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ON AN OPEN PROBLEM OF ZHANG AND XU

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Abstract. Taking an open problem in [25] into background we employ the idea of normal family to investigate the uniqueness problem of meromorphic functions sharing a non-zero polynomial which improves a number of existing results. Specially we rectify some errors and gaps in a recent result of P. Sahoo [15].

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1. Introduction, definitions and results

In this paper by meromorphic functions we always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let $a \in \mathbb{C}$. We say that f and g share a CM if $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM if $f - a$ and $g - a$ have the same zeros ignoring multiplicities.

We adopt the standard notation of value distribution theory (see [8]). For a non-constant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly except a set of finite linear measure. A meromorphic function a is said to be a small function of f if $T(r, a) = S(r, f)$.

Throughout the paper, we denote by $\mu(f)$ and $\rho(f)$ the lower order and the order of f, respectively (see [8, 19]). Let f be a transcendental meromorphic function such that $\rho(f) = \rho \leq \infty$. A complex number a is said to be a Borel exceptional value (see [19]) if

$$\limsup_{r \longrightarrow \infty} \frac{\log^+ N(r, a; f)}{\log r} < \rho.$$

A finite value z_0 is said to be a fixed point of $f(z)$ if $f(z_0) = z_0$. We will use the following definition:

$$\Theta(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where a is a value in the extended complex plane.

In 1959, W. K. Hayman (see [7], Corollary of Theorem 9) proved the following assertion.

Theorem A. [7] *Let f be a transcendental meromorphic function and let $n \in \mathbb{N}$ with $n \geq 3$. Then $f^n f' = 1$ has infinitely many solutions.*

In 1997, C. C. Yang and X. H. Hua [20] obtained the following uniqueness result corresponding to Theorem A.

Theorem B. [20] *Let f and g be two non-constant meromorphic functions, and let $n \in \mathbb{N}$ with $n \geq 11$. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c, c_1, c_2 \in \mathbb{C}\backslash\{0\}$ and $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for $t \in \mathbb{C}\backslash\{0\}$ such that $t^{n+1} = 1$.*

In 2002, using the idea of sharing fixed points, M. L. Fang and H. L. Qiu [5] further generalized and improved Theorem B by proving the following theorem.

Theorem C. [5] *Let f and g be two non-constant meromorphic functions, and let $n \in \mathbb{N}$ with $n \geq 11$. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$.*

For the last couple of years a number of astonishing results have been obtained regarding the value sharing of nonlinear differential polynomials, which are mainly the k-th derivative of some linear expression of f and g.

In 2010, J. F. Xu, F. Lü and H. X. Yi [17] studied the analogous problem corresponding to Theorem C, where in addition to the fixed point sharing problem, sharing of poles are also taken under consideration. More precisely, they proved the following theorems.

Theorem D. [17] *Let f and g be two non-constant meromorphic functions, and let $n, k \in \mathbb{N}$ such that $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, and f and g share*

∞ IM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where $c, c_1, c_2 \in \mathbb{C}\setminus\{0\}$ satisfying $4n^2(c_1c_2)^nc^2 = -1$, or $f \equiv tg$ for $t \in \mathbb{C}\setminus\{0\}$ such that $t^n = 1$.

Theorem E. [17] *Let f and g be two non-constant meromorphic functions such that $\Theta(\infty; f) > \frac{2}{n}$, and let $n, k \in \mathbb{N}$ such that $n \geq 3k + 12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, and f and g share ∞ IM, then $f \equiv g$.*

Recently X. B. Zhang and J. F. Xu [25] further generalized and improved the results of [17] as follows (see [25], Theorem 1.3).

Theorem F. [25] *Let f and g be two transcendental meromorphic functions, p be a non-zero polynomial with $\deg(p) = l \leq 5$, $k, n \in \mathbb{N}$, $m \in \mathbb{N}\cup\{0\}$ such that $n > 3k + m + 7$, and let $P(w) = a_mw^m + a_{m-1}w^{m-1} + \ldots + a_1w + a_0$ be a non-zero polynomial. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share p CM, and f and g share ∞ IM, then one of the following three cases hold:*

(1) $f(z) \equiv tg(z)$ for $t \in \mathbb{C}\setminus\{0\}$ such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,

(2) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$;

(3) $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_iz^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$;
if $p(z)$ is not a constant, then $f(z) = c_1e^{cQ(z)}$, $g(z) = c_2e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, $c, c_1, c_2 \in \mathbb{C}\setminus\{0\}$ such that $a_i^2(c_1c_2)^{n+i}[(n+i)c]^2 = -1$,
if $p(z)$ is a non-zero constant b, then $f(z) = c_3e^{cz}$, $g(z) = c_4e^{-cz}$, where $c, c_3, c_4 \in \mathbb{C}\setminus\{0\}$ such that $(-1)^ka_i^2(c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2$.

Zhang and Xu made the following observation in Remark 1.2 of [25]:
"From the proof of Theorem 1.3, we can see that the computation will be very complicated when $\deg(p)$ becomes large, so we are not sure whether Theorem 1.3 holds for the general polynomial p."

Also, at the end of the paper [25], the authors posed the following problem.

Open problem. What happens to Theorem 1.3 [25] if the condition "$l \leq 5$" is removed ?

Let us define $m^* = m$ if $P(z) \not\equiv c_0$, and $m^* = 0$ if $P(z) \equiv c_0$.
Regarding the above problem, P. Sahoo [15] proved the following result.

Theorem G. [15] *Let f and g be two transcendental meromorphic functions, p be a non-constant polynomial of degree l, and let $k, n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ such that $n > \max\{3k + m^* + 6, k + 2l\}$. In addition, we suppose that either k, l are co-prime or $k > l$, when $l \geq 2$. Let $P(w)$ be as in Theorem F. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share p CM, and f and g share ∞ IM, then the following conclusions hold.*

(i) If $P(z) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0$ is not a monomial, then either $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^d = 1$, where $d = (n+m, \ldots n+m-i \ldots, n)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, 2, \ldots, m\}$, or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by $R(w_1, w_2) = w_1^n(a_m w_1^m + \ldots + a_1 w_1 + a_0) - w_2^n(a_m w_2^m + \ldots + a_1 w_2 + a_0)$. In particular, when $m = 1$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$, then $f \equiv g$,

(ii) When $P(z) = c_0$ or $P(w) = a_m w^m$, then either $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+m^*} = 1$, or $f(z) = b_1 e^{bQ(z)}$, $g(z) = b_2 e^{-bQ(z)}$, where $Q(z)$ is a polynomial without constant such that $Q'(z) = p(z)$, $b, b_1, b_2 \in \mathbb{C} \setminus \{0\}$, and $c_0^2(nb)^2(b_1 b_2)^n = -1$ or $a_m^2((n+m)b)^2(b_1 b_2)^{n+m} = -1$.

