]^2 du_1\,du_2. \tag{5.13}$$

We set

$$p_1(\lambda_1,\lambda_2,\mu) = a(\lambda_1)a(\lambda_2)g(\mu), \tag{5.14}$$

$$p_2(\lambda_1,\lambda_2,\mu) = a_1(\lambda_1)a_1(\lambda_2)\delta(\mu) = a(\lambda_1)a(\lambda_2)[g(\lambda_1)]^{1/2}[g(\lambda_2)]^{1/2}. \tag{5.15}$$

By Parseval-Plancherel's identity we have

$$\begin{aligned} &\int_{\mathbb{R}^2} d_{iT}^2(u_1,u_2)\,du_1\,du_2 \\ &= (2\pi)^2 \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} H_{1,T}(\lambda_1-\mu)H_{1,T}(\mu-\lambda_2)p_i(\lambda_1,\lambda_2,\mu)\,d\mu \right|^2 d\lambda_1\,d\lambda_2 \\ &= (2\pi)^2 T||p_i||_T^2, \qquad i=1,2, \end{aligned} \tag{5.16}$$

where $H_{1,T}(u)$ is given by (1.7), $||p||_T^2 = (p,p)_T$,

$$(p,p')_T = \int_{\mathbb{R}^4} p(\lambda_1,\lambda_2,\lambda_3)\overline{p'(\lambda_1,\lambda_2,\lambda_4)}\,d\mu_T, \tag{5.17}$$

and the measure μ_T is defined by (4.7).

As in (5.16) (see also (5.13)), we have

$$\operatorname{Var}(Q_T - L_T) = 8\pi^2 T \|p_1 - p_2\|_T^2. \tag{5.18}$$

For any $K > 0$ we consider the sets

$$E_1^K = \{u \in \mathbb{R} : |a(u)| < K\}, \qquad E_2^K = \{u \in \mathbb{R} : g(u) < K\}, \tag{5.19}$$

and denote

$$\begin{aligned} p_1^K(\mathbf{u}) &= p_1(\mathbf{u})\chi_1^K(u_1)\chi_1^K(u_2)\chi_2^K(u_3), \\ p_2^K(\mathbf{u}) &= p_2(\mathbf{u})\chi_1^K(u_1)\chi_1^K(u_2)\chi_2^K(u_1)\chi_2^K(u_2), \end{aligned} \tag{5.20}$$

where $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $\chi_j^K(u)$ is the characteristic function of the set E_j^K, $j = 1, 2$. Then

$$\|p_1 - p_2\|_T^2 \le 3\left(\|p_1^K - p_2^K\|_T^2 + \|p_1 - p_1^K\|_T^2 + \|p_2 - p_2^K\|_T^2\right). \tag{5.21}$$

Now, (5.14), (5.15) and (5.20) imply that $\|p_1^K - p_2^K\|_T^2 = \int_{\mathbb{R}^4} \Gamma d\mu_T$, where $\Gamma = \Gamma(u_1, u_2, u_3, u_4)$ is a sum of four functions satisfying the conditions of Lemma 4.4. Since $\Gamma(u, u, u, u) = 0$ for $u \in \mathbb{R}$, applying Lemma 4.4 we get

$$\lim_{T\to\infty} \|p_1^K - p_2^K\|_T = \int_{\mathbb{R}} \Gamma(u, u, u, u)du = 0. \tag{5.22}$$

Next, according to (5.17) we have

$$\|p_1\|_T^2 = \|p_1^K + (p_1 - p_1^K)\|_T^2 = \|p_1\|_T^2 + 2(p_1^K, p_1 - p_1^K)_T + \|p_1 - p_1^K\|_T^2.$$

Therefore

$$\|p_1 - p_1^K\|_T^2 \le \left|\|p_1\|_T^2 - \|p_1^K\|_T^2\right| + 2\left|(p_1^K, p_1 - p_1^K)_T\right|. \tag{5.23}$$

