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# CONVERGENCE OF A SUBSEQUENCE OF TRIANGULAR PARTIAL SUMS OF DOUBLE WALSH-FOURIER SERIES

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Abstract. In 1987 Harris proved-among others that for each $1 \leq p < 2$ there exists a two-dimensional function $f \in L_p$ such that its triangular Walsh-Fourier series does not converge almost everywhere. In this paper we prove that the set of functions from the space $L_p(\mathbb{I}^2)$, $1 \leq p < 2$, with subsequence of triangular partial means $S_{2^A}^{\triangle}(f)$ of the double Walsh-Fourier series convergent in measure on $\mathbb{I}^2$ is of first Baire category in $L_p(\mathbb{I}^2)$. We also prove that for each function $f \in L_2(\mathbb{I}^2)$ a.e. convergence $S_{a(n)}^{\triangle}(f) \to f$ holds, where $a(n)$ is a lacunary sequence of positive integers.

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## 1. Introduction

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I} = [0,1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \leq p \leq 2^n$.

Denote the dyadic expension of $n \in \mathbb{N}$ and $x \in \mathbb{I}$ by

$$n = \sum_{j=0}^{\infty} n_j 2^j, n_j = 0,1 \quad \text{and} \quad x = \sum_{j=0}^{\infty} \frac{x_j}{2^{j+1}}, x_j = 0,1.$$

In the case of $x \in \mathbb{Q}$ chose the expension which terminates in zeros. $n_i, x_i$ are the $i$-th coordinates of $n$, $x$, respectively. Define the dyadic addition $\dot{+}$ as

$$x \dot{+} y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

Denote by $\oplus$ the dyadic (or logical) addition. That is,

$$k \oplus n = \sum_{i=0}^{\infty} |k_i - n_i| \, 2^i,$$

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where $k_i, n_i$ are the $i$th coordinate of natural numbers $k, n$ with respect to number system based 2.

The sets $I_n(x) = \{y \in \mathbb{I} : y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$ for $x \in \mathbb{I}, I_n = I_n(0)$ for $0 < n \in \mathbb{N}$ and $I_0(x) = \mathbb{I}$ are the dyadic intervals of $\mathbb{I}$. For $0 < n \in \mathbb{N}$ denote by $|n| = \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$. Set $e_j = 1/2^{j+1}$, the $i$-th coordinate of $e_i$ is 1, the rest are zeros $(i \in \mathbb{N})$.

The Rademacher system is defined by

$$r_n(x) = (-1)^{x_n}, \quad x \in \mathbb{I}, n \in \mathbb{N}.$$

The Walsh-Paley system is defined as the sequence of the Walsh-Paley functions:

$$w_n(x) = \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum\limits_{k=0}^{|n|} n_k x_k}, \ x \in \mathbb{I}, n \in \mathbb{N}.$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [13])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 2^{-n}) \\ 0, & \text{if } x \in [2^{-n}, 1) \end{cases}, \tag{1.1}$$

We consider the double system $\{w_n(x^1) \times w_m(x^2) : n, m \in \mathbb{N}\}$ on the unit square $\mathbb{I}^2 = [0, 1) \times [0, 1)$.

We denote by $L_0(\mathbb{I}^2)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbb{I}^2$. $\mu(A)$ is the Lebesgue measure of $A \subset \mathbb{I}^d$.

We denote by $L_p(\mathbb{I}^2)$ the class of all measurable functions $f$ that are 1-periodic with respect to all variable and satisfy

$$\|f\|_p = \left( \int_{\mathbb{I}^2} |f(y^1, y^2)|^p dy^1 dy^2 \right)^{1/p} < \infty.$$

If $f \in L_1(\mathbb{I}^2)$, then

$$\hat{f}(n^1, n^2) = \int_{\mathbb{I}^2} f(y^1, y^2) w_{n^1}(y^1) w_{n^2}(y^2) dy^1 dy^2$$

is the $(n^1, n^2)$-th Fourier coefficient of $f$.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{N^1,N^2}\left(x^1,x^2;f\right)=\sum_{n^1=0}^{N^1-1}\sum_{n^2=0}^{N^2-1}\hat{f}\left(n^1,n^2\right)w_{n^1}(x^1)w_{n^2}(x^2).$$

