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MEROMORPHIC FUNCTIONS SHARING THREE VALUES WITH THEIR LINEAR DIFFERENTIAL POLYNOMIALS IN AN ANGULAR DOMAIN

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Abstract. Let f be a nonconstant meromorphic function of lower order $\mu(f) > 1/2$ in $\mathbb{C}$, and let a_j $(j = 1, 2, 3)$ be three distinct finite complex numbers. We show that there exists an angular domain $D = \{z : \alpha \leq \arg z \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$, such that if f share a_j $(j = 1, 2, 3)$ CM with its k-th linear differential polynomial $L[f]$ in D, then $f \equiv L[f]$. This generalizes the corresponding results from Frank and Schwick [Results. Math. 22 (1992) 679-684], Zheng [Canad. Math. Bull. 47 (2004) 152-160] and Li-Liu-Yi [Results. Math. 68 (2015) 441-453].

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1. Introduction

We use $\mathbb{C}$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to denote the whole complex plane and the extended complex plane, respectively. In what follows, we shall suppose that the reader is familiar with standard notations and fundamental results of the Nevanlinna theory (see [7, 14, 15]). For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic function of f and by $\delta(a, f)$ the Nevanlinna deficiency of f. Also, by $\lambda(f)$ and $\mu(f)$ we denote the order and the lower order of a meromorphic function f, respectively.

Let f and g be nonconstant meromorphic functions in the domain $D \subset \mathbb{C}$, and let $c \in \overline{\mathbb{C}}$. If $f - c$ and $g - c$ have the same zeros with the same multiplicities in D, then we say that f and g share c CM in D. If $f - c$ and $g - c$ only have the same zeros, then we say that f and g share c IM in D. The zeros of $f - c$ imply the poles of f when $c = \infty$.

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In 1979, Gundersen [6] and Mues-Steinmetz [10] have considered the uniqueness of a meromorphic function f and its derivative f' and obtained the following result.

Theorem A (see [6, 10]). *Let f be a nonconstant meromorphic function in $\mathbb{C}$, and let a_j $(j = 1, 2, 3)$ be three distinct finite complex numbers. If f and f' share a_j $(j = 1, 2, 3)$ IM in $\mathbb{C}$, then $f \equiv f'$.*

In 1992, Frank and Schwick [3] generalized Theorem A and proved the following result.

Theorem B (see [3]). *Let f be a nonconstant meromorphic function in $\mathbb{C}$ and a_j $(j = 1, 2, 3)$ be three distinct finite complex numbers, and let k be a positive integer. If f and $f^{(k)}$ share a_j $(j = 1, 2, 3)$ IM in $\mathbb{C}$, then $f \equiv f^{(k)}$.*

Remark 1.1. *Three IM shared values in Theorem B can be replaced by two CM shared values (see Frank and Weissenborn [4]).*

In 2004, Zheng [16] has extended Theorem B from complex plane to an angular domain, and proved the following theorem.

Theorem C (see [16]). *Let f be a transcendental meromorphic function of finite lower order $\mu(f)$ in $\mathbb{C}$ such that $\delta(a, f^{(p)}) > 0$ for some $a \in \overline{\mathbb{C}}$ and an integer $p \geq 0$. Let the pairs of real numbers $\{\alpha_j, \beta_j\}$ $(j = 1, \ldots, q)$ be such that*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 < \cdots \leq \alpha_q < \beta_q \leq \pi$$

with $\omega = \max\{\pi/(\beta_j - \alpha_j) : 1 \leq j \leq q\}$, and

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma} \arcsin\sqrt{\frac{\delta\left(a, f^{(p)}\right)}{2}},$$

where $\sigma = \max\{\omega, \mu\}$. For a positive integer k, assume that f and $f^{(k)}$ share a_j $(j = 1, 2, 3)$ IM in $X := \bigcup_{j=1}^{q} \{z : \alpha_j \leq \arg z \leq \beta_j\}$, where a_j $(j = 1, 2, 3)$ are three distinct finite complex numbers such that $a \neq a_j$ $(j = 1, 2, 3)$. If $\lambda(f) > \omega$, then $f \equiv f^{(k)}$.

In 2015, Li, Liu, and Yi [9] observed that Theorem C is invalid for $q \geq 2$, and proved the following more general result, which extends Theorem C (see [9, p. 443]).

