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ON EXCESS OF RETRO BANACH FRAMES

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Abstract. The excess of a frame is the greatest number of elements that can be removed from a given frame, yet leave a set which is a frame for the underlying space. We present a characterization of retro Banach frames in Banach spaces with finite excess. A sufficient condition for the existence of a retro Banach frame with infinite excess is obtained.

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1. Introduction and preliminaries

Duffin and Schaeffer [6], while addressing some difficult problems from the theory of nonharmonic Fourier series introduced *frames* (or *Hilbert frames*) for Hilbert spaces. Daubechies, Grossmann and Meyer [5] found a fundamental new application to wavelets and Gabor transforms in which frames continue to play an important role. For utility of frames in applied mathematics, see [1, 4].

A sequence (finite or countable) $\{f_k\} \subset \mathcal{H}$ is called a *frame* (or a *Hilbert frame*) for a separable Hilbert space $\mathcal{H}$, if there exist constants $0 < \mathrm{A} \leq \mathrm{B} < \infty$ such that

$$\mathrm{A}\|f\|^2 \leq \sum_k |\langle f, f_k\rangle|^2 \leq \mathrm{B}\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

Gröchenig [9] generalized Hilbert frames to Banach spaces. Before the concept of Banach frames was formalized, it appeared in the foundational work of Feichtinger and Gröchenig [7, 8] related to *atomic decompositions*. An atomic decomposition allow a representation of every vector of the space via a series expansion in terms of a fixed sequence of vectors which we call *atoms*. On the other hand, a Banach frame for a Banach space ensure reconstruction via a bounded linear operator or the *synthesis operator*. Casazza, Han and Larson studied atomic decompositions and Banach frames

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in [3]. Han and Larson [10] defined a Schauder frame for a Banach space $\mathcal{X}$ to be an inner direct summand (that is, a compression) of a Schauder basis of $\mathcal{X}$. Retro Banach frames were introduced in [11] and were further studied in [14, 15].

In this paper, motivated by the recent work of Balan et al. [2] in the direction of excess of frames in Hilbert spaces, we present necessary and sufficient conditions for excess of retro Banach frames in Banach spaces.

In the remaining part of this section we recall some basic definitions and results which will be used throughout this paper. Let $\mathcal{X}$ be an infinite dimensional separable real (or complex) Banach space, and let $\mathcal{X}^*$ be the dual space (topological) of $\mathcal{X}$. For a sequence $\{f_k\} \subset \mathcal{X}$, by $[f_k]$ we denote the closure of $\text{span}\{f_k\}_k$ in the norm topology of $\mathcal{X}$. For a set L, let $|L|$ denote the number of elements in L. The set of positive integers is denoted by $\mathbb{N}$.

Definition 1.1 ([11]). A system $\mathcal{F} \equiv (\{x_n\}, \Theta)$ $(\{x_n\} \subset \mathcal{X}, \Theta : \mathcal{Z}_d \to \mathcal{X}^*)$ is called a *retro Banach frame* for $\mathcal{X}^*$ with respect to an associated Banach space of scalar valued sequences $\mathcal{Z}_d$, if the following conditions are fulfilled:

(*i*) $\{f(x_n)\} \in \mathcal{Z}_d$ for each $f \in \mathcal{X}^*$,

(*ii*) there exist positive constants $0 < \text{A}_0 \leq \text{B}_0 < \infty$ such that

$$\text{A}_0\|f\| \leq \|\{f(x_n)\}\|_{\mathcal{Z}_d} \leq \text{B}_0\|f\| \text{ for all } f \in \mathcal{X}^*,$$

(*iii*) Θ is a bounded linear operator such that $\Theta(\{f(x_n)\}) = f, \ f \in \mathcal{X}^*$.

The positive constants A_0 and B_0 are called the *lower* and *upper retro frame bounds* of the frame $\mathcal{F}$, respectively. The operator $\Theta : \mathcal{Z}_d \to \mathcal{X}^*$ is called the *retro pre-frame operator* (or simply *a reconstruction operator*) associated with $\mathcal{F}$. If there exists no reconstruction operator Θ_m such that $(\{x_n\}_{k \neq m}, \Theta_m)$ ($m \in \mathbb{N}$ is arbitrary) is a retro Banach frame for $\mathcal{X}^*$, then $\mathcal{J}$ is called an *exact* retro Banach frame for $\mathcal{X}^*$.

