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ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATORS FROM ANALYTIC BESOV SPACES INTO ZYGMUND TYPE SPACES

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Abstract. In this paper, we give some estimates for the essential norm of weighted composition operators from analytic Besov spaces into Zygmund type spaces. In particular, a new characterization for the boundedness and compactness of the weighted composition operators uC_φ is obtained.

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1. Introduction

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Recall that the essential norm of $T : X \to Y$ is its distance to the set of compact operators $K : X \to Y$, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K \text{ is compact } \}.$$

Here $\|T\|_{X\to Y}$ denotes the operator norm of $T : X \to Y$.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denote the space of analytic functions on $\mathbb{D}$, $S(\mathbb{D})$ denote the set of all analytic self-maps of $\mathbb{D}$, and let $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ, is defined as follows:

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

When $u = 1$, we get the composition operator, denoted by C_φ. When $\varphi(z) = z$, we get the multiplication operator, denoted by M_u. A basic and interesting problem concerning concrete operators (such as composition operator, weighted composition

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operator, Toeplitz operator and Hankel operator) is to relate operator theoretic properties to their function theoretic properties of their symbols (for more information, we refer the reader to [2] and [28]).

For $0 < \alpha < \infty$, the Bloch type space $\mathcal{B}^\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

If $\alpha = 1$, then $\mathcal{B}^\alpha$ is the Bloch space $\mathcal{B}$ (see [28] for more details of the Bloch spaces).

For $0 < \alpha < \infty$, the Zygmund type space, denoted by $\mathcal{Z}^\alpha$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f''(z)| < \infty.$$

It is easy to see that the space $\mathcal{Z}^\alpha$ is a Banach space with the above norm. If $\alpha = 1$, then $\mathcal{Z}^\alpha$ is the classical Zygmund space, denoted by $\mathcal{Z}$. When $1 < \alpha < \infty$, the space $\mathcal{Z}^\alpha$ coincides with the space $\mathcal{B}^{\alpha-1}$. In particular, we have $\mathcal{Z}^2 = \mathcal{B}$.

For $p \in (1, \infty)$, the analytic Besov space B_p is the space consisting of all $f \in H(\mathbb{D})$ such that

$$b_p(f)^p := \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$. The quantity b_p is a seminorm and the norm is defined by $\|f\|_{B_p} = |f(0)| + b_p(f)$. In particular, B_2 is the classical Dirichlet space.

The compactness and essential norm of the operator $C_\varphi : \mathcal{B} \to \mathcal{B}$ were studied in [19, 20, 24, 25, 27]. The boundedness, compactness and essential norm of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ were studied in [3, 11, 17, 18, 22, 29, 30]. See [1, 5 – 9, 12–16, 23, 26] for some results of composition operators, weighted composition operators and related operators mapping into the Zygmund space. In [5], the authors characterized the boundedness and compactness of the operator $uC_\varphi : B_p \to \mathcal{Z}$. In fact, under the assumption that $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded, $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is compact if and only if

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^\alpha |u(z)| |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} = 0,$$

$$\lim_{|\varphi(z)| \to 1} (1 - |z|^2)^\alpha |u''(z)| \Big(\log \frac{e}{1 - |\varphi(z)|^2} \Big)^{1 - \frac{1}{p}} = 0$$

and

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^\alpha |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{1 - |\varphi(z)|^2} = 0.$$

The purpose of this paper is to give some estimates for the essential norm of the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$, in particular, by using $u\varphi^n$. Moreover, we give a new characterization of the boundedness and compactness for the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$. Throughout the paper, we will write $A \lesssim B$ if there exists a constant C such that $A \leq CB$. The notation $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Essential norm of $uC_\varphi : B_p \to \mathcal{Z}^\alpha$

In this section, we give some estimates for the essential norm of the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$. For this purpose, we state some lemmas which will be used in the proofs of the main results.

Lemma 2.1 ([24]). *Let X and Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that the following conditions are satisfied.*

(1) The point evaluation functionals on Y are continuous.

(2) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.

(3) The operator $T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Lemma 2.2 ([4]). *Let $1 < p < \infty$ and $f \in B_p$. Then the following statements hold.*

(i) $|f(z)| \lesssim \|f\|_{B_p}(\log \frac{2}{1-|z|^2})^{1-\frac{1}{p}}$ *for every* $z \in \mathbb{D}$.

(ii) $|f'(z)| \lesssim \frac{1}{1-|z|^2}\|f\|_{B_p}$ *for every* $z \in \mathbb{D}$.