Theorem D (see [9]). *Let f be a transcendental meromorphic function of finite lower order $\mu(f)$ in $\mathbb{C}$ and such that $\delta(a, f) > 0$ for some $a \in \mathbb{C}$. Assume that $q \geq 2$ pairs of real numbers $\{\alpha_j, \beta_j\}$ satisfy the conditions:*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 < \cdots \leq \alpha_q < \beta_q \leq \pi$$

with $\omega = \max\{\pi/(\beta_j - \alpha_j) : 1 \le j \le q\}$, *and*

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta(a,f)}{2}},$$

where $\sigma = \max\{\omega, \mu\}$. *For a k-th order linear differential polynomial* $L[f]$ *in* f *with constant coefficients given by*

$$L[f] = b_k f^{(k)} + b_{k-1} f^{(k-1)} + \cdots + b_1 f', \tag{1.1}$$

where k *is a positive integer,* $b_k, b_{k-1}, \cdots, b_1$ *are constants and* $b_k \neq 0$, *assume that* f *and* $L[f]$ *share* a_j $(j = 1,2,3)$ *IM in* $X = \bigcup_{j=1}^{q}\{z : \alpha_j \le \arg z \le \beta_j\}$, *where* a_j $(j = 1,2,3)$ *are three distinct finite complex numbers such that* $a \neq a_j$ $(j = 1,2,3)$. *If* $\lambda(f) \neq \omega$, *then* $f = L[f]$.

Based on Theorem D, we naturally arise the following question.

Question 1.1. Does there exist an angular domain $D = \{z : \alpha \le \arg z \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$, such that if f and $L[f]$ share a_j $(j = 1,2,3)$ CM or IM in D, then $f = L[f]$ in Theorem D?

In this paper, we investigate the above question and prove the following result, which generalizes Theorems C and D.

Theorem 1.1. *Let* f *be a nonconstant meromorphic function of lower order* $\mu(f) > 1/2$ *in* $\mathbb{C}$, a_j $(j = 1,2,3)$ *be three distinct finite complex numbers, and let* $L[f]$ *be given by (1.1). Then there exists an angular domain* $D = \{z : \alpha \le \arg z \le \beta\}$, *where* $0 < \beta - \alpha \le 2\pi$, *such that if* f *and* $L[f]$ *share* a_j $(j = 1,2,3)$ *CM in* D, *then* $f = L[f]$.

As an immediate consequence of Theorem 1.1, we have the following result.

Corollary 1.1. *Let* f *be a nonconstant meromorphic function of lower order* $\mu(f) > 1/2$ *in* $\mathbb{C}$, a_j $(j = 1,2,3)$ *be three distinct finite complex numbers, and let* k *be a positive integer. Then there exists an angular domain* $D = \{z : \alpha \le \arg z \le \beta\}$, *where* $0 < \beta - \alpha \le 2\pi$, *such that if* f *and* $f^{(k)}$ *share* a_j $(j = 1,2,3)$ *CM in* D, *then* $f = f^{(k)}$.

In order to prove our results, we recall the Nevanlinna theory on an angular domain. Let f be a meromorphic function in $D = \{z : \alpha \le \arg z \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$. Nevanlinna [5, 11] defined the following symbols.

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right| + \log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t}, \tag{1.2}$$

$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^{\omega}}\int_{\alpha}^{\beta}\log^{+}\left|f\left(re^{i\theta}\right)\right|\sin\omega\left(\theta-\alpha\right)d\theta, \tag{1.3}$$

$$C_{\alpha,\beta}(r,f)=2\sum_{1<|b_m|<r}\left(\frac{1}{|b_m|^{\omega}}-\frac{|b_m|^{\omega}}{r^{2\omega}}\right)\sin\omega\left(\theta_m-\alpha\right), \tag{1.4}$$

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f), \tag{1.5}$$

where $\omega=\pi/\left(\beta-\alpha\right)$, and $b_m=|b_m|e^{i\theta_m}$ are the poles of f in D counting multiplicities. If we ignore their multiplicities, then we replace $C_{\alpha,\beta}(r,f)$ by $\overline{C}_{\alpha,\beta}(r,f)$. Also, $S_{\alpha,\beta}(r,f)$ will stand for the Nevanlinna's angular characteristic function in D.

Throughout the paper, we denote by $R(r,*)$ a quantity satisfying the following relation:

$$R(r,*)=O\left\{\log\left(rT(r,*)\right)\right\},\qquad \forall r\notin E, \tag{1.6}$$

where E denotes a set of positive real numbers with finite linear measure, which will not necessarily be the same in each occurrence.