The triangular partial sums defined as

$$S_k^{\triangle}\left(x^1,x^2;f\right)=\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}\hat{f}\left(i,j\right)w_i(x^1)w_j(x^2).$$

Let $a=(a(n))$ be a lacunary sequence of positive integers with quotient $q$. That is, $a(n+1)/a(n)\geq q>1$ for any $n\in\mathbb{N}$. Now, set the maximal function

$$S_{a,*}^{\triangle}f=\sup_n\left|S_{a(n)}^{\triangle}\left(f\right)\right|.$$

In 1971 Fefferman proved [2] the following result with respect to the trigonometric system. Let $P$ be an open polygonal region in $\mathbb{R}^2$, containing the origin. Set

$$\lambda P=\left\{\left(\lambda x^1,\lambda x^2\right):\left(x^1,x^2\right)\in P\right\}$$

for $\lambda>0$. Then for every $p>1, f\in L_p\left([-\pi,\pi]^2\right)$ it holds the relation

$$\sum_{(n^1,n^2)\in\lambda P}\widehat{f}\left(n^1,n^2\right)\exp\left(i\left(n^1y^1+n^2y^2\right)\right)\to f\left(y^1,y^2\right)\text{ as }\lambda\to\infty$$

for a. e. $\left(y^1,y^2\right)\in[-\pi,\pi]^2$. That is, $S_{\lambda P}f\to f$ a.e. Sjulin gave [14] a better result in the case when $P$ is a rectangle. He proved a.e. convergence for the class $f\in L\left(\log^+L\right)^3\log\log L$ and for functions $f\in L\left(\log^+L\right)^2\log\log L$ when $P$ is a square. This result for squares is improved by Antonov [1]. There is a sharp constrant between the trigonometric and the Walsh case. In 1987 Harris proved [8] for the Walsh system that if $S$ is a region in $[0,\infty)\times[0,\infty)$ with piecewise $C^1$ boundary not always paralled to the axes and $1\leq p<2$, then there exists an $f\in L_p\left(\mathbb{I}^2\right)$ such that $S_{\lambda P}f$ does not converges a. e. and in $L_p$ norms as $\lambda\to\infty$. In particular, from theorem of Harris follows that for any $1\leq p<2$ there exists an $f\in L_p\left(\mathbb{I}^2\right)$ such that $S_{2^A}^{\triangle}f$ does not converges a. e. as $A\to\infty$.

In this paper we improve this result of Harris for tringular partial sums ($P=\triangle$), In particular, let $1\leq p<2$, then we prove that the set of the functions from the space $L_p(\mathbb{I}^2)$ with subsequence of triangular partial means $S_{2^A}^{\triangle}\left(f\right)$ of the double Walsh-Fourier series convergent in measure on $\mathbb{I}^2$ is of first Baire category in $L_p(\mathbb{I}^2)$. We also prove that for each function $f\in L_2(\mathbb{I}^2)$ a.e. convergence $S_{a(n)}^{\triangle}\left(f\right)\to f$ holds, where $a(n)$ is a lacunary sequence of positive integers.

For results with respect to convergence of rectangular and triangular partial sums of Walsh-Fourier series see [6, 12, 15, 9, 10, 11, 7].

## 2. The main results

The following results are the main statements of the paper.

**Theorem 2.1.** *Let $1 \leq p < 2$. The set of the functions from the space $L_p(\mathbb{I}^2)$ with subsequence of triangular partial sums $S_{2^A}^{\triangle}(f)$ of the double Walsh-Fourier series convergent in measure on $\mathbb{I}^2$ is of first Baire category in $L_p(\mathbb{I}^2)$.*

**Theorem 2.2.** *The operator $S_{a,*}^{\triangle}$ is of strong type $(L_2, L_2)$. More precisely,*

$$\|S_{a,*}^{\triangle} f\|_2 \leq C_q \|f\|_2.$$

By Theorem 2.2 and by the usual density argument we obtain the following result.