By (2.5), (5.16) and Lemma 4.2 we have

$$\|p_1\|_T^2 = (2\pi)^{-2}\frac{1}{T}\operatorname{tr}\left[W_T^h(f)W_T^h(g)\right]^2 \to 2\pi H_4 \int_{\mathbb{R}} f^2(u)g^2(u)du, \tag{5.24}$$

while according to Lemma 4.4 and (5.16)

$$\|p_1^K\|_T^2 \to 2\pi H_4 \int_{F_K} f^2(u)g^2(u)du, \tag{5.25}$$

where $F_K := \{u \in \mathbb{R} : f(u) < K,\ g(u) < K\}$. From (5.24) and (5.25) we get

$$\lim_{T\to\infty}\left(\|p_1\|_T^2 - \|p_1^K\|_T^2\right) = 2\pi H_4 \int_{\mathbb{R}\setminus F_K} f^2(u)g^2(u)du = o(1) \quad \text{as} \quad K \to \infty. \tag{5.26}$$

To estimate the last term on the right hind side of (5.23) we denote

$$\Gamma_K(u_1, u_2, u_3, u_4) = p_1^K(u_1, u_2, u_3)\left[p_1(u_1, u_2, u_4) - p_1^K(u_1, u_2, u_4)\right].$$

From (5.19) and (5.20) for $\Gamma_K(u_1, u_2, u_3, u_4) \ne 0$ we have

$$|a(u_1)| < K,\ |a(u_2)| < K,\ g(u_3) < K,\ g(u_4) > K, \tag{5.27}$$

Next, for any $L > K$ and $\mathbf{u} = (u_1, u_2, u_3, u_4)$ we have

$$(p_1^K, p_1 - p_1^K)_T = \int_{\mathbb{R}^4} \Gamma_K(\mathbf{u}) d\mu_T = \int_{\mathbb{R}^4} \Gamma_K(\mathbf{u}) \chi_2^L(u_4) d\mu_T + I, \tag{5.28}$$

where with some constant C_K depending on K

$$|I| \le C_K \int_{\mathbb{R}^4} g(u_4)\left(1 - \chi_2^L(u_4)\right) d|\mu_T|. \tag{5.29}$$

It follows from (5.14), (5.15) and (5.20) that $\Gamma_K(\mathbf{u})\chi_2^L(u_4)$ is a linear combination of functions satisfying the conditions of Lemma 4.4. Applying Lemma 4.4 and taking into account that $\Gamma_K(u,u,u,u) = 0$ for $u \in \mathbb{R}$ (see (5.27)), we obtain

$$\lim_{T\to\infty} \int_{\mathbb{R}^4} \Gamma_K(\mathbf{u}) \chi_2^L(u_4) d\mu_T = \int_{\mathbb{R}} \Gamma_K(u,u,u,u)\chi_2^L(u) du = 0. \tag{5.30}$$

For given $\varepsilon > 0$ and sufficiently large L by (4.16) we get

$$C_K \int_{\mathbb{R}^4} g(u)\left(1 - \chi_2^L(u)\right) d|\mu_T| \le C_K \int_{\{u:\ g(u) > L\}} g(u) du \le \varepsilon. \tag{5.31}$$

From (5.28) – (5.31) we obtain

$$\lim_{T\to\infty} (p_1^K, p_1 - p_1^K)_T = 0.$$

This combined with (5.23) and (5.26) yields

$$\lim_{T\to\infty} \|p_1 - p_1^K\|_T = 0. \tag{5.32}$$

Finally, we prove that

$$\lim_{T\to\infty} \|p_2 - p_2^K\|_T = 0. \tag{5.33}$$

Indeed, according to (5.15), (5.20) and (4.17), we have

$$\begin{aligned}
\|p_2 - p_2^K\|_T &\le \int_{\mathbb{R}^4} [1 - \chi_1^K(u_1)] f(u_1) g(u_1) f(u_2) g(u_2) d|\mu_T| \\
&\quad + \int_{\mathbb{R}^4} [1 - \chi_1^K(u_2)] f(u_1) g(u_1) f(u_2) g(u_2) d|\mu_T| \\
&\le \int_{\{u:|f(u)| > \sqrt{K}\}} f^2(u) g^2(u) du + \int_{\{u:|g(u)|>K\}} f^2(u) g^2(u) du = o(1),
\end{aligned}$$

when $K \to \infty$ (uniformly on T). A combination of (5.18), (5.22), (5.32) and (5.33) yields (5.7). This completes the proof of Lemma 5.1. Theorem 2.1 is proved.