Lemma 1.1 ([12]). *Let $\mathcal{X}$ be a Banach space and $\{f_n\} \subset \mathcal{X}^*$ be a sequence such that*

$$\{x \in \mathcal{X} : f_n(x) = 0 \ \textit{for all } n \in \mathbb{N}\} = \{0\}.$$

Then $\mathcal{X}$ is linearly isometric to the Banach space $\mathcal{Z}_d = \{\{f_n(x)\} : x \in \mathcal{X}\}$, where the norm of $\mathcal{Z}_d$ is given by

$$\|\{f_n(x)\}\|_{\mathcal{Z}_d} = \|x\|_{\mathcal{X}}, x \in \mathcal{X}.$$

2. Main results

We start by the definition of excess of a retro Banach frame.

Definition 2.1. Let $\mathcal{F} \equiv (\{x_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_d$. The *excess* of $\mathcal{F}$, denoted by $\mathrm{e}(\mathcal{F})$, is defined as follows:

$$\mathrm{e}(\mathcal{F}) = \sup\Big\{|J| : (\{x_k\}_{k\notin J}, \Theta_0) \text{ is a retro Banach frame for } \mathcal{X}^*, J \subset \mathbb{N}\Big\}.$$

Remark 2.1. One may observe that $\mathrm{e}(\mathcal{F})$ is an integer-valued (extended) function, and if $\mathcal{F}$ is an exact retro Banach frame, then $\mathrm{e}(\mathcal{F}) = 0$.

Example 2.1. Let $\mathcal{X} = \ell^p$ $(1 \leq p < \infty)$ and let $\{\chi_k\}$ be the sequence of canonical unit vectors in $\mathcal{X}$.

Define $\{x_k\} \subset \mathcal{X}$ as follows:

$$x_k = \chi_1 \ (1 \leq k \leq n), \quad \text{and} \quad x_k = \chi_{k-n}, \ k > n \quad (n \in \mathbb{N}). \tag{2.1}$$

Then, in view of Lemma 1.1, $\mathcal{Z}_d = \{\{f(x_k)\} : f \in \mathcal{X}^*\}$ is a Banach space with the norm

$$\|\{f(x_k)\}\|_{\mathcal{Z}_d} = \|f\|_{\mathcal{X}^*}, \ f \in \mathcal{X}^*.$$

Define $\Theta : \mathcal{Z}_d \to \mathcal{X}^*$ by $\Theta(\{f(x_k)\}) = f$, $f \in \mathcal{X}^*$. Then, Θ is a bounded linear operator such that $\mathcal{F} \equiv (\{x_k\}, \Theta)$ is a retro Banach frame for $\mathcal{X}^*$ with bounds $A = B = 1$.

Choose $J = \{1, 2, ..., k\} \subset \mathbb{N}$, where k is given in (2.1). Then, there exists a reconstruction operator Θ_o such that $\mathcal{F}_o \equiv (\{x_k\}_{k\notin J}, \Theta_o)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to the sequence space $\mathcal{Z}_{d_o} = \{\{f(x_k)\}_{k\notin J} : f \in \mathcal{X}^*\}$. Furthermore, $\mathcal{F}_o$ is exact. Therefore, $\mathrm{e}(\mathcal{F}) = k$, which is finite.

Example 2.2. Let $\{z_k\} \subset \mathcal{X}$ be a sequence given by

$$z_{2k} = z_{2k-1} = \chi_k, \ k \in \mathbb{N}.$$

Then, there exists a retro pre-frame operator U such that $\Omega \equiv (\{z_k\}, U)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to $\Omega_d = \{\{f(z_k)\} : f \in \mathcal{X}^*\}$.

Choose $J = \{1, 3, 5, ..., 2k-1, ...\} \subset \mathbb{N}$. Then, there exists a reconstruction operator U_o such that $\Omega_o \equiv (\{z_k\}_{k\notin J}, U_o)$ is a retro Banach frame for $\mathcal{X}^*$. Hence $\mathrm{e}(\Omega)$ is infinite.

Next, we characterize the finite excess of a retro Banach frame.