**Corollary 2.1.** *As $n \to \infty$ we have $S_{a(n)}^{\triangle}(f) \to f$ a.e. for every $f \in L_2(\mathbb{I}^2)$, where $a(n)$ is a lacunary sequence of positive integers.*

The following theorem is proved in [4, 5].

**Theorem GGT.** *Let $\{T_m\}_{m=1}^{\infty}$ be a sequence of linear continues operators, acting from space $L_p(\mathbb{I}^2)$ in to the space $L_0(\mathbb{I}^2)$. Suppose that there exists the sequence of functions $\{\xi_k\}_{k=1}^{\infty}$ from unit bull $S_p(0,1)$ of space $L_p(\mathbb{I}^2)$, sequences of integers $\{m_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ increasing to infinity such that*

$$\varepsilon_0 = \inf_k \mu\{\left(x^1, x^2\right) \in \mathbb{I}^2 : |T_{m_k}\xi_k\left(x^1, x^2\right)| > \lambda_k\} > 0.$$

*Then the set of functions $f$ from space $L_p(\mathbb{I}^2))$, for which the sequence $\{T_m f\}$ converges in measure to an a. e. finite function is of first Baire category in space $L_p(\mathbb{I}^2)$.*

*Proof of Theorem 2.1.* First we prove that there exists a function $h_A$ for which

$$\left\|h_A\right\|_p \leq 1 \tag{2.1}$$

and

$$\mu\left\{\left(x^1, x^2\right) \in \mathbb{I}^2 : \left|S_{2^A}^{\triangle}\left(x^1, x^2; h_A\right)\right| > \frac{2^{A/p}}{\sqrt{A}}\right\} \geq \frac{A}{2^{A+3}}. \tag{2.2}$$

Let

$$f_A\left(x^1, x^2\right) = \sum_{k=0}^{A-1} \sum_{l=0}^{2^A-1} w_{2^k \oplus l}\left(x^1\right) w_l\left(x^2\right)$$

and

$$h_A\left(x^1,x^2\right)=\frac{w_{2^A-1}\left(x^1\right)}{2^{A(1-1/p)}\sqrt{A}}f_A\left(x^1,x^2\right).$$

We can write

$$\begin{aligned}
\|f_A\|_p &= \left(\int\limits_{\mathbb{I}^2}\left|\sum_{k=0}^{A-1}w_{2^k}\left(x^1\right)D_{2^A}\left(x^1\dotplus x^2\right)\right|^p dx^1dx^2\right)^{1/p}\\
&= \left(\int\limits_{\mathbb{I}}\left|\sum_{k=0}^{A-1}w_{2^k}\left(x^1\right)\right|^p\left(\int\limits_{\mathbb{I}}D_{2^A}^p\left(x^1\dotplus x^2\right)dx^2\right)dx^1\right)^{1/p}\\
&= \left(\int\limits_{\mathbb{I}}\left|\sum_{k=0}^{A-1}w_{2^k}\left(x^1\right)\right|^p dx^1\left(\int\limits_{\mathbb{I}}D_{2^A}^p\left(x^2\right)dx^2\right)\right)^{1/p}\\
&\le \left(\int\limits_{\mathbb{I}}\left(\sum_{k=0}^{A-1}w_{2^k}\left(x^1\right)\right)^2 dx^1\right)^{1/2}2^{A(1-1/p)}=\sqrt{A}2^{A(1-1/p)}.
\end{aligned}$$

Hence (2.1) is proved.