Remark 1.1. Observing Theorem 1.1 of [15], it seems that the condition "$l \leq 5$" was removed. But unfortunately it is not the case. Actually the condition "$l \leq 5$" is replaced by the condition "$n > k + 2l$", with n depending on l. In the same paper the author claims that "Theorem 1.1 of [15] improves Theorem F by reducing the lower bound of n", but this is not true. For example, if we assume that $k = 1$, $m = 1$ and $l = 5$, then from Theorem F we get $n > 11$, while in Theorem G we have $n > 11$. On the other hand, we see that Theorem F holds for $k = l \leq 5$ but Theorem G does not hold.

Therefore, by the best knowledge of the authors, the above open problem is still open. Consequently one of the goals of this paper is to solve the above open problem without imposing any other conditions.

Remark 1.2. In the proof of Lemma 2.7 of [15], one can easily point out a gap. Indeed, from the relation

$$a_m^2(n+m)^2\alpha'\beta' e^{(n+m)(\alpha+\beta)} \equiv p^2$$

the authors conclude that α and β are polynomials. A question arises when $\alpha' = pe^{\gamma}$ and $\beta' = pe^{\delta}$. Actually the authors did not consider this case.

The above discussion is enough to make oneself inquisitive to investigate the accurate form of Theorem G. To state our main result we need the following definition, which also will be used throughout the paper.

Definition 1.1. [9, 10] *Let $k \in \mathbb{N} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, then we say that f and g share the value a with weight k. We write f, g share (a, k) to mean that f and g share the value a with weight k. Also, we say that f, g share a value a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞), respectively.*

Also, it is quite natural to ask the following questions.

Question 1. Can one remove the condition "Suppose that either k, l are co-prime or $k > l$, when $l \geq 2$" in Theorem G ?

Question 2. Can "CM" sharing in Theorems F and G be reduced to a finite weight sharing ?

In this paper, taking the possible answers of the above questions into background, we obtain the following result.

Theorem 1.1. *Let f, g be two transcendental meromorphic functions, and let $k, n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ be such that $n > 3k + m + 6$. Let p be a non-zero polynomial and $P(w)$ be defined as in Theorem F. If $[f^n P(f)]^{(k)} - p$, $[g^n P(g)]^{(k)} - p$ share $(0, k_1)$, where $k_1 = \left[\frac{3+k}{n+m-k-1}\right] + 3$ and f, g share $(\infty, 0)$, then one of the following three cases hold:*

(1) $f(z) \equiv tg(z)$ for $t \in \mathbb{C}\setminus\{0\}$ such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,

(2) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \ldots + a_0)$. In particular, when $m = 1$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$, then $f \equiv g$;

(3) $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$;
if $p(z)$ is not a constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, and c_1, c, c_1, $c_2 \in \mathbb{C}\setminus\{0\}$ are such that $a_i^2(c_1c_2)^{n+i}[(n+i)c]^2 = -1$,

if $p(z)$ is a non-zero constant b, then $f(z) = c_3e^{cz}$, $g(z) = c_4e^{-cz}$, where $c, c_3, c_4 \in \mathbb{C}\setminus\{0\}$ are such that $(-1)^k a_i^2(c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2$.

Remark 1.3. Clearly Theorem 1.1 improves Theorems F and G. Also, in this paper we can remove the condition "$l \leq 5$" in Theorem F without imposing any other conditions and keeping all the conclusions intact.

The following definitions and notations will be used in the paper.

Definition 1.2. [11] *Let $a \in \mathbb{C}\cup\{\infty\}$. For $p \in \mathbb{N}$ we denote by $N(r,a;f\mid\leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f\mid\leq p)$ we denote the corresponding reduced counting function. In an analogous manner we can define $N(r,a;f\mid\geq p)$ and $\overline{N}(r,a;f\mid\geq p)$.*

Definition 1.3. [10] *Let $k \in \mathbb{N}\cup\{\infty\}$. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then $N_k(r,a;f) = \overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+\ldots+\overline{N}(r,a;f\mid\geq k)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.*

Definition 1.4. [2] *Let f and g be two non-constant meromorphic functions such that f and g share the value a IM for $a \in \mathbb{C}\cup\{\infty\}$. Let z_0 be an a-point of f with multiplicity p and also an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ ($\overline{N}_L(r,a;g)$) the reduced counting function of those a-points of f and g, where $p > q \geq 1$ ($q > p \geq 1$). Also, we denote by $\overline{N}_E^{(1}(r,a;f)$ the reduced counting function of those a-points of f and g, where $p = q \geq 1$.*

Definition 1.5. [9, 10] *Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly, $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.*

Definition 1.6. [13] *Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.*

2. Lemmas

Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M \ \forall \ z \in \mathbb{C}$, where $h^{\#}(z) = \frac{|h'(z)|}{1+|h(z)|^2}$ denotes the spherical derivative of h.

Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathfrak{F}$ is a normal family in D if every sequence $\{f_n\}_n \subseteq \mathfrak{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [16]).

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H and V the functions defined as follows:

$$(2.1)\quad H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),\ V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right).$$

Lemma 2.1 ([18]). *Let f be a non-constant meromorphic function, and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z), \ldots, a_0(z)$ be meromorphic functions such that $T(r, a_i) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2 ([24]). *Let f be a non-constant meromorphic function and $k, p \in \mathbb{N}$. Then*

$$(2.2)\qquad N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f),$$

$$(2.3)\qquad N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).$$

Lemma 2.3 ([12]). *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid < k) + k\overline{N}(r, 0; f \mid \geq k) + S(r, f).$$

Lemma 2.4 ([25]). *Let f and g be two non-constant meromorphic functions, $P(w)$ be defined as in Theorem F, and let $k, n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ be such that $n > 2k+m+1$. If $[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}$, then $f^n P(f) \equiv g^n P(g)$.*

Lemma 2.5 ([21], Lemma 6). *If $H \equiv 0$, then F, G share 1 CM. If further F, G share ∞ IM then F, G share ∞ CM.*

Lemma 2.6 ([25]). *Let f, g be non-constant meromorphic functions, $k, n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ be such that $n > k + 2$, and let $P(w)$ be defined as in Theorem F. Let $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to f with finitely many zeros and poles. If $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv \alpha^2$, f and g share ∞ IM, then $P(w)$ is reduced to a non-zero monomial, namely $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$.*

Lemma 2.7 ([6]). *Let $f(z)$ be a non-constant entire function and let $k \in \mathbb{N} \setminus \{1\}$. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{az+b}$, where $a(\neq 0), b \in \mathbb{C}$.*

Lemma 2.8 ([8], Theorem 3.10). *Suppose that f is a non-constant meromorphic function and $k \in \mathbb{N} \setminus \{1\}$. If*

$$N(r, \infty, f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, \frac{f'}{f}),$$

then $f(z) = e^{az+b}$, where $a(\neq 0), b \in \mathbb{C}$.

Lemma 2.9 ([8], Lemma 3.5). *Suppose that F is meromorphic in a domain D, and set $f = \frac{F'}{F}$. Then for $n \in \mathbb{N}$, we have*

$$\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4} (f')^2 + P_{n-3}(f),$$

where $a_n = \frac{1}{6} n(n-1)(n-2)$, $b_n = \frac{1}{8} n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n - 3$ when $n > 3$.