Also, we will need the following definitions.

Definition 1.1. *(see [8, cf.1]). Assume that f is a meromorphic function of infinite order in $\mathbb{C}$. Then there exists a proximate order $\rho(r)$ of f such that:*

(i) $\rho(r)$ is continuous and nondecreasing for $r\geq r_0$, and $\rho(r)\to+\infty$ as $r\to+\infty$;

(ii) $U(r)=r^{\rho(r)}$ $(r\geq r_0)$ satisfies the condition:

$$\overline{\lim}_{r\to+\infty}\frac{\log U(R)}{\log U(r)}=1,\qquad R=r+\frac{r}{\log U(r)};$$

(iii) the following relation holds:

$$\overline{\lim}_{r\to+\infty}\frac{\log T(r,f)}{\rho(r)\log r}=1.$$

Definition 1.2. *(see [13, cf.1, 8]). Let f be a meromorphic function of infinite order in $\mathbb{C}$, and let $\rho(r)$ be the proximate order of f. A direction $\arg z=\theta_0$ is called a Borel direction of proximate order $\rho(r)$ of f if for arbitrarily small $\varepsilon>0$ the following relation holds:*

$$\overline{\lim}_{r\to+\infty}\frac{\log n(r,\theta_0,\varepsilon,f=a)}{\rho(r)\log r}=1$$

for all $a\in\overline{\mathbb{C}}$ except at most two exceptional values, where $n(r,\theta_0,\varepsilon,f=a)$ denotes the number of the zeros of $f-a$ counting multiplicities in the sector $|\arg z-\theta_0|<\varepsilon$, $|z|\leq r$.

Definition 1.3. *(see [12]). Let f be a meromorphic function of finite order $\lambda(f) > 0$ in $\mathbb{C}$. A direction $\arg z = \theta_0$ is called a Borel direction of order $\lambda(f)$ if for arbitrarily small $\varepsilon > 0$ the following relation holds:*

$$\overline{\lim}_{r \to +\infty} \frac{\log n(r, \theta_0, \varepsilon, f = a)}{\log r} = \lambda(f)$$

for all $a \in \overline{\mathbb{C}}$ except at most two exceptional values, where $n(r, \theta_0, \varepsilon, f = a)$ is as in Definition 1.2.

2. Some lemmas

Lemma 2.1. *(see [5, 11]). Let f be a meromorphic function in $\mathbb{C}$. Then for any $a \in \mathbb{C}$ the following relation holds:*

$$S_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) = S_{\alpha,\beta}(r, f) + O(1).$$

Lemma 2.2. *(see [5, 11, cf.2]). Let f be a meromorphic function in $\mathbb{C}$. Then the following assertions hold:*

(i) for q (≥ 3) distinct complex numbers $a_j \in \overline{\mathbb{C}}$ $(j = 1, 2, \cdots, q)$ we have

$$(q-2) S_{\alpha,\beta}(r, f) \leq \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) + R(r, f);$$

(ii) for a positive integer k we have

$$A_{\alpha,\beta}\left(r, \frac{f^{(k)}}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f^{(k)}}{f}\right) = R(r, f);$$

(iii) if f is of finite order, then $R(r, f) = O(1)$;

(iv) if f is of infinite order and of proximate order $\rho(r)$, then $R(r, f) = O(\log U(r))$, where $U(r) = r^{\rho(r)}$ is as in Definition 1.

Lemma 2.3. *(see [7]). Let f be a meromorphic function in $\mathbb{C}$, and let $L[f]$ be given by (1.1). Then $T(r, L[f]) \leq (k+1) T(r, f) + O(\log r T(r, f))$.*

Lemma 2.4. *(see [12]). Let f be a meromorphic function of finite order $\lambda(f) > 0$ in $\mathbb{C}$. Then f has at least one Borel direction $\arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ of order $\lambda(f)$.*

Using the same arguments applied in Lemma 1.3 of [15, p.14], we can easily obtain the following result.