Let $a \in \mathbb{D}$. Define the functions:

$$f_a(z) = \frac{\log \frac{e}{1-\bar{a}z}}{(\log \frac{e}{1-|a|^2})^{\frac{1}{p}}}, \qquad g_a(z) = \frac{(\log \frac{e}{1-\bar{a}z})^2}{(\log \frac{e}{1-|a|^2})^{1+\frac{1}{p}}}, \qquad h_a(z) = \frac{(\log \frac{e}{1-\bar{a}z})^3}{(\log \frac{e}{1-|a|^2})^{2+\frac{1}{p}}},$$

$$x_a(z) = \frac{(1-|a|^2)(a-z)}{(1-\bar{a}z)^2}, \qquad y_a(z) = \bar{a}\frac{(1-|a|^2)(a-z)^2}{(1-\bar{a}z)^3}, \quad z \in \mathbb{D}.$$

Now we are in position to state and prove our main results in this section.

Theorem 2.1. *Let $1 < p < \infty, 0 < \alpha < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ with $\|\varphi\|_\infty = 1$ be such that $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded. Then*

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \approx \max\{A, B, C, P, Q\} \approx \max\{E, F, G\}.$$

Here

$$A=\limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{Z}^\alpha},\qquad B=\limsup_{|a|\to 1}\|uC_\varphi g_a\|_{\mathcal{Z}^\alpha},C=\limsup_{|a|\to 1}\|uC_\varphi h_a\|_{\mathcal{Z}^\alpha},$$

$$P=\limsup_{|a|\to 1}\|uC_\varphi x_a\|_{\mathcal{Z}^\alpha},Q=\limsup_{|a|\to 1}\|uC_\varphi y_a\|_{\mathcal{Z}^\alpha},$$

$$G=\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2},$$

$$E=\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}}$$

and

$$F=\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}.$$

Proof. First we prove that

$$\max\{A,B,C,P,Q\}\lesssim\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha}.$$

In [4] it was shown that $f_a,g_a,h_a,x_a,y_a\in B_p$, the norms $\|f_a\|_{B_p}$, $\|g_a\|_{B_p}$, $\|h_a\|_{B_p}$, $\|x_a\|_{B_p}$, $\|y_a\|_{B_p}$ are bounded by a constant independent of a, and f_a,g_a,h_a,x_a,y_a converge to zero uniformly on compact subsets of $\mathbb{D}$ as $|a|\to 1$. Thus, by Lemma 2.1, for any compact operator $K:B_p\to\mathcal{Z}^\alpha$, we have

$$\lim_{|a|\to 1}\|Kf_a\|_{\mathcal{Z}^\alpha}=0,\ \lim_{|a|\to 1}\|Kg_a\|_{\mathcal{Z}^\alpha}=0,\ \lim_{|a|\to 1}\|Kh_a\|_{\mathcal{Z}^\alpha}=0,$$

$$\lim_{|a|\to 1}\|Kx_a\|_{\mathcal{Z}^\alpha}=0\quad\text{and}\quad\lim_{|a|\to 1}\|Ky_a\|_{\mathcal{Z}^\alpha}=0.$$

Since

$$\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha}\gtrsim\|(uC_\varphi-K)f_a\|_{\mathcal{Z}^\alpha}\geq\|uC_\varphi f_a\|_{\mathcal{Z}^\alpha}-\|Kf_a\|_{\mathcal{Z}^\alpha},$$

$$\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha}\gtrsim\|(uC_\varphi-K)g_a\|_{\mathcal{Z}^\alpha}\geq\|uC_\varphi g_a\|_{\mathcal{Z}^\alpha}-\|Kg_a\|_{\mathcal{Z}^\alpha},$$

$$\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha}\gtrsim\|(uC_\varphi-K)h_a\|_{\mathcal{Z}^\alpha}\geq\|uC_\varphi h_a\|_{\mathcal{Z}^\alpha}-\|Kh_a\|_{\mathcal{Z}^\alpha},$$

$$\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha}\gtrsim\|(uC_\varphi-K)x_a\|_{\mathcal{Z}^\alpha}\geq\|uC_\varphi x_a\|_{\mathcal{Z}^\alpha}-\|Kx_a\|_{\mathcal{Z}^\alpha}$$

and

$$\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha}\gtrsim\|(uC_\varphi-K)y_a\|_{\mathcal{Z}^\alpha}\geq\|uC_\varphi y_a\|_{\mathcal{Z}^\alpha}-\|Ky_a\|_{\mathcal{Z}^\alpha},$$

by taking $\limsup_{|a|\to 1}$ on both sides of these inequalities, we obtain

$$\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha}\gtrsim\max\Big\{A,B,C,P,Q\Big\}.$$

Therefore,

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha}=\inf_K\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha}\gtrsim\max\Big\{A,B,C,P,Q\Big\}.$$