Lemma 2.10 ([4]). *Let f be a meromorphic function on $\mathbb{C}$ with finitely many poles. If f has bounded spherical derivative on $\mathbb{C}$, then f is of order at most 1.*

Lemma 2.11 ([19], Theorem 2.11). *Let f be a transcendental meromorphic function in the complex plane such that $\rho(f) > 0$. If f has two distinct Borel exceptional values in the extended complex plane, then $\mu(f) = \rho(f)$ and $\rho(f)$ is a positive integer or ∞.*

Lemma 2.12 ([23]). *Let F be a family of meromorphic functions in the unit disc Δ such that all zeros of functions in F have multiplicity greater than or equal to l and all poles of functions in F have multiplicity greater than or equal to j, and let α be a real number satisfying $-l < \alpha < j$. Then F is not normal in any neighborhood of $z_0 \in \Delta$ if and only if there exist*

(i) points $z_n \in \Delta$, $z_n \to z_0$,

(ii) positive numbers ρ_n, $\rho_n \to 0^+$, and

(iii) functions $f_n \in F$,

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ *spherically locally uniformly in* $\mathbb{C}$, *where* g *is a non-constant meromorphic function. The function* g *may be taken to satisfy the normalization condition:* $g^{\#}(\zeta) \leq g^{\#}(0) = 1 (\zeta \in \mathbb{C})$.

Remark 2.1. Suppose that in Lemma 2.12, F is a family of holomorphic functions in the domain D and there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$, whenever $f = 0$. Then the real number α in Lemma 2.12 can be chosen to satisfy $0 \leq \alpha \leq k$. In that case, we also have $f_n(z_n + \rho_n\zeta) \to g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where g is a non-constant holomorphic function. The function g may be taken to satisfy the normalization condition: $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1 (\zeta \in \mathbb{C})$.

Lemma 2.13 ([19]). *Let* f_j $(j = 1, 2, 3)$ *be a meromorphic and* f_1 *be a non-constant functions. Suppose that* $\sum_{j=1}^{3} f_j = 1$ *and*

$$\sum_{j=1}^{3} N(r, 0; f_j) + 2\sum_{j=1}^{3} \overline{N}(r, \infty; f_j) < (\lambda + o(1))T(r),$$

as $r \to +\infty$, $r \in I$, *where* I *is a set of* $r \in (0, \infty)$ *with infinite linear measure,* $\lambda < 1$ *and* $T(r) = \max_{1\leq j\leq 3} T(r, f_j)$. *Then* $f_2 = 1$ *or* $f_3 = 1$.

Lemma 2.14 ([19], Theorem 1.24). *Let* f *be a non-constant meromorphic function, and let* $k \in \mathbb{N}$. *Suppose that* $f^{(k)} \not\equiv 0$, *then*

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.15. *Let* f, g *be two transcendental entire functions such that* f *and* g *have no zeros, and let* p *be a non-zero polynomial. Suppose that* $(f^n)'(g^n)' \equiv p^2$, *where* $n \in \mathbb{N}$. *Then the following assertions hold:*

(i) if $p(z)$ *is not a constant, then* $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$; *where* $Q(z) = \int_0^z p(t)dt$, *and* $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ *are such that* $(nc)^2(c_1c_2)^n = -1$,

(ii) if $p(z) = b \in \mathbb{C} \setminus \{0\}$, *then* $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{\ dz}$, *where* $c_3, c_4, d \in \mathbb{C} \setminus \{0\}$ *are such that* $(-1)^k (c_3c_4)^n (nd)^{2k} = b^2$.

The proof follows from that of Theorem 1.3 of [25].

Lemma 2.16. *Let* f, g *be two transcendental meromorphic functions,* p *be a non-zero polynomial, and let* $k, n \in \mathbb{N}$ *be such that* $n > k$. *Suppose that* $(f^n)^{(k)}(g^n)^{(k)} \equiv p^2$, *where $(f^n)^{(k)} - p$ and $(g^n)^{(k)} - p$ share 0 CM and f, g share ∞ IM. Then the following assertions hold:*

(i) if $p(z)$ is not a constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, and $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ are such that $(nc)^2(c_1c_2)^n = -1$,

(ii) if $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C} \setminus \{0\}$ are such that $(-1)^k (c_3c_4)^n (nd)^{2k} = b^2$.

Proof. Suppose

$$(f^n)^{(k)}(g^n)^{(k)} \equiv p^2. \tag{2.4}$$

Since f and g share ∞ IM, from (2.4) one can easily infer that f and g are transcendental entire functions. Let $F_1 = \frac{(f^n)^{(k)}}{p}$ and $G_1 = \frac{(g^n)^{(k)}}{p}$. From (2.4) we get

$$F_1 G_1 \equiv 1. \tag{2.5}$$

If $F_1 \equiv c_1^* G_1$, where $c_1^* \in \mathbb{C} \setminus \{0\}$, then by (2.5), F_1 is a constant and so f is a polynomial, which contradicts our assumption. Hence $F_1 \not\equiv c_1^* G_1$.

Let

$$\Phi = \frac{(f^n)^{(k)} - p}{(g^n)^{(k)} - p}. \tag{2.6}$$

Then from (2.6) we have

$$\Phi = e^{\gamma_1}, \tag{2.7}$$

where γ_1 is an entire function. Let $f_1 = F_1$, $f_2 = e^{\gamma_1} G_1$ and $f_3 = e^{\gamma_1}$. Here f_1 is transcendental. Now from (2.7), we have $f_1 + f_2 + f_3 = 1$. Hence by Lemma 2.14 we get

$$\begin{aligned} \sum_{j=1}^{3} N(r, 0; f_j) + 2\sum_{j=1}^{3} \overline{N}(r, \infty; f_j) &\leq N(r, 0; F_1) + N(r, 0; e^{\gamma_1} G_1) + O(\log r) \\ &\leq (\lambda + o(1))T(r), \end{aligned}$$

as $r \to +\infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max_{1 \leq j \leq 3} T(r, f_j)$.

So, by Lemma 2.13, we infer that either $e^{\gamma_1} G_1 = -1$ or $e^{\gamma_1} = 1$. But here the only possibility is that $e^{\gamma_1} G_1 = -1$, that is, $(g^n)^{(k)} = -e^{-\gamma_1} p$, and so from (2.4) we get

$$(f^n)^{(k)} = c_2^* e^{\gamma_1} p \text{ and } (g^n)^{(k)} = c_2^* e^{-\gamma_1} p, \tag{2.8}$$

where $c_2^* = \pm 1$. This shows that $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM. Let z_p be a zero of $f(z)$ of multiplicity p and z_q be a zero of $g(z)$ of multiplicity q. Since $n > k$, it follows that z_p will be a zero of $(f^n(z))^{(k)}$ of multiplicity $np - k$ and z_q will be a zero of $(g^n(z))^{(k)}$ of multiplicity $nq-k$. Since $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share 0 CM, it follows that $z_p = z_q$ and $p = q$. Consequently f and g share 0 CM. Since $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$, we can take

$$f(z) = h_1(z)e^{\alpha(z)} \text{ and } g(z) = h_1(z)e^{\beta(z)}, \tag{2.9}$$

where $h_1(z)$ is a non-zero polynomial and α, β are two non-constant entire functions. We consider the following cases.