Lemma 2.5. *Let f be a nonconstant meromorphic function in $\mathbb{C}$, and let $a_j \in \overline{\mathbb{C}}$ $(j = 1, 2, \cdots, q)$ be q distinct complex numbers. Then we have*

$$\sum_{j=1}^{q} (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{1}{f - a_j}\right) = (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) + O(1).$$

Lemma 2.6. *Let f be a nonconstant meromorphic function in $\mathbb{C}$, a_j $(j = 1, 2, 3)$ be three distinct finite complex numbers, and let $L[f]$ be given by (1.1). Suppose that f and $L[f]$ share a_j $(j = 1, 2, 3)$ CM in $D = \{z : \alpha \le \arg z \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$. If $f \not\equiv L[f]$, then $S_{\alpha,\beta}(r, f) = R(r, f)$.*

Proof. By Lemma 2.5, the Nevanlinna basic reasoning (see [7], p. 5), the definition (1.5) of $S_{\alpha,\beta}(r, *)$, Lemma 2.1, and Lemma 2.2(ii), we can write

$$\begin{aligned}
\textstyle\sum_{j=1}^{3} (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{1}{f-a_j}\right) &= (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \textstyle\sum_{j=1}^{3} \frac{1}{f-a_j}\right) + O(1) \\
&\le (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \textstyle\sum_{j=1}^{3} \frac{L[f]}{f-a_j}\right) + (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{1}{L[f]}\right) + O(1) \\
\le \textstyle\sum_{j=1}^{3} (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{L[f]}{f-a_j}\right) &+ S_{\alpha,\beta} \left(r, \frac{1}{L[f]}\right) + O(1) \le S_{\alpha,\beta}(r, L[f]) + R(r, f).
\end{aligned}$$

that is,

$$\sum_{j=1}^{3} (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{1}{f - a_j}\right) \le S_{\alpha,\beta}(r, L[f]) + R(r, f).$$

Therefore, we have

$$\sum_{j=1}^{3} (A_{\alpha,\beta} + B_{\alpha,\beta}) \left(r, \frac{1}{f - a_j}\right) + \sum_{j=1}^{3} C_{\alpha,\beta} \left(r, \frac{1}{f - a_j}\right) \le$$

$$\le S_{\alpha,\beta}(r, L[f]) + \sum_{j=1}^{3} C_{\alpha,\beta} \left(r, \frac{1}{f - a_j}\right) + R(r, f),$$

which together with definition (1.5) of $S_{\alpha,\beta}(r, *)$ and Lemma 2.1 implies that

$$3S_{\alpha,\beta}(r, f) \le S_{\alpha,\beta}(r, L[f]) + \sum_{j=1}^{3} C_{\alpha,\beta} \left(r, \frac{1}{f - a_j}\right) + R(r, f). \tag{2.1}$$

Next, since f and $L[f]$ share a_j $(j = 1, 2, 3)$ CM in D, by the Nevanlinna basic reasoning [7, p. 5], Lemma 2.1, the definition (1.5) of $S_{\alpha,\beta}(r, *)$, and Lemma 2.2(ii), we can write

$$\sum_{j=1}^{3} C_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) \le C_{\alpha,\beta}\left(r, \frac{1}{f-L[f]}\right) \le S_{\alpha,\beta}(r, f-L[f]) + O(1)$$
$$\le (A_{\alpha,\beta} + B_{\alpha,\beta})(r, f-L[f]) + C_{\alpha,\beta}(r, f-L[f]) + O(1)$$
$$\le (A_{\alpha,\beta} + B_{\alpha,\beta})\left(r, \frac{f-L[f]}{f}\right) + (A_{\alpha,\beta} + B_{\alpha,\beta})(r, f) + C_{\alpha,\beta}(r, L[f]) + O(1)$$
$$\le A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f) + k\overline{C}_{\alpha,\beta}(r, f) + R(r, f)$$
$$\le S_{\alpha,\beta}(r, f) + \frac{k}{k+1} C_{\alpha,\beta}\left(r, f^{(k)}\right) + R(r, f) \le S_{\alpha,\beta}(r, f) + \frac{k}{k+1} C_{\alpha,\beta}(r, L[f]) + R(r, f)$$
$$\le S_{\alpha,\beta}(r, f) + \frac{k}{k+1} S_{\alpha,\beta}(r, L[f]) + R(r, f).$$

Combining this with (2.1), we get

$$2S_{\alpha,\beta}(r, f) \le \frac{2k+1}{k+1} S_{\alpha,\beta}(r, L[f]) + R(r, f). \tag{2.2}$$