Next, we prove that

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \gtrsim \max\Big\{E,F,G\Big\}.$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)|\to 1$ as $j\to\infty$. Define

$$k_j(z)=\frac{\log\frac{e}{1-\overline{\varphi(z_j)}z}}{(\log\frac{e}{1-|\varphi(z_j)|^2})^{\frac{1}{p}}}-\frac{(\log\frac{e}{1-\overline{\varphi(z_j)}z})^2}{(\log\frac{e}{1-|\varphi(z_j)|^2})^{1+\frac{1}{p}}}+\frac{1}{3}\frac{(\log\frac{e}{1-\overline{\varphi(z_j)}z})^3}{(\log\frac{e}{1-|\varphi(z_j)|^2})^{2+\frac{1}{p}}},$$

$$l_j(z)=\frac{(1-|\varphi(z_j)|^2)(\varphi(z_j)-z)}{(1-\overline{\varphi(z_j)}z)^2}-2\overline{\varphi(z_j)}\frac{(1-|\varphi(z_j)|^2)(\varphi(z_j)-z)^2}{(1-\overline{\varphi(z_j)}z)^3}$$

and

$$m_j(z)=\overline{\varphi(z_j)}\frac{(1-|\varphi(z_j)|^2)(\varphi(z_j)-z)^2}{(1-\overline{\varphi(z_j)}z)^3},$$

and observe that k_j, l_j and m_j belong to B_p and converge to zero uniformly on compact subsets of $\mathbb{D}$. Moreover, we have

$$|k_j(\varphi(z_j))|=\frac{1}{3}\Big(\log\frac{e}{1-|\varphi(z_j)|^2}\Big)^{1-\frac{1}{p}},\qquad k_j'(\varphi(z_j))=0,\qquad k_j''(\varphi(z_j))=0,$$

$$l_j(\varphi(z_j))=0,\qquad |l_j'(\varphi(z_j))|=\frac{1}{1-|\varphi(z_j)|^2},\qquad l_j''(\varphi(z_j))=0,$$

$$m_j(\varphi(z_j))=0,\qquad m_j'(\varphi(z_j))=0,\qquad |m_j''(\varphi(z_j))|=\frac{2|\varphi(z_j)|}{(1-|\varphi(z_j)|^2)^2}.$$

Then for any compact operator $K: B_p\to\mathcal{Z}^\alpha$, we get

$$\begin{aligned}
\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha} &\gtrsim \limsup_{j\to\infty}\|uC_\varphi k_j\|_{\mathcal{Z}^\alpha}-\limsup_{j\to\infty}\|Kk_j\|_{\mathcal{Z}^\alpha}\\
&\gtrsim \limsup_{j\to\infty}(1-|z_j|^2)^\alpha|u''(z_j)|\Big(\log\frac{e}{1-|\varphi(z_j)|^2}\Big)^{1-\frac{1}{p}}\\
&= \limsup_{|\varphi(z)|\to1}(1-|z|^2)^\alpha|u''(z)|\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}}=E,
\end{aligned}$$

$$\begin{aligned}
\|uC_\varphi-K\|_{B_p\to\mathcal{Z}^\alpha} &\gtrsim \limsup_{j\to\infty}\|uC_\varphi l_j\|_{\mathcal{Z}^\alpha}-\limsup_{j\to\infty}\|Kl_j\|_{\mathcal{Z}^\alpha}\\
&\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)^\alpha|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)|}{1-|\varphi(z_j)|^2}\\
&= \limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}=F
\end{aligned}$$

and

$$\begin{aligned}\|uC_\varphi - K\|_{B_p\to\mathcal{Z}^\alpha} &\gtrsim \limsup_{j\to\infty}\|uC_\varphi m_j\|_{\mathcal{Z}^\alpha} - \limsup_{j\to\infty}\|Km_j\|_{\mathcal{Z}^\alpha}\\ &\gtrsim \limsup_{j\to\infty}\frac{(1-|z_j|^2)^\alpha|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|}{(1-|\varphi(z_j)|^2)^2}\\ &= \limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} = G.\end{aligned}$$

Hence, we have

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} = \inf_K \|uC_\varphi - K\|_{B_p\to\mathcal{Z}^\alpha} \gtrsim \max\Big\{E,F,G\Big\}.$$

Now we prove that

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \lesssim \max\{E,F,G\},\qquad \|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \lesssim \max\{A,B,C,P,Q\}.$$