Proof of Theorem 2.2. By a change of variables $x_1 = u$, $x_1 - x_2 = u_1$, $x_2 - x_3 = u_2$, $x_3 - x_4 = u_3$, in view of (4.1), (4.2), (4.5) and (4.6), we can write

$$\begin{aligned}
\operatorname{tr}\left[W_T^h(f) W_T^h(g)\right]^2 &= \int_{\mathbb{R}^3} \int_{\mathbb{R}} f(u) g(u - u_1) f(u - u_1 - u_2) g(u - u_1 - u_2 - u_3) du \\
&\times\ H_{1,T}(u_1) H_{1,T}(u_2) H_{1,T}(u_3) H_{1,T}(-u_1 - u_2 - u_3) du_1 du_2 du_3 \\
&=:\ 8\pi^3 H_4 \int_{\mathbb{R}^3} \Psi(\mathbf{u}) \Phi_T^h(\mathbf{u}) d\mathbf{u}, \qquad (5.34)
\end{aligned}$$

where $\mathbf{u} = (u_1, u_2, u_3)$, $\Phi_T^h(\mathbf{u}) := \Phi_{4,T}^h(\mathbf{u})$ is defined by (4.1), $\Psi(\mathbf{u}) := \varphi(u_1, u_1 + u_2, u_1 + u_2 + u_3)$ and $\varphi(u_1, u_2, u_3)$ is defined by (2.6). By Theorem 2.1 and (5.34) we need to prove that

$$\lim_{T\to\infty} \int_{\mathbb{R}^3} \Psi(\mathbf{u})\Phi_T^h(\mathbf{u})d\mathbf{u} = \int_{\mathbb{R}} f^2(x)g^2(x)dx. \tag{5.35}$$

Now, since both functions $\varphi(u_1, u_2, u_3)$ and $\Psi(u_1, u_2, u_3) = \varphi(u_1, u_1+u_2, u_1+u_2+u_3)$ are square integrable and continuous at $(0, 0, 0)$, and

$$\Psi(0, 0, 0) = \int_{\mathbb{R}} f^2(x)g^2(x)dx,$$

from Lemma 4.1 we obtain (5.35). Theorem 2.2 is proved.

Proof of Theorem 2.5. In view of (4.5) and (4.7), we need to prove that (2.8) and (2.9) imply

$$\lim_{T\to\infty} \frac{1}{T} \int_{\mathbb{R}^4} f(x_1)f(x_2)g(x_3)g(x_4)d\mu_T = 8\pi^3 H_4 \int_{\mathbb{R}} f^2(x)g^2(x)dx. \tag{5.36}$$

If $\alpha, \beta \geq 0$, then (2.8), (2.9) imply $f \in L^{1/\alpha}(\mathbb{R})$, $g \in L^{1/\beta}(\mathbb{R})$, and the result follows from Theorem 2.3. Assuming $\beta < 0$, from (2.8) we have $g \in L^\infty(\mathbb{R})$.

Denote

$$\overline{f}(x) = \begin{cases} 0, & \text{if} \quad x \in \left[-\frac{a}{2}, \frac{a}{2}\right] \\ f(x), & \text{otherwise} \end{cases}, \qquad \overline{g}(x) = \begin{cases} 0, & \text{if} \quad x \in [-a, a] \\ g(x), & \text{otherwise,} \end{cases}$$

where the number $a > 0$ is as in the statement of the theorem, and let $\underline{f} = f - \overline{f}$, $\underline{g} = g - \overline{g}$. Then we have

$$\begin{aligned} \frac{1}{T} \int_{\mathbb{R}^4} f(x_1)f(x_2)g(x_3)g(x_4)d\mu_T &= \frac{1}{T} \int_{\mathbb{R}^4} \overline{f}(x_1)f(x_2)g(x_3)g(x_4)d\mu_T \\ &+ \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1)\overline{f}(x_2)g(x_3)g(x_4)d\mu_T + \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)g(x_3)g(x_4)d\mu_T \\ &= I_T^1 + I_T^2 + I_T^3. \end{aligned} \tag{5.37}$$