Case 1. Suppose 0 is a Picard exceptional value of both f and g. We consider the following sub-cases.

Sub-case 1.1. Let $\deg(p) = l \in \mathbb{N}$.

Since $N(r,0;f) = 0$ and $N(r,0;g) = 0$, we can take

$$f(z) = e^{\alpha(z)} \text{ and } g(z) = e^{\beta(z)}, \tag{2.10}$$

where α and β are two non-constant entire functions.

We deduce from (2.4) and (2.10) that either both α and β are transcendental entire functions, or both are polynomials. We consider the following sub-cases.

Sub-case 1.1.1. Let $k \in \mathbb{N}\backslash\{1\}$. We first suppose that both α and β are transcendental entire functions. Note that

$$S(r, n\alpha') = S(r, \frac{(f^n)'}{f^n}) \text{ and } S(r, n\beta') = S(r, \frac{(g^n)'}{g^n}).$$

Moreover we see that

$$N(r,0;(f^n)^{(k)}) \leq N(r,0;p^2) = O(\log r), \quad N(r,0;(g^n)^{(k)}) \leq N(r,0;p^2) = O(\log r).$$

From these and using (2.10) we have

$$N(r,\infty;f^n) + N(r,0;f^n) + N(r,0;(f^n)^{(k)}) = S(r,n\alpha') = S(r,\frac{(f^n)'}{f^n})$$

$$N(r,\infty;g^n) + N(r,0;g^n) + N(r,0;(g^n)^{(k)}) = S(r,n\beta') = S(r,\frac{(g^n)'}{g^n}). \tag{2.11}$$

Then from (2.11) and Lemma 2.8 we must have $f(z) = e^{a_3^* z + b_3^*}$ and $g(z) = e^{c_3^* z + d_3^*}$, where $a_3^*(\neq 0), b_3^*, c_3^*(\neq 0), d_3^* \in \mathbb{C}$. But these types of f and g do not agree with the relation (2.4).

Next, we suppose that α and β both are non-constant polynomials, since otherwise f and g reduce to polynomials contradicting that they are transcendental. Also, from (2.4) we get $\alpha + \beta = C_1 \in \mathbb{C}$, that is, $\alpha' = -\beta'$. Therefore $\deg(\alpha) = \deg(\beta)$. If

$\deg(\alpha) = \deg(\beta) = 1$, then we again get a contradiction from (2.4). Next, we suppose that $\deg(\alpha) = \deg(\beta) \geq 2$. Now from (2.10) and Lemma 2.9 we see that

$$(f^n)^{(k)} = \Big(n^k(\alpha')^k + \frac{k(k-1)}{2}n^{k-1}(\alpha')^{k-2}\alpha'' + P_{k-2}(\alpha')\Big)e^{n\alpha}.$$

Similarly we have

$$\begin{aligned}(g^n)^{(k)} &= \Big(n^k(\beta')^k + \frac{k(k-1)}{2}n^{k-1}(\beta')^{k-2}\beta'' + P_{k-2}(\beta')\Big)e^{n\beta}\\ &= \Big((-1)^k n^k(\alpha')^k - \frac{k(k-1)}{2}n^{k-1}(-1)^{k-2}(\alpha')^{k-2}\alpha'' + P_{k-2}(-\alpha')\Big)e^{n\beta}.\end{aligned}$$

Since $\deg(\alpha) \geq 2$, we observe that $\deg((\alpha')^k) \geq k\ \deg(\alpha')$, and so $(\alpha')^{k-2}\alpha''$ is either a non-zero constant or $\deg((\alpha')^{k-2}\alpha'') \geq (k-1)\ \deg(\alpha') - 1$. Also, we see that

$$\deg\left((\alpha')^k\right) > \deg\left((\alpha')^{k-2}\alpha''\right) > \deg\left(P_{k-2}(\alpha')\right) \text{ (or } \deg\left(P_{k-2}(-\alpha')\right)).$$

Let

$$(\alpha(z))' = e_t z^t + e_{t-1}z^{t-1} + \ldots + e_0,$$

where $e_0, e_1, \ldots, e_t (\neq 0) \in \mathbb{C}$. Then we have

$$((\alpha)')^i = e_t^i z^{it} + i e_t^{i-1} e_{t-1} z^{it-1} + \ldots\ldots,$$

where $i \in \mathbb{N}$. Therefore we have

$$(f^n)^{(k)} = \Big(n^k e_t^k z^{kt} + kn^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots + (D_1 + D_2) z^{kt-t-1} + \ldots\Big)e^{n\alpha}$$

and

$$\begin{aligned}(g^n)^{(k)} &= \Big((-1)^k n^k e_t^k z^{kt} + (-1)^k kn^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots\\ &\quad + ((-1)^k D_1 + (-1)^{k-1} D_2) z^{kt-t-1} + \ldots\Big)e^{n\beta},\end{aligned}$$

where $D_1, D_2 \in \mathbb{C}$ are such that $D_2 = \frac{k(k-1)}{2} t n^{k-1} e_t^{k-1}$. Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM, we have

$$\begin{aligned}& n^k e_t^k z^{kt} + kn^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots + (D_1 + D_2) z^{kt-t-1} + \ldots\\ =\ & d_1^* \Big((-1)^k n^k e_t^k z^{kt} + (-1)^k kn^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots\\ & + ((-1)^k D_1 + (-1)^{k-1} D_2) z^{kt-t-1} + \ldots\Big)\end{aligned}$$

where $d_1^* \in \mathbb{C} \setminus \{0\}$. From (2.12) we get $D_2 = 0$, that is, $\frac{k(k-1)}{2} t n^{k-1} e_t^{k-1} = 0$, which is impossible for $k \geq 2$.

Sub-case 1.1.2. Let $k = 1$. The result follows from Lemma 2.15.

Sub-case 1.2. Let $p(z) = b \in \mathbb{C} \setminus \{0\}$. Since $n > k$, we have $f \neq 0$ and $g \neq 0$.

Now using Sub-case 1.1 we can show that $f = e^{\alpha}$ and $g = e^{\beta}$, where α and β are non-constant entire functions. We now consider the following two sub-cases.