For $r\in[0,1)$, define $K_r : H(\mathbb{D})\to H(\mathbb{D})$ by

$$(K_rf)(z) = f_r(z) = f(rz),\quad f\in H(\mathbb{D}).$$

It is obvious that $f_r\to f$ uniformly on compact subsets of $\mathbb{D}$ as $r\to1$. Moreover, the operator K_r is compact on B_p and $\|K_r\|_{B_p\to B_p}\le 1$. Let $\{r_j\}\subset(0,1)$ be a sequence such that $r_j\to1$ as $j\to\infty$. Then for any positive integer j, the operator $uC_\varphi K_{r_j} : B_p\to\mathcal{Z}^\alpha$ is compact. Hence, we have

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \le \limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{Z}^\alpha}. \tag{2.1}$$

Thus, we have only to show that

$$\limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{Z}^\alpha} \lesssim \max\{A,B,C,P,Q\}$$

and

$$\limsup_{j\to\infty}\|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{Z}^\alpha} \lesssim \max\{E,F,G\}.$$

For any $f\in B_p$ such that $\|f\|_{B_p}\le1$, we can write

$$\begin{aligned}&\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{Z}^\alpha}\\ =\ & |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| + \|u\cdot(f-f_{r_j})\circ\varphi\|_*\\ &+|u'(0)(f-f_{r_j})(\varphi(0)) + u(0)(f-f_{r_j})'(\varphi(0))\varphi'(0)|.\end{aligned}\tag{2.2}$$

Here $\|g\|_* = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g''(z)|$. It is obvious that

$$\lim_{j\to\infty}|u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| = 0 \tag{2.3}$$

and

$$\lim_{j\to\infty} |u'(0)(f-f_{r_j})(\varphi(0)) + u(0)(f-f_{r_j})'(\varphi(0))\varphi'(0)| = 0. \tag{2.4}$$

Next, we can write

$$\begin{aligned}
&\limsup_{j\to\infty} \|u\cdot(f-f_{r_j})\circ\varphi\|_* \\
=\ & \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f-f_{r_j})(\varphi(z))||u''(z)| \\
&+ \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f-f_{r_j})(\varphi(z))||u''(z)| \\
&+ \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f-f_{r_j})'(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)| \\
&+ \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f-f_{r_j})'(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)| \\
&+ \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f-f_{r_j})''(\varphi(z))||\varphi'(z)|^2|u(z)| \\
&+ \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f-f_{r_j})''(\varphi(z))||\varphi'(z)|^2|u(z)| \\
=\ & Q_1+Q_2+Q_3+Q_4+Q_5+Q_6, \qquad (2.5)
\end{aligned}$$

where $N\in\mathbb{N}$ is large enough such that $r_j \ge \frac{1}{2}$ for all $j\ge N$,

$$Q_1 = \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f-f_{r_j})(\varphi(z))||u''(z)|,$$

$$Q_2 = \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f-f_{r_j})(\varphi(z))||u''(z)|,$$

$$Q_3 = \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f-f_{r_j})'(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)|,$$

$$Q_4 = \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f-f_{r_j})'(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)|,$$

$$Q_5 = \limsup_{j\to\infty} \sup_{|\varphi(z)|\le r_N} (1-|z|^2)^\alpha |(f-f_{r_j})''(\varphi(z))||\varphi'(z)|^2|u(z)|$$

and

$$Q_6 = \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\alpha |(f-f_{r_j})''(\varphi(z))||\varphi'(z)|^2|u(z)|.$$

Since $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded, applying the operator uC_φ to $1, z$ and z^2, we see that $u\in\mathcal{Z}^\alpha$, $u\varphi\in\mathcal{Z}^\alpha$ and $u\varphi^2\in\mathcal{Z}^\alpha$. Hence, using the boundedness of φ and the triangle inequality, we get

$$\widetilde{K_1} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |2u'(z)\varphi'(z)+u(z)\varphi''(z)| < \infty$$

and

$$\widetilde{K_2} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\varphi'(z)|^2|u(z)| < \infty.$$

Next, since $f_{r_j} \to f$, $r_j f'_{r_j} \to f'$, as well as $r_j^2 f''_{r_j} \to f''$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, we have

$$Q_1 \leq \|u\|_{\mathcal{Z}^\alpha} \limsup_{j\to\infty} \sup_{|w|\leq r_N} |f(w) - f(r_j w)| = 0, \tag{2.6}$$

$$Q_3 \leq \widetilde{K_1} \limsup_{j\to\infty} \sup_{|w|\leq r_N} |f'(w) - r_j f'(r_j w)| = 0 \tag{2.7}$$

and

$$Q_5 \leq \widetilde{K_2} \limsup_{j\to\infty} \sup_{|w|\leq r_N} |f''(w) - r_j^2 f''(r_j w)| = 0. \