Set $F = 1/(f-c)$ and $L_1[f] = 1/(L[f]-c)$, where $c \in \mathbb{C}$ ($c \notin \{a_1, a_2, a_3\}$), and observe that f and $L[f]$ share a_j ($j = 1, 2, 3$) CM in D. Since f and $L[f]$ always share ∞ IM in D, F and $L_1[f]$ share 0 IM, and $1/(a_j - c)$ ($j = 1, 2, 3$) CM in D, then by Lemma 2.1, Lemma 2.2(i), the definition (1.6) of $R_{\alpha,\beta}(r, *)$, and Lemma 2.3 we get

$$\begin{aligned} 2S_{\alpha,\beta}(r, L_1[f]) &\le \sum_{j=1}^{3} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{L_1[f] - 1/(a_j-c)}\right) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{L_1[f]}\right) + R(r, L_1[f]) \\ &\le C_{\alpha,\beta}\left(r, \frac{1}{F - L_1[f]}\right) + R(r, f) \le S_{\alpha,\beta}(r, F - L_1[f]) + R(r, f) \\ &\le S_{\alpha,\beta}(r, F) + S_{\alpha,\beta}(r, L_1[f]) + R(r, f), \end{aligned}$$

implying that

$$S_{\alpha,\beta}(r, L_1[f]) \le S_{\alpha,\beta}(r, F) + R(r, f).$$

Hence, by Lemma 2.1 we have

$$S_{\alpha,\beta}(r, L[f]) \le S_{\alpha,\beta}(r, f) + R(r, f). \tag{2.3}$$

In view of (2.2) and (2.3) we obtain the conclusion of the lemma. Lemma 2.6 is proved. □

Lemma 2.7. *Let f be a meromorphic function in $D = \{z : \alpha \le \arg z \le \beta\}$ ($0 < \beta - \alpha \le 2\pi$), and $\omega = \pi/(\beta-\alpha)$. Then for any $c \in \overline{\mathbb{C}}$ and arbitrarily small $\nu > 0$, we have*

$$n(r, D_\nu, f = c) \le K r^{\omega} C_{\alpha,\beta}\left(2r, \frac{1}{f-c}\right),$$

where K is a positive constant, $D_\nu = \{z : \alpha+\nu \le \arg z \le \beta-\nu\}$, and $n(r, D_\nu, f = c)$ denotes the number of zeros of $f - c$ counting multiplicities in $D_\nu \cap \{z : |z| \le r\}$.

Proof. Let η_m be the zeros of $f - c$ counting multiplicities in D. Put $n(*) := n(*, D_\nu, f = c)$ for the sake of simplicity. Then for arbitrarily small $\nu > 0$ we can write

$$\begin{aligned}
C_{\alpha,\beta}\left(2r, \frac{1}{f-c}\right) &= 2 \sum_{1<|\eta_m|<2r, \alpha<\theta_m<\beta} \left(\frac{1}{|\eta_m|^\omega} - \frac{|\eta_m|^\omega}{(2r)^{2\omega}}\right) \sin\omega(\theta_m - \alpha) \\
&\ge 2 \sum_{1<|\eta_m|<r, \alpha+\nu<\theta_m<\beta-\nu} \left(\frac{1}{|\eta_m|^\omega} - \frac{|\eta_m|^\omega}{(2r)^{2\omega}}\right) \sin\omega(\theta_m - \alpha) \\
&\ge 2\sin(\omega\nu) \sum_{1<|\eta_m|<r, \alpha+\nu<\theta_m<\beta-\nu} \left(\frac{1}{|\eta_m|^\omega} - \frac{|\eta_m|^\omega}{(2r)^{2\omega}}\right) \\
&= 2\sin(\omega\nu)\left(\int_1^r \frac{dn(t)}{t^\omega} - \int_1^r \frac{t^\omega}{(2r)^{2\omega}} dn(t)\right) \\
&= 2\sin(\omega\nu)\left(\frac{n(r)}{r^\omega} + \omega\int_1^r \frac{n(t)}{t^{\omega+1}}dt - \frac{n(r)}{4^\omega r^\omega} + \frac{\omega}{(2r)^{2\omega}}\int_1^r t^{\omega-1} n(t)\, dt\right) \\
&\ge 2\sin(\omega\nu)\left(\frac{n(r)}{r^\omega} - \frac{n(r)}{4^\omega r^\omega}\right) = 2\sin(\omega\nu)\frac{n(r)}{r^\omega}\frac{4^\omega - 1}{4^\omega} \ge K\frac{n(r)}{r^\omega}.
\end{aligned}$$