Sub-case 1.2.1. Let $k \geq 2$. We see that $N(r, 0; (f^n)^{(k)}) = 0$. It is clear that

$$f^n(z)(f^n(z))^{(k)} \neq 0 \text{ and } g^n(z)(g^n(z))^{(k)} \neq 0. \tag{2.12}$$

Then from (2.12) and Lemma 2.7 we must have $f(z) = e^{a_4^* z + b_4^*}$, $g(z) = e^{c_4^* z + d_4^*}$, where $a_4^*(\neq 0), b_4^*, c_4^*(\neq 0), d_4^* \in \mathbb{C}$. In view of (2.4) it is clear that $a_4^* + c_4^* = 0$. Finally, by (2.4) we take $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where c_3, c_4 and $d \in \mathbb{C} \setminus \{0\}$ are such that $(-1)^k (c_3 c_4)^n (nd)^{2k} = b^2$.

Sub-case 1.2.2. Let $k = 1$. The result follows from Lemma 2.15.

Case 2. Suppose 0 is not a Picard exceptional value of f and g.

Let $H = f^n$, $\hat{H} = g^n$, $F = \frac{H}{p}$ and $G = \frac{\hat{H}}{p}$, and let $\mathcal{F} = \{F_\omega\}$ and $\mathcal{G} = \{G_\omega\}$, where $F_\omega(z) = F(z+\omega) = \frac{H(z+\omega)}{p(z+\omega)}$ and $G_\omega(z) = G(z+\omega) = \frac{\hat{H}(z+\omega)}{p(z+\omega)}$, $z \in \mathbb{C}$. Clearly $\mathcal{F}$ and $\mathcal{G}$ are two families of meromorphic functions defined on $\mathbb{C}$. We now consider following two sub-cases.

Sub-case 2.1. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$, is normal on $\mathbb{C}$. Then by Marty's theorem $F^{\#}(\omega) = F^{\#}_{\omega}(0) \leq M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. Hence by Lemma 2.10 we have that $F(= \frac{f^n}{p})$ is of order at most 1. Now from (2.4) we have

(2.13)
$$\rho(f) = \rho(\frac{f^n}{p}) = \rho(f^n) = \rho((f^n)^{(k)}) = \rho((g^n)^{(k)}) = \rho(g^n) = \rho(\frac{g^n}{p}) = \rho(g) \leq 1.$$

Since f and g are transcendental entire functions, from (2.9) we have $\rho(f) > 0$ and $\rho(g) > 0$. We observe from (2.13) and Lemma 2.11 that $\mu(f) = \rho(f) = 1$ and $\mu(g) = \rho(g) = 1$. Now from (2.9) we get

$$f = h_1 e^{\alpha}, \quad g = h_1 e^{\beta}, \tag{2.14}$$

where α and β are non-constant polynomials of degree 1. From (2.4) we see that $\alpha + \beta = C_2 \in \mathbb{C}$, and so $\alpha' + \beta' = 0$. Again, from (2.14) we have

$$(f^n)^{(k)} = e^{n\alpha} \sum_{i=0}^{k} {}^kC_i (n\alpha')^{k-i} (h_1^n)^{(i)},$$

where we define $(h_1^n)^{(0)} = h_1^n$. Similarly we have

$$(g^n)^{(k)} = e^{n\beta} \sum_{i=0}^{k} {}^kC_i (-1)^{k-i} (n\alpha')^{k-i} (h_1^n)^{(i)}.$$

Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM, it follows that

$$\sum_{i=0}^{k} {}^kC_i(n\alpha')^{k-i}(h_1^n)^{(i)} \equiv d_2^* \sum_{i=0}^{k} {}^kC_i(-1)^{k-i}(n\alpha')^{k-i}(h_1^n)^{(i)}, \tag{2.15}$$

where $d_2^* \in \mathbb{C} \setminus \{0\}$. But from (2.15) we arrive at a contradiction.

Sub-case 2.2. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$ is not normal on $\mathbb{C}$. Then there exists at least one $z_0 \in \Delta$ such that $\mathcal{T}$ is not normal z_0, we assume that $z_0 = 0$. Now by Marty's theorem there exists a sequence of meromorphic functions $\{F(z+\omega_j)\} \subset \mathcal{F}$, where $z \in \{z : |z| < 1\}$ and $\{\omega_j\} \subset C$ is some sequence of complex numbers, such that $F^{\#}(\omega_j) \to \infty$ as $|\omega_j| \to \infty$.

Note that p has only finitely many zeros. So there exists a number $r > 0$ such that $p(z) \neq 0$ in $D = \{z : |z| \geq r\}$. Since p is a polynomial, for all $z \in \mathbb{C}$ satisfying $|z| \geq r$, we have

$$0 \leftarrow |\frac{p'(z)}{p(z)}| \leq \frac{M_1}{|z|} < 1, \quad p(z) \neq 0. \tag{2.16}$$

Also, since $w_j \to \infty$ as $j \to \infty$, without loss of generality we may assume that $|w_j| \geq r + 1$ for all j. Let $D_1 = \{z : |z| < 1\}$ and

$$F(w_j + z) = \frac{H(w_j + z)}{p(w_j + z)}.$$

Since $|w_j + z| \geq |w_j| - |z|$, it follows that $w_j + z \in D$ for all $z \in D_1$. Also, since $p(z) \neq 0$ in D, it follows that $p(\omega_j + z) \neq 0$ in D_1 for all j. Observing that $F(z)$ is analytic in D, we conclude that $F(\omega_j + z)$ is analytic in D_1. Therefore, all $F(\omega_j + z)$ are analytic in D_1. Also, from (2.8) we see that every zero of h_1 must be a zero of p. Thus, we have structured a family $\{F(\omega_j + z)\}$ of holomorphic functions such that $F(\omega_j + z) \neq 0$ in D_1 for all j.

Then by Lemma 2.12 there exist:

(i) points z_j, $|z_j| < 1$,
(ii) positive numbers ρ_j, $\rho_j \to 0^+$,
(iii) a subsequence $\{F(\omega_j+z_j+\rho_j\zeta)\}$ of $\{F(\omega_j+z)\}$, such that $h_j(\zeta) = \rho_j^{-k}F(\omega_j + z_j + \rho_j\zeta) \to h(\zeta)$, that is,

$$h_j(\zeta) = \rho_j^{-k}\frac{H(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h(\zeta) \tag{2.17}$$

spherically locally uniformly in $\mathbb{C}$, where $h(\zeta)$ is some non-constant holomorphic function such that $h^{\#}(\zeta) \leq h^{\#}(0) = 1$.