Theorem 2.1. *Let $\mathcal{F} \equiv (\{x_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_d$. Then $\mathcal{F}$ has finite excess if and only if for every infinite subset $J \subset \mathbb{N}$, we have*

$$[x_k]_{k\notin J} \neq \mathcal{X}.$$

Proof. Suppose first that $e(\mathcal{F})$ is finite. Assume, on the contrary, that there exists an infinite subset $J_o \subset \mathbb{N}$ such that

$$[x_k]_{k \notin J_o} = \mathcal{X}.$$

Then, there exists a reconstruction operator Θ_o such that $(\{x_k\}_{k \notin J_o}, \Theta_o)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to the Banach space of scalar-valued sequences $\mathcal{Z}_{d_o} = \{\{f(x_k)\}_{k \notin J_o} : f \in \mathcal{X}^*\}$. But $|J_o|$ is infinite. Hence $e(\mathcal{F})$ is infinite, which is impossible. Thus, the forward part is proved.

To prove the converse part, assume, on the contrary, that $e(\mathcal{F})$ is not finite. Then, for some suitable choice of J, there exists a number $k_1 \in J$ such that $(\{x_k\}_{k \neq k_1}, \Theta_{k_1})$ is a retro Banach frame for $\mathcal{X}^*$ with respect to some associated Banach space $\mathcal{Z}_{d_1}$. Therefore, there exist positive constants A_1 and B_1 such that

$$A_1\|f\| \leq \|\{f(x_k)\}_{k \neq k_1}\|_{\mathcal{Z}_{d_1}} \leq B_1\|f\| \text{ for each } f \in \mathcal{X}^*. \tag{2.2}$$

By using lower frame inequality in (2.2), we have $[x_k]_{k \neq k_1} = \mathcal{X}$. Hence

$$x_{k_1} \in [x_k]_{k \neq k_1} = \mathcal{X}.$$

Therefore, we can find a positive integer $n_1 \geq k_1$ such that

$$\text{dist}\left(x_{k_1}, [x_k]^{n_1}_{\substack{k=1 \\ k \neq k_1}}\right) < \frac{1}{2}.$$

Since $e(\mathcal{F})$ is not finite, there exist a number $k_2 \in J$ and a reconstruction operator Θ_{k_2} such that $(\{x_k\}_{k \notin \{1,\ldots,n_1\} \cup \{k_2\}}, \Theta_{k_2})$ is a retro Banach frame for $\mathcal{X}^*$ with respect to some $\mathcal{Z}_{d_2}$. Thus, we can find positive constants A_2 and B_2 such that

$$A_2\|f\| \leq \|\{f(x_k)\}_{k \notin \{1,\ldots,n_1\} \cup \{k_2\}}\|_{\mathcal{Z}_{d_2}} \leq B_2\|f\| \text{ for each } f \in \mathcal{X}^*. \tag{2.3}$$

By using lower frame inequality in (2.3), we obtain

$$x_{k_2} \in [x_k]_{\substack{k \geq n_1+1 \\ k \neq k_2}} = \mathcal{X}.$$

Also, there exists a positive integer $n_2 \geq k_2$ such that

$$\text{dist}\left(x_{k_2}, [x_k]^{n_2}_{\substack{k=1 \\ k \neq k_1, k_2}}\right) < \frac{1}{2^2}, \quad \text{dist}\left(x_{k_1}, [x_k]^{n_2}_{\substack{k=1 \\ k \neq k_1, k_2}}\right) < \frac{1}{2^2}.$$

Continuing this process, we obtain a sequence $\{k_j\} \subset J$ and $\{n_j\} \subset \mathbb{N}$ such that

$$\text{dist}\left(x_{k_j}, [x_k]^{n_l}_{\substack{k=1 \\ k \notin \{k_1, k_2, \ldots, k_l\}}}\right) < \frac{1}{2^l}, \ 1 \leq j \leq l, \ l \in \mathbb{N}. \tag{2.4}$$

By using (2.4), we get

$$x_{k_j} \in [x_k]_{k \notin J} = \mathcal{X},$$

which is a contradiction. Hence $e(\mathcal{F})$ must be finite. Theorem 2.1 is proved. □

To conclude the paper, we show that if a given retro Banach frame for $\mathcal{X}^*$ has finite excess associated with a certain nested sequence, then we can construct a retro Banach frame with infinite excess.