From simple calculation we obtain that

$$\begin{aligned}
\widehat{h}_A(i,j) &= \int\limits_{\mathbb{I}^2}h_A\left(y^1,y^2\right)w_i\left(y^1\right)w_j\left(y^2\right)dy^1dy^2\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\int\limits_{\mathbb{I}^2}f_A\left(y^1,y^2\right)w_{2^A-1}\left(y^1\right)w_i\left(y^1\right)w_j\left(y^2\right)dy^1dy^2\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\int\limits_{\mathbb{I}^2}f_A\left(y^1,y^2\right)w_{2^A-1-i}\left(y^1\right)w_j\left(y^2\right)dy^1dy^2\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\widehat{f}_A\left(2^A-1-i,j\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)=\sum_{i+j<2^A}\widehat{h}_A(i,j)\,w_i\left(x^1\right)w_j\left(x^2\right)\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\sum_{i+j<2^A}\widehat{f}_A\left(2^A-1-i,j\right)w_i\left(x^1\right)w_j\left(x^2\right)\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\sum_{i=0}^{2^A-1}\sum_{j=0}^{2^A-i-1}\widehat{f}_A\left(2^A-1-i,j\right)w_i\left(x^1\right)w_j\left(x^2\right)\\
&= \frac{1}{2^{A(1-1/p)}\sqrt{A}}\sum_{i=0}^{2^A-1}\sum_{j=0}^{i}\widehat{f}_A(i,j)\,w_{2^A-1-i}\left(x^1\right)w_j\left(x^2\right).
\end{aligned}$$

Consequently,

$$S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)=\frac{w_{2^A-1}\left(x^1\right)}{2^{A(1-1/p)}\sqrt{A}}\sum_{k=0}^{A-1}\sum_{l\leq 2^k\oplus l}w_{2^k\oplus l}\left(x^1\right)w_l\left(x^2\right).$$

We see that $l\leq 2^k\oplus l$ holds if and only if $l_k=0$. Hence, we have

$$S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)=\frac{w_{2^A-1}\left(x^1\right)}{2^{A(1-1/p)}\sqrt{A}}\sum_{k=0}^{A-1}w_{2^k}\left(x^1\right)\sum_{l\in\{l=0,1,\ldots,2^A-1:l_k=0\}}w_l\left(x^1\dotplus x^2\right).$$

Let

$$\left(x^1,x^2\right)\in G_{A,s}=I_A\left(t_0,...,t_{s-1},1,t_{s+1},...,t_{A-1}\right)\times I_A\left(t_0,...,t_{s-1},0,t_{s+1},...,t_{A-1}\right).$$

Since $x^1\dotplus x^2=I_A\left(e_s\right)$, we can write

$$\begin{aligned}\sum_{l\in\{l=0,1,\ldots,2^A-1:l_k=0\}}w_l\left(x^1\dotplus x^2\right) &= \sum_{l_0=0}^{1}\cdots\sum_{l_{k-1}=0}^{1}\sum_{l_{k+1}=0}^{1}\cdots\sum_{l_{A-1}=0}^{1}\left(-1\right)^{l_s}\\ &= \left\{\begin{array}{l}2^{A-1},\ \text{ if } k=s\\ 0,\ k\neq s\end{array}\right..\end{aligned}$$

Hence,

$$\left|S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)\right|\geq\frac{2^{A-1}}{2^{A(1-1/p)}\sqrt{A}}\sum_{s=0}^{A-1}\mathbb{I}_{G_{A,s}}\left(x^1,x^2\right)=\frac{2^{A/p}}{2\sqrt{A}}\sum_{s=0}^{A-1}\mathbb{I}_{G_{A,s}}\left(x^1,x^2\right).$$

Set

$$\Omega_A=\bigcup_{s=0}^{A-1}\bigcup_{t_0=0}^{1}\cdots\bigcup_{t_{s-1}=0}^{1}\bigcup_{t_{s+1}=0}^{1}\cdots\bigcup_{t_{A-1}=0}^{1}G_{A,s}$$

From estimation (2.3) we get

$$\begin{aligned}&\mu\left\{\left(x^1,x^2\right)\in\mathbb{I}^2:\left|S_{2^A}^{\triangle}\left(x^1,x^2;h_A\right)\right|>\frac{2^{A/p}}{2\sqrt{A}}\right\}\\ \geq\ &\mu\left(\Omega_A\right)=\frac{1}{2^{2A}}\sum_{s=0}^{A-1}\sum_{x_0=0}^{1}\cdots\sum_{x_{s-1}=0}^{1}\sum_{x_{s+1}=0}^{1}\cdots\sum_{x_{A-1}=0}^{1}=\frac{A}{2^{A+1}}.\end{aligned}$$