Now from Lemma 2.10 we see that $\rho(h) \leq 1$. In view of the proof of Zalcman's lemma (see [14, 22]), we see that $\rho_j = \frac{1}{F^{\#}(b_j)}$ and $F^{\#}(b_j) \geq F^{\#}(\omega_j)$, where $b_j = \omega_j + z_j$. By Hurwitz's theorem we see that $h(\zeta) \neq 0$. Note that

$$\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to 0 \quad \text{as} \quad j \to \infty. \tag{2.18}$$

Now we prove that

$$(h_j(\zeta))^{(k)} = \frac{H^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h^{(k)}(\zeta). \tag{2.19}$$

To this end, note first that by (2.17) we have

$$\begin{aligned} \rho_j^{-k+1}\frac{H'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} &= h_j'(\zeta) + \rho_j^{-k+1}\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p^2(\omega_j + z_j + \rho_j\zeta)}H(\omega_j + z_j + \rho_j\zeta) \\ &= h_j'(\zeta) + \rho_j\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}h_j(\zeta). \end{aligned} \tag{2.20}$$

Now from (2.22), (2.18) and (2.20) we observe that

$$\rho_j^{-k+1}\frac{H'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h'(\zeta).$$

Suppose

$$\rho_j^{-k+l}\frac{H^{(l)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h^{(l)}(\zeta) \quad \text{and let} \quad G_j(\zeta) = \rho_j^{-k+l}\frac{H^{(l)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}.$$

Then $G_j(\zeta) \to h^{(l)}(\zeta)$. Note that

$$\begin{aligned} &\rho_j^{-k+l+1}\frac{H^{(l+1)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \\ = \;& G_j'(\zeta) + \rho_j^{-k+l+1}\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p^2(\omega_j + z_j + \rho_j\zeta)}H^{(l)}(\omega_j + z_j + \rho_j\zeta) \\ = \;& G_j'(\zeta) + \rho_j\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}G_j(\zeta). \end{aligned} \tag{2.21}$$

So, from (2.18) and (2.21), we see that

$$\rho_j^{-k+l+1}\frac{H^{(l+1)}(\omega_j + z_n + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h_j^{(l+1)}(\zeta).$$

Then by mathematical induction we get desired result (2.19). Let

$$(\hat{h}_j(\zeta))^{(k)} = \frac{\hat{H}^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}. \tag{2.22}$$

From (2.4) we have

$$\frac{H^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}\frac{\hat{H}^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} = 1,$$

and so, from (2.19) and (2.22), we get

$$(h_j(\zeta))^{(k)}(\hat{h}_j(\zeta))^{(k)} = 1. \tag{2.23}$$

Now from (2.19), (2.23) and the formula of higher derivatives we can deduce that $\hat{h}_j(\zeta) \to \hat{h}(\zeta)$, that is,

$$\frac{\hat{H}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to \hat{h}(\zeta), \tag{2.24}$$

spherically locally uniformly in C, where $\hat{h}(\zeta)$ is some non-constant holomorphic function in the complex plane. By Hurwitz's theorem we see that $\hat{h}(\zeta) \neq 0$. Therefore, by (2.24) we have

$$(\hat{h}_j(\zeta))^{(k)} \to (\hat{h}(\zeta))^{(k)} \tag{2.25}$$

spherically locally uniformly in C. From (2.19), (2.23) and (2.25) we get

$$(h(\zeta))^{(k)}(\hat{h}(\zeta))^{(k)} \equiv 1. \tag{2.26}$$

Since $\rho(h) \leq 1$, from (2.26) we see that

$$\rho(h) = \rho(h^{(k)}) = \rho(\hat{h}^{(k)}) = \rho(\hat{h}) \leq 1. \tag{2.27}$$

Since h and $\hat{h}$ are non-constant entire functions such that $h \neq 0$ and $\hat{h} \neq 0$, we can take $h = e^{\alpha_1}$ and $\hat{h} = e^{\beta_1}$, where α_1 and β_1 are non-constants entire functions. Consequently, $\rho(h) > 0$ and $\rho(\hat{h}) > 0$. Now we observe from (2.27) and Lemma 2.11 that $\mu(h) = \rho(h) = 1$ and $\mu(\hat{h}) = \rho(\hat{h}) = 1$. Therefore, we have

$$h(z) = \hat{c}_1 e^{\hat{c}z}, \;\; \hat{h}(z) = \hat{c}_2 e^{-\hat{c}z}, \tag{2.28}$$

where $\hat{c}, \hat{c}_1, \hat{c}_2 \in \mathbb{C}\setminus\{0\}$ are such that $(-1)^k(\hat{c}_1\hat{c}_2)(\hat{c})^{2k} = 1$. Also, from (2.28) we have

$$\frac{h_j'(\zeta)}{h_j(\zeta)} = \rho_j \frac{F'(w_j + z_j + \rho_j\zeta)}{F(w_j + z_j + \rho_j\zeta)} \to \frac{h'(\zeta)}{h(\zeta)} = \hat{c}, \tag{2.29}$$

spherically locally uniformly in $\mathbb{C}$. From (2.28) and (2.29) we get

$$\rho_j\left|\frac{F'(\omega_j + z_j)}{F(\omega_j + z_j)}\right| = \frac{1 + |F(\omega_j + z_j)|^2}{|F'(\omega_j + z_j)|}\frac{|F'(\omega_j + z_j)|}{|F(\omega_j + z_j)|} = \frac{1 + |F(\omega_j + z_j)|^2}{|F(\omega_j + z_j)|} \to \left|\frac{h'(0)}{h(0)}\right| = |\hat{c}|,$$

which implies that

$$\lim_{j\to\infty} F(\omega_j + z_j) \neq 0, \infty. \tag{2.30}$$

From (2.29) and (2.30) we see that

$$h_j(0) = \rho_j^{-k} F(\omega_j + z_j) \to \infty. \tag{2.31}$$

Again, from (2.29) and (2.28) we have

$$h_j(0) \to h(0) = \hat{c}_1. \tag{2.32}$$

Now from (2.31) and (2.32) we arrive at a contradiction. Lemma 2.16 is proved. □

Lemma 2.17. *Let f, g be two transcendental meromorphic functions, and let $P(w)$ be defined as in Theorem F. Let $F = \frac{[f^nP(f)]^{(k)}}{p}$, $G = \frac{[g^nP(g)]^{(k)}}{p}$, where p is a non-zero polynomial, and $k, n \in \mathbb{N}$ and $m \in \mathbb{N}\setminus\{0\}$ are such that $n > 3k+m+3$. If f, g share ∞ IM and $H \equiv 0$, then either $[f^nP(f)]^{(k)}[g^nP(f)]^{(k)} \equiv p^2$, where $[f^nP(f)]^{(k)} - p$ and $[g^nP(g)]^{(k)} - p$ share 0 CM, or $f^nP(f) \equiv g^nP(g)$.*

Proof. Since $H \equiv 0$, by Lemma 2.5 we conclude that F and G share 1 CM. By integration we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1}, \tag{2.33}$$

where $a(\neq 0), b \in \mathbb{C}$. Now we consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$.