Therefore

$$n(r) \le K r^\omega C_{\alpha,\beta}\left(2r, \frac{1}{f-c}\right),$$

where K is a positive constant not necessarily the same for each occurrence. This completes the proof of the lemma. Lemma 2.7 is proved. □

Lemma 2.8. *(see [13]). Let f be a meromorphic function of infinite order in $\mathbb{C}$, and let $\rho(r)$ be a proximate order of f. Then f has at least one Borel direction $\arg z = \theta_0$ $(0 \le \theta_0 < 2\pi)$ of proximate order $\rho(r)$.*

Lemma 2.9. *(see [12]). Let f be a meromorphic function of infinite order in $\mathbb{C}$, and let $\rho(r)$ be a proximate order of f. Then a direction $\arg z = \theta_0$ is a Borel direction of proximate order $\rho(r)$ of f, if and only if for arbitrarily small $\varepsilon > 0$ we have*

$$\limsup_{r\to+\infty} \frac{\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r, f)}{\rho(r)\log r} = 1.$$

3. Proof of Theorem 1.1

Suppose that $f \not\equiv L[f]$. Since $\lambda(f) \ge \mu(f)$ and $\mu(f) > 1/2$, it follows that $\lambda(f) > 1/2$. Now we consider the following two cases.

Case 1. Assume that $1/2 < \lambda(f) < +\infty$. Choose ω such that $1/2 < \omega < \lambda(f)$, where $\omega = \pi/(\beta-\alpha)$ and $0 < \beta - \alpha \leq 2\pi$. Then for one given angular domain $D = \{z : \alpha \leq \arg z \leq \beta\}$, we have $\lambda(f) > \omega$. Thus, by Lemma 2.4, we can assume that f has at least one Borel direction $\arg z = \theta_0$ in D of order $\lambda(f)$. Therefore, in view of Definition 2.3 there exists a finite complex number c such that for arbitrarily small $\varepsilon > 0$,

$$\limsup_{r\to+\infty} \frac{\log n(r,\theta_0,\varepsilon,f=c)}{\log r} = \lambda(f) > \omega. \tag{3.1}$$

Next, since f and $L[f]$ share a_j $(j = 1,2,3)$ CM in D, by Lemma 2.6 and Lemma 2.2(iii), we have

$$S_{\alpha,\beta}(r,f) = R(r,f) = O(1). \tag{3.2}$$

On the other hand, for arbitrarily small $\nu > 0$, by Lemma 2.7 we get

$$n(r,D_\nu,f=c) \leq Kr^\omega C_{\alpha,\beta}\left(2r,\frac{1}{f-c}\right), \tag{3.3}$$

where K is a positive constant, $D_\nu = \{z : \alpha+\nu \leq \arg z \leq \beta-\nu\}$, and $n(r,D_\nu,f=c)$ denotes the number of zeros of $f-c$ counting multiplicities in $D_\nu \cap \{z : |z| \leq r\}$. Thus, by (3.2), (3.3), and Lemma 2.1 it follows that

$$n(r,\theta_0,\varepsilon,f=c) \leq n(r,D_\nu,f=c) \leq$$

$$\leq Kr^\omega C_{\alpha,\beta}\left(2r,\frac{1}{f-c}\right) \leq Kr^\omega\left(S_{\alpha,\beta}(2r,f)+O(1)\right) \leq O(r^\omega),$$

and hence, we have

$$n(r,\theta_0,\varepsilon,f=c) = O(r^\omega).$$

This contradicts (3.1) and so we obtain $f \equiv L[f]$.

Case 2. Assume that $\lambda(f) = +\infty$ and $\rho(r)$ is a proximate order of f. Then in view of Lemma 8 we can assume that f has at least one Borel direction $\arg z = \theta_0$ in D of proximate order $\rho(r)$. Moreover, by Lemma 2.2(iv) and Lemma 2.6 we have

$$S_{\alpha,\beta}(r,f) = R(r,f) = O(\log U(r)), \qquad U(r) = r^{\rho(r)},$$

implying that

$$S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r,f) = O(\log U(r)), \qquad U(r) = r^{\rho(r)}. \tag{3.4}$$

Now by Lemma 2.9, for arbitrarily small $\varepsilon > 0$, we have

$$\limsup_{r\to+\infty} \frac{\log S_{\theta_0-\varepsilon,\theta_0+\varepsilon}(r,f)}{\rho(r)\log r} = 1. \tag{3.5}$$

Combining (3.4) and (3.5) we arrive at a contradiction. This completes the proof of Theorem 1.1.

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