Now, we prove that there exists $\left(x_1^1,x_1^2\right),...,\left(x_{p(A)}^1,x_{p(A)}^2\right)\in\mathbb{I}^2$, $p\left(A\right):=\left[2^{A+3}/A\right]+1$, such that

$$\mu\left(\bigcup_{j=1}^{p(A)}\left(\Omega_A\dotplus\left(x_j^1,x_j^2\right)\right)\right)\geq\frac{1}{2}. \tag{2.3}$$

Indeed,

$$\mu\left(\bigcup_{j=1}^{p(A)}\left(\Omega_A\dotplus\left(x_j^1,x_j^2\right)\right)\right)=1-\mu\left(\bigcap_{j=1}^{p(A)}\left(\overline{\Omega_A\dotplus\left(x_j^1,x_j^2\right)}\right)\right)$$

$$=\quad 1-\int\limits_{\mathbb{I}^2}\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x_1^1,t^2\dotplus x_1^2\right)\cdots\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x_{p(A)}^1,t^2\dotplus x_{p(A)}^2\right)dt^1dt^2.$$

Interpreting $\mathbb{I}_{\overline{\Omega_A}}\left(t^1\dotplus x_1^1,t^2\dotplus x_1^2\right)\cdots\mathbb{I}_{\overline{\Omega_A}}\left(t^1\dotplus x_{p(A)}^1,t^2\dotplus x_{p(A)}^2\right)$ as a function of the $2p\left(A\right)+2$ variables $t^1,t^2$, $\left(x_1^1,x_1^2\right),...,\left(x_{p(A)}^1,x_{p(A)}^2\right)$ and integrating over all variables, each over $\mathbb{I}^2$, we note that

$$\int\limits_{\mathbb{I}^2}\cdots\int\limits_{\mathbb{I}^2}\int\limits_{\mathbb{I}^2}\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x_1^1,t^2\dotplus x_1^2\right)\cdots\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x_{p(A)}^1,t^2\dotplus x_{p(1)}^2\right)$$
$$dt^1dt^2dx_1^1dx_1^2\cdots dx_{p(A)}^1dx_{p(A)}^2$$
$$=\quad\int\limits_{\mathbb{I}^2}\left(\int\limits_{\mathbb{I}^2}\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x_1^1,t^2\dotplus x_1^2\right)dx_1^1dx_1^2\right)\cdots$$
$$\left(\int\limits_{\mathbb{I}^2}\mathbb{I}_{\overline{\Omega}_A}\left(t^1\dotplus x_{p(A)}^1,t^2\dotplus x_{p(A)}^2\right)dx_{p(A)}^1dx_{p(A)}^2\right)dt^1dt^2$$
$$=\quad\left(\mu\left(\overline{\Omega}_A\right)\right)^{p(A)}=\left(1-\mu\left(\Omega_A\right)\right)^{p(A)}\leq\left(1-\frac{1}{p\left(A\right)}\right)^{p(A)}\leq\frac{1}{2}.$$

Consequently, there exists $\left(x_1^1,x_1^2\right),...,\left(x_{p(A)}^1,x_{p(A)}^2\right)\in\mathbb{I}^2$ such that

$$\int\limits_{\mathbb{I}^2}\mathbb{I}_{\overline{\Omega_A}}\left(t^1\dotplus x_1^1,t^2\dotplus x_1^2\right)\cdots\mathbb{I}_{\overline{\Omega_A}}\left(t^1\dotplus x_{p(A)}^1,t^2\dotplus x_{p(A)}^2\right)dt^1dt^2\leq\frac{1}{2}. \tag{2.4}$$

Combining (2.4) and (2.4) we conclude that

$$\mu\left(\bigcup_{j=1}^{p(A)}\left(\Omega_A\dotplus\left(x_j^1,x_j^2\right)\right)\right)\geq 1-\frac{1}{2}=\frac{1}{2}.$$