If $b = -1$, then from (2.33) we obtain

$$F \equiv \frac{-a}{G-a-1}.$$

Therefore

$$\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

So, in view of Lemmas 2.1 and 2.2 with $p = 1$ and the second fundamental theorem, we can write

$$\begin{aligned}
&(n+m)\,T(r,g) \le T(r,G) + N_{k+1}(r,0;g^nP(g)) - \overline{N}(r,0;G)\\
\le\;& \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) + N_{k+1}(r,0;g^nP(g)) - \overline{N}(r,0;G) + S(r,g)\\
\le\;& \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^nP(g)) + \overline{N}(r,\infty;f) + S(r,g)\\
\le\;& \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^n) + N_{k+1}(r,0;P(g)) + S(r,g)\\
\le\;& 2\,\overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + S(r,g)\\
\le\;& (k+3+m)\,T(r,g) + S(r,g),
\end{aligned}$$

which is a contradiction since $n > k+3$. If $b \neq -1$, then from (2.33) we obtain

$$F - (1+\frac{1}{b}) \equiv \frac{-a}{b^2[G+\frac{a-b}{b}]},$$

and hence

$$\overline{N}(r, \frac{b-a}{b}; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

Using Lemmas 2.1 and 2.2 and the same argument as used in the case when $b = -1$ we can get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$. If $b = -1$, then from (2.33) we have $FG \equiv 1$, that is, $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv p^2$, where $[f^n P(f)]^{(k)} - p$ and $[g^n P(g)]^{(k)} - p$ share 0 CM. If $b \neq -1$, then from (2.33) we obtain

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G - 1}.$$

Therefore

$$\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r, 0; F).$$

So, in view of Lemmas 2.1 and 2.2 with $p = 1$ and the second fundamental theorem, we can write

$$\begin{aligned}
& (n+m)\, T(r,g) \\
\leq\ & \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\frac{1}{1+b};G) + N_{k+1}(r,0;g^n P(g)) - \overline{N}(r,0;G) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + \overline{N}(r,0;F) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + (k+1)\overline{N}(r,0;f) + T(r,P(f)) \\
& +k\overline{N}(r,\infty;f) + S(r,f) + S(r,g) \\
\leq\ & (k+2+m)\, T(r,g) + (2k+1+m)\, T(r,f) + S(r,f) + S(r,g).
\end{aligned}$$

Without loss of generality, we can assume that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$. So, for $r \in I$, we have

$$(n - 3k - 3 - m)\, T(r,g) \leq S(r,g),$$

which is a contradiction since $n > 3k + 3 + m$.

Case 3. Let $b = 0$. From (2.33) we obtain

$$F \equiv \frac{G + a - 1}{a}. \tag{2.34}$$

If $a \neq 1$, then from (2.34) we obtain $\overline{N}(r, 1-a; G) = \overline{N}(r, 0; F)$. Similarly we can get a contradiction as in Case 2. Therefore $a = 1$ and from (2.34) we obtain $F \equiv G$, that is, $[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}$. Then by Lemma 2.4 we have $f^n P(f) \equiv g^n P(g)$. This completes the proof. Lemma 2.17 is proved. □

Lemma 2.18. *Let f and g be two transcendental meromorphic functions, $n, k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ such that $n > k + 2$, and let p be a non-zero polynomial. Suppose that $[f^n P(f)]^{(k)} - p$, $[g^n P(g)]^{(k)} - p$ share 0 CM, and f, g share ∞ IM, where $P(w)$ is defined as in Theorem F. If $[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv p^2$, then $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$; if $p(z)$ is not a constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, and $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ are such that $a_i^2 (c_1 c_2)^{n+i} [(n+i)c]^2 = -1$, if $p(z)$ is a non-zero constant b, then $f(z) = c_3 e^{cz}$, $g(z) = c_4 e^{-cz}$, where $c, c_3, c_4 \in \mathbb{C} \setminus \{0\}$ are such that $(-1)^k a_i^2 (c_3 c_4)^{n+i} [(n+i)c]^{2k} = b^2$.*

The proof follows from Lemmas 2.6 and 2.16.

Lemma 2.19 ([1]). *Let f and g be two non-constant meromorphic functions sharing $(1, k_1)$, where $2 \le k_1 \le \infty$. Then*

$$\overline{N}(r, 1; f| = 2) + 2\,\overline{N}(r, 1; f| = 3) + \ldots + (k_1 - 1)\,\overline{N}(r, 1; f| = k_1) + k_1\,\overline{N}_L(r, 1; f)$$
$$+(k_1 + 1)\,\overline{N}_L(r, 1; g) + k_1\,\overline{N}_E^{(k_1+1}(r, 1; g) \le N(r, 1; g) - \overline{N}(r, 1; g).$$

Lemma 2.20. *Let f and g be two transcendental meromorphic functions, p be a non-zero polynomial, and let $F = [f^n P(f)]^{(k)}/p$, $G = [g^n P(g)]^{(k)}/p$, where $n, k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and $P(w)$ is defined as in Theorem F. Suppose $H \not\equiv 0$. If f, g share $(\infty, 0)$ and F, G share $(1, k_1)$, where $0 \le k_1 \le \infty$, then $(n + m - k - 1)\overline{N}(r, \infty; f) \le (k + m + 1)(T(r, f) + T(r, g)) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g)$.*

Proof. Suppose that ∞ is an e.v.P of f and g, then the result follows immediately. Next, suppose that ∞ is not an e.v.P of f and g. Since $H \not\equiv 0$, we have $F \not\equiv G$. We claim that $V \not\equiv 0$. Suppose the opposite $V \equiv 0$. Then by integration we obtain $1 - \frac{1}{F} = A(1 - \frac{1}{G})$,
where A is a constant such that $A \neq 0, 1$. Note that if z_0 $(p(z_0) \neq 0)$ is a pole of f, then it is a pole of g as well. Hence, from the definition of F and G we have $\frac{1}{F(z_0)} = 0$ and $\frac{1}{G(z_0)} = 0$. So $A = 1$, which is a contradiction.

Next, suppose that z_0 is a pole of f with multiplicity q and a pole of g with multiplicity r such that $p(z_0) \neq 0$. Clearly z_0 is a pole of F with multiplicity $(n + m)q + k$ and a pole of G with multiplicity $(n+m)r+k$. Noting that f, g share $(\infty, 0)$ from the definition of V it follows that z_0 is a zero of V with multiplicity at least $n + m + k - 1$. Now using the Milloux theorem (see [8], p. 55), and Lemma 2.1, we obtain from the definition of V that $m(r,V) = S(r,f) + S(r,g)$. Thus, using Lemma 2.1 and (2.3), we can write

$$\begin{aligned}
&(n+m+k-1)\overline{N}(r,\infty;f) \leq N(r,0;V) + O(\log r) \leq T(r,V) + O(\log r)\\
\leq\ & N(r,\infty;V) + m(r,V) + O(\log r)\\
\leq\ & \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
\leq\ & N_{k+1}(r,0;f^nP(f)) + N_{k+1}(r,0;g^nP(g)) + k\overline{N}(r,\infty;f)\\
&+k\overline{N}(r,\infty;g) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
\leq\ & N_{k+1}(r,0;f^n) + N_{k+1}(r,0;P(f)) + N_{k+1}(r,0;g^n)\\
&+N_{k+1}(r,0;P(g)) + 2k\overline{N}(r,\infty;f) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
\leq\ & (k+1)\overline{N}(r,0;f) + N(r,0;P(f)) + (k+1)\overline{N}(r,0;g)\\
&+N(r,0;P(g)) + 2k\overline{N}(r,\infty;f) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g),
\end{aligned}$$

implying that

$$\begin{aligned}
(n+m-k-1)\overline{N}(r,\infty;f) \leq\ & (k+m+1)(T(r,f) + T(r,g)) + \overline{N}_*(r,1;F,G)\\
&+S(r,f) + S(r,g).
\end{aligned}$$

Lemma 2.20 is proved. □

3. Proof of the theorem

Let $F = \frac{[f^nP(f)]^{(k)}}{p}$ and $G = \frac{[g^nP(g)]^{(k)}}{p}$. Note that since f and g are transcendental meromorphic functions, p is a small function with respect to both $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$. Also, F, G share $(1,k_1)$ except the zeros of p, and f, g share $(\infty,0)$. Now we consider two cases.