Theorem 2.2. *Let $\mathcal{F} \equiv (\{x_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_d$, and let $\mathbb{J}_1 \subset \mathbb{J}_2 \subset ... \subset \mathbb{J}_t \subset ...$ be a nested sequence of subsets of $\mathbb{N}$, where each $\mathbb{J}_n$ is finite. Assume that for each n, $\mathcal{F}_n \equiv (\{x_k\}_{k\notin\mathbb{J}_n}, \Theta_n)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_{d_n}$, that is, for each n, $\mathrm{e}(\mathcal{F}_n)$ is finite. Then, there exist an infinite subset $J \subset \mathbb{N}$ and a reconstruction operator Θ such that $\mathcal{F} \equiv (\{x_k\}_{k\notin J}, \Theta_n)$ is a retro Banach frame for $\mathcal{X}^*$, and hence $\mathrm{e}(\mathcal{F})$ is infinite.*

Proof. Without loss of generality, let us write $\mathbb{J}_t = \{1, 2, .., t\}$, $t \in \mathbb{N}$. By hypothesis, $\mathcal{F}_1 \equiv (\{x_k\}_{k\notin\mathbb{J}_1}, \Theta_1)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{Z}_{d_n}$. Hence, there exist finite positive constants a_1 and b_1 such that

$$a_1\|f\| \leq \|\{f(x_k\}_{k\neq 1}\|_{\mathcal{Z}_{d_1}} \leq b_1\|f\| \text{ for each } f \in \mathcal{X}^*. \tag{2.5}$$

By using lower inequality in (2.5), we can find a positive integer $k_2 > k_1 = 1$ such that

$$\mathrm{dist}\left(x_{k_1}, \left[\{x_k\}_{k\notin\bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=2}^{k_2-1}\right]\right) < \frac{1}{2}.$$

Again, by hypothesis, there exists a reconstruction operator Θ_{k_2} such that $\mathcal{F}_{k_2} \equiv (\{x_k\}_{k\notin\mathbb{J}_{k_2}}, \Theta_{k_2})$ is a retro Banach frame for $\mathcal{X}^*$. By using lower frame inequality for $\mathcal{F}_{k_2}$, we can find a positive integer k_3 such that

$$\mathrm{dist}\left(x_{k_1}, \left[\{x_k\}_{k\notin\bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=k_2+1}^{k_3-1}\right]\right) < \frac{1}{3}$$

and

$$\mathrm{dist}\left(x_{k_2}, \left[\{x_k\}_{k\notin\bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=k_2+1}^{k_3-1}\right]\right) < \frac{1}{3}.$$

By induction, we can find a monotone increasing sequence of positive integers $\{k_j\}$ such that

$$\mathrm{dist}\left(x_{k_j}, \left[\{x_k\}_{k\notin\bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=k_\ell+1}^{k_{\ell+1}-1}\right]\right) < \frac{1}{\ell+1},\ j = 1, 2.., \ell,\ (\ell \in \mathbb{N}). \tag{2.6}$$

Choose $\mathbb{I} = \{k_1, k_2, k_3, ...\}$. Then, since $\{x_k\}_{k\notin\bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=k_\ell+1}^{k_{\ell+1}-1} \subset \{x_k\}_{k\notin\mathbb{I}}$, by using (2.6), we obtain

$$\mathrm{dist}(x_{k_j}, \{x_k\}_{k\notin\mathbb{I}}) \leq \mathrm{dist}\left(x_{k_j}, \left[\{x_k\}_{k\notin\bigcup_t \mathbb{J}_t} \bigcup \{x_n\}_{n=k_\ell+1}^{k_{\ell+1}-1}\right]\right) < \frac{1}{\ell+1} \text{ for all } \ell \geq j.$$

Therefore, $\mathcal{Z}_\infty = \{\{f(x_k)\}_{k\notin\mathbb{I}} : f \in \mathcal{X}^*\}$ is a Banach space of scalar valued sequences with norm given by $\|\{f(x_k)\}_{k\notin\mathbb{I}}\|_{\mathcal{Z}_\infty} = \|f\|_{\mathcal{X}^*}$, $f \in \mathcal{X}^*$. Define Θ_∞ :

$\mathcal{Z}_\infty \to \mathcal{X}^*$ by $\Theta_\infty(\{f(x_k)\}_{k\notin\mathbb{I}}) = f$ $f \in \mathcal{X}^*$, and observe that Θ_∞ is a bounded linear operator such that $F = (\{x_k\}_{k\notin\mathbb{I}}), \Theta_\infty)$ is a retro Banach frame for $\mathcal{X}^*$ with bounds $A = B = 1$. Since $|\mathbb{I}|$ is infinite, $e(\mathcal{F})$ is also infinite. □

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