Hence (2.3) is proved. Let $\left(t:=t^1\dotplus t^2\in\mathbb{I}\right)$

$$F_A\left(x^1,x^2,t\right)\quad=\quad\frac{1}{\left(4p\left(A\right)\right)^{1/p}}\sum_{j=1}^{p(A)}r_j\left(t^1\dotplus t^2\right)h_A\left(x^1\dotplus x_j^1,x^2\dotplus x_j^2\right)$$
$$=\quad\frac{1}{\left(4p\left(A\right)\right)^{1/p}}\sum_{j=1}^{p(A)}r_j\left(t\right)h_A\left(x^1\dotplus x_j^1,x^2\dotplus x_j^2\right).$$

Then it is proved in ([3], pp. 7-12) that there exists $t_0 \in \mathbb{I}$, such that

$$\int_{\mathbb{I}} \left|F_A\left(x^1, x^2, t_0\right)\right|^p dx^1 dx^2 \leq 1 \tag{2.5}$$

and

$$\mu\left\{\left(x^1, x^2\right) \in \mathbb{I}^2 : \left|S_{2^A}^{\triangle}\left(x^1, x^2; F_A\right)\right| > \frac{2^{A/p} / \left(2\sqrt{A}\right)}{\left(p\left(A\right)\right)^{1/p}}\right\} \geq \frac{1}{8}. \tag{2.6}$$

Set $\xi_A\left(x^1, x^2\right) := F_A\left(x^1, x^2, t_0\right)$. Then from (2.5) and (2.6) we have $\left\|\xi_A\right\|_p \leq 1$ and

$$\mu\left\{\left(x^1, x^2\right) \in \mathbb{I}^2 : \left|S_{2^A}^{\triangle}\left(x^1, x^2; \xi_A\right)\right| > 2^{1-3/p} A^{1/p-1/2}\right\} \geq \frac{1}{8}$$

and using Theorem GGT we complete the proof of Theorem 2.1. $\square$

*Proof of Theorem 2.2.* First, we suppose that $q \geq 2$. Let $S_n^{\square}(f)$ be $n$-th square partial sums of the two-dimensional Walsh-Fourier series. It is easy to see that the spectrums of the polynomials

$$S_{a(n)}^{\square}(f) - S_{a(n)}^{\triangle}(f), \quad n = 1, 2, ...$$

are pairwise disjoint that implies

$$\begin{aligned}
\left\|\sup_n \left|S_{a(n)}^{\triangle}(f)\right|\right\|_2^2 &\leq 2\left\|\sup_n \left|S_{a(n)}^{\square}(f)\right|\right\|_2^2 + 2\left\|\sup_n \left|S_{a(n)}^{\triangle}(f) - S_{a(n)}^{\square}(f)\right|\right\|_2^2 \\
&\leq 2\left\|\sup_n \left|S_{a(n)}^{\square}(f)\right|\right\|_2^2 + 2\sum_n \left\|S_{a(n)}^{\triangle}(f) - S_{a(n)}^{\square}(f)\right\|_2^2 \\
&\leq 2\left\|\sup_n \left|S_{a(n)}^{\square}(f)\right|\right\|_2^2 + 2\left\|f\right\|_2^2 \leq c\left\|f\right\|_2^2,
\end{aligned}$$

where the last inequality is obtained from the $L_2$ boundedness of the square partial sums majorant operator (see [13]). This completes the proof of Theorem 2.2 in the case of $q \geq 2$. If $2 > q > 1$, then let $Q$ the least natural number for which $q^Q \geq 2$. For any fixed $j = 0, \ldots, Q-1$ we have that the quotient of lacunary sequence n integers $(a(Qn+j))$ is at least 2 since $a(Q(n+1)+j) \geq q^Q a(Qn+j)$. From the above written we have

$$\left\|\sup_n \left|S_{a(Qn+j)}^{\triangle} f\right|\right\|_2^2 \leq C\left\|f\right\|_2^2$$

and consequently we also have $\left\|S_{a,*}^{\triangle} f\right\|_2^2 \leq C_q \left\|f\right\|_2^2$. $\square$

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