Case 1. Let $H \not\equiv 0$.

From (2.1) it can easily be deduced that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different, (iv) zeros of $F'(G')$ which are not the zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$\begin{aligned}
&N(r,\infty;H) \leq \overline{N}_*(r,\infty;F,G) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2)\\
&+\overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g). \qquad (3.1)
\end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$, and $\overline{N}_0(r,0;G')$ is defined similarly.

Let z_0 be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So, we have

$$N(r,1;F|=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,f)+S(r,g). \tag{3.2}$$

Using (3.2) and (3.3) we get

$$\begin{aligned}
\overline{N}(r,1;F) &\leq N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2) \\
&\leq \overline{N}_*(r,\infty;f,g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G) \\
&\quad +\overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g) \\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G) \\
&\quad +\overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g).
\end{aligned} \tag{3.3}$$

Now in view of Lemmas 2.3 and 2.19 we get

$$\begin{aligned}
&\overline{N}_0(r,0;G')+\overline{N}(r,1;F|\geq 2)+\overline{N}_*(r,1;F,G) \\
&\leq \overline{N}_0(r,0;G')+\overline{N}(r,1;F|=2)+\overline{N}(r,1;F|=3)+\ldots+\overline{N}(r,1;F|=k_1) \\
&\quad +\overline{N}_E^{(k_1+1}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,1;F,G) \\
&\leq \overline{N}_0(r,0;G')-\overline{N}(r,1;F|=3)-\ldots-(k_1-2)\overline{N}(r,1;F|=k_1) \\
&\quad -(k_1-1)\overline{N}_L(r,1;F)-k_1\overline{N}_L(r,1;G)-(k_1-1)\overline{N}_E^{(k_1+1}(r,1;F) \\
&\quad +N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_*(r,1;F,G) \\
&\leq \overline{N}_0(r,0;G')+N(r,1;G)-\overline{N}(r,1;G)-(k_1-2)\overline{N}_L(r,1;F) \\
&\quad -(k_1-1)\overline{N}_L(r,1;G) \\
&\leq N(r,0;G'\mid G\neq 0)-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G) \\
&\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;g)-(k_1-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G).
\end{aligned} \tag{3.4}$$

Hence, using (3.3), (3.4), Lemmas 2.2 and 2.20, and the second fundamental theorem, we can write

$$(n+m)T(r,f)\leq T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)$$

$$\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)-N_0(r,0;F')+S(r,f)$$

$$\begin{aligned}
&\le \overline{N}(r,\infty,f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))+\overline{N}(r,0;F\mid\ge 2)\\
&\quad+\overline{N}(r,0;G\mid\ge 2)+\overline{N}(r,1;F\mid\ge 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;G')\\
&\quad-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\le 3\,\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)-(k_1-2)\,\overline{N}_*(r,1;F,G)\\
&\quad-\overline{N}_L(r,1;G)+S(r,f)+S(r,g)\\
&\le 3\,\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g)+N_{k+2}(r,0;g^nP(g))\\
&\quad-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\le (3+k)\,\overline{N}(r,\infty;f)+(k+2)\,\overline{N}(r,0;f)+T(r,P(f))+(k+2)\,\overline{N}(r,0;g)\\
&\quad+T(r,P(g))-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\le (k+m+2)\,\big(T(r,f)+T(r,g)\big)+(3+k)\overline{N}(r,\infty;f)\\
&\quad-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)
\end{aligned}$$

$$\begin{aligned}
&\le (k+m+2)\,\big(T(r,f)+T(r,g)\big)+\frac{(3+k)(k+m+1)}{n+m-k-1}\,\big(T(r,f)+T(r,g)\big)\\
&+\frac{3+k}{n+m-k-1}\,\overline{N}_*(r,1;F,G)-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\le \left[k+m+2+\frac{(3+k)(k+m+1)}{n+m-k-1}\right]\,\big(T(r,f)+T(r,g)\big)+S(r,f)+S(r,g).
\end{aligned}$$

In a similar way we can obtain

$$\begin{aligned}
(3.5)\qquad &(n+m)T(r,g)\\
\le\ &\left[k+m+2+\frac{(3+k)(k+m+1)}{n+m-k-1}\right]\,\big(T(r,f)+T(r,g)\big)+S(r,f)+S(r,g).
\end{aligned}$$

Adding (3.4) and (3.5) we get

$$\left[n-m-2k-4-\frac{(6+2k)(k+m+1)}{n+m-k-1}\right]\,\Big(T(r,f)+T(r,g)\Big)\le S(r,f)+S(r,g).$$

Since the quantity in the third bracket can be written as

$$(3.6)\left[\frac{(n+m-k-1)^2-(2m+k+3)(n+m-k-1)-2(k+3)(k+m+1)}{n+m-k-1}\right],$$

by a simple computation one can easily verify that when

$$\begin{aligned}
n+m-k-1 &> 2m+2k+5\\
&> \frac{2m+k+3+\sqrt{(2m+k+3)^2+8(k+3)(k+m+1)}}{2},
\end{aligned}$$

that is, when $n > 3k + m + 6$, we obtain a contradiction from (3.6).

Case 2. Let $H \equiv 0$. Then by Lemma 2.17 we have either

$$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv p^2, \tag{3.7}$$

or

$$f^n P(f) \equiv g^n P(g). \tag{3.8}$$

From (3.8) we get

$$f^n(a_m f^m + a_{m-1} f^{m-1} + \ldots + a_0) = g^n(a_m g^m + a_{m-1} g^{m-1} + \ldots + a_0). \tag{3.9}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ into (3.9) we deduce that

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \ldots + a_0 g^n(h^n - 1) = 0,$$

which implies $h^d = 1$, where $d = GCD(n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus, $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^d = 1$, where $d = GCD(n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If h is non-constant, then by (3.9) f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \ldots + a_0) - \omega_2^n(a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \ldots + a_0)$. In particular, when $P(w) = a_1 w + a_2$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$, then by Lemma 2.12 of [3], we have $f \equiv g$. Note that when $P(w) \equiv a_0$, then we must have $f \equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^n = 1$. The remaining part of the proof follows from (3.7) and Lemma 2.18. □

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