Since $\overline{f},\ g \in L^\infty(\mathbb{R})$ and $f \in L^1(\mathbb{R})$ we obtain

$$\begin{aligned} \lim_{T\to\infty} I_T^1 &= 8\pi^3 H_4 \int_{\mathbb{R}} \overline{f}(x)f(x)g^2(x)dx = 8\pi^3 H_4 \int_{|x|>\frac{a}{2}} f^2(x)g^2(x)dx, \\ \lim_{T\to\infty} I_T^2 &= 8\pi^3 H_4 \int_{\mathbb{R}} \underline{f}(x)\overline{f}(x)g^2(x)dx = 0. \end{aligned} \tag{5.38}$$

Next, we can write

$$\begin{aligned} I_T^3 = \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\underline{g}(x_3)\underline{g}(x_4)d\mu_T + \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\underline{g}(x_3)\overline{g}(x_4)d\mu_T \\ + \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\overline{g}(x_3)\underline{g}(x_4)d\mu_T + \frac{1}{T} \int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\overline{g}(x_3)\overline{g}(x_4)d\mu_T = \sum_{i=1}^4 J_T^i. \end{aligned} \tag{5.39}$$

We have

$$J_T^1 = \frac{1}{T}\int_{[-a,a]^4} \underline{f}(x_1)\underline{f}(x_2)\underline{g}(x_3)\underline{g}(x_4)d\mu_T.$$

Arguments similar to those leading to equality (4.3) from [16] may be used to prove that

$$\lim_{T\to\infty} J_T^1 = 8\pi^3 H_4 \int_{-a}^{a} \underline{f}^2(x)g^2(x)dx = 8\pi^3 H_4 \int_{-a/2}^{a/2} f^2(x)g^2(x)dx. \tag{5.40}$$

Since $f(x_1)f(x_2)\in L^1(\mathbb{R}^2)$ for any $\varepsilon>0$ we can find $\delta>0$ satisfying

$$\int_{|x_1-x_2|<\delta} |f(x_1)f(x_2)|dx_1dx_2 < \varepsilon.$$

Because $g\in L^\infty(\mathbb{R})$, in view of (4.18) and (4.19) for sufficiently large T we obtain

$$\begin{aligned}|J_T^2| &\le C\cdot T \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} |f(x_1)f(x_2)| \int_{-a/2}^{a/2} \psi_T(x_1-x_3)\psi_T(x_2-x_3)\\ &\quad\times \int_{|x_4|>a} x_4^{-2}dx_4dx_3dx_1dx_2\\ &\le C\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} |f(x_1)f(x_2)|(1+T|x_1-x_2|)^{-1/2}dx_1dx_2\\ &\le C\int_{|x_1-x_2|<\delta} |f(x_1)f(x_2)|dx_1dx_2\\ &\quad+(1+T\delta)^{-1/2}\int_{|x_1-x_2|\ge\delta} |f(x_1)f(x_2)|dx_1dx_2 \le 2\varepsilon.\end{aligned}$$

This means that

$$\lim_{T\to\infty} J_T^2 = 0. \tag{5.41}$$

Likewise, we get

$$\lim_{T\to\infty} J_T^3 = 0. \tag{5.42}$$

To estimate the integral J_T^4 in (5.39) note that in this case $|x_i-x_j|>\frac{a}{2}$, $i=1,2$, $j=3,4$. Therefore

$$\begin{aligned}|J_T^4| &\le \frac{C}{T}\int_{\mathbb{R}^4} \underline{f}(x_1)\underline{f}(x_2)\overline{g}(x_3)\overline{g}(x_4)dx_1dx_2dx_3dx_4\\ &\le \frac{C}{T}\|f\|^2_{L^1(\mathbb{R})}\|g\|^2_{L^1(\mathbb{R})} \to 0 \quad \text{as} \quad T\to\infty. \end{aligned}\tag{5.43}$$

From (5.39) – (5.43) we obtain

$$\lim_{T\to\infty} I_T^3 = 8\pi^3 H_4 \int_{-a/2}^{a/2} f^2(x)g^2(x)dx,$$

which combined with (5.37) and (5.38) yields (5.36). Theorem 2.5 is proved.

Proof of Theorem 3.1. Taking into account the equality

$$T^{1/2}\left[J_T^h - J\right] = T^{1/2}\left[\mathbb{E}(J_T^h) - J\right] + T^{1/2}\left[J_T^h - \mathbb{E}(J_T^h)\right], \tag{5.44}$$

to prove the theorem we have to establish the following two asymptotic relations:

$$T^{1/2}\left[\mathbb{E}(J_T^h) - J\right] \to 0 \quad \text{as} \quad T \to \infty, \tag{5.45}$$

$$T^{1/2}\left[J_T^h - E(J_T^h)\right] \xrightarrow{d} \eta \sim N\left(0, \sigma_h^2(J)\right) \quad \text{as} \quad T \to \infty, \tag{5.46}$$

where $\sigma_h^2(J)$ is given by (3.2).

Observe first that the relation (5.45) is an immediate consequence of Theorem 2.1 of Ginovyan and Sahakyan [18], since under each of the conditions a)-d) in assumption (A), we have $\beta_1 + \beta_2 > 1/2$.

Now we proceed to show that the relation (5.46) follows from Theorem 2.3. To do this we need to show that, under the assumption (A), there exist numbers p_1 ($p_1 > p$) and q_1 ($q_1 > q$), such that $H_p(\beta_1) \subset L_{p_1}$, $H_q(\beta_2) \subset L_{q_1}$ and $1/p_1 + 1/q_1 \leq 1/2$.

The case $\beta_1 > 1/p$, $\beta_2 > 1/q$ is obvious, since in view of Lemma 4.9 b) we have $H_p(\beta_1) \subset L_\infty$ and $H_q(\beta_2) \subset L_\infty$.

Let $\beta_1 \leq 1/p$, $\beta_2 \leq 1/q$ and $\beta_1 + \beta_2 > 1/2$. For an arbitrary number $\varepsilon > 0$ satisfying $\beta_1 > \varepsilon$ and $\beta_2 > \varepsilon$, we set

$$\frac{1}{p_1} = \frac{1}{p} - \beta_1 + \varepsilon \quad \text{and} \quad \frac{1}{q_1} = \frac{1}{q} - \beta_2 + \varepsilon.$$

It is easy to see that $p < p_1 < p/(1 - \beta_1 p)$ and $q < q_1 < q/(1 - \beta_2 q)$. Hence by Lemma 4.9 a) we obtain $H_p(\beta_1) \subset L_{p_1}$ and $H_q(\beta_2) \subset L_{q_1}$. On the other hand, we have

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p} + \frac{1}{q} - (\beta_1 + \beta_2) + 2\varepsilon = 1 - (\beta_1 + \beta_2) + 2\varepsilon.$$

Since $\beta_1 + \beta_2 > 1/2$, choosing ε sufficiently small, we obtain $1/p_1 + 1/q_1 \leq 1/2$.

Now let $\beta_1 > 1/p$ and $1/q - 1/2 < \beta_2 \leq 1/q$. By Lemma 4.9 b) we have $H_p(\beta_1) \subset L_\infty$. For an arbitrary number $\varepsilon > 0$ satisfying $\beta_2 > \varepsilon$, we set

$$\frac{1}{q_1} = \frac{1}{q} - \beta_2 + \varepsilon.$$

Obviously $q < q_1 < q/(1 - \beta_2 q)$, and hence $H_q(\beta_2) \subset L_{q_1}$ by Lemma 4.9 a).
Further, we have

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{q} - \beta_2 + \varepsilon.$$

Since $1/q - \beta_2 < 1/2$, choosing ε sufficiently small we obtain $1/p_1 + 1/q_1 \leq 1/2$.

The case $\beta_2 > 1/q$ and $1/p - 1/2 < \beta_1 \leq 1/p$ can be treated similarly.

Thus, we can apply Theorem 2.3, to obtain that

$$Q_T^h = T^{-1/2}\left(Q_T^h - \mathbb{E}[Q_T^h]\right) \xrightarrow{d} \eta \sim N(0, \sigma_h^2) \quad \text{as} \quad T \to \infty, \tag{5.47}$$

where Q_T^h and $\sigma_h^2 = \sigma_h^2(Q)$ are given by (1.2) and (2.3), respectively.

Also, in view of (1.10), (1.12) and (2.3), we have

$$\sigma_h^2(J) = \frac{1}{4\pi^2 H_2^2}\sigma_h^2(Q) = 4\pi e(h)\int_{\mathbb{R}} f^2(\lambda)g^2(\lambda)d\lambda, \quad e(h) = H_4 H_2^{-2}. \tag{5.48}$$

Putting together (5.47) and (5.48), we obtain the relation (5.46). Theorem 3.1 is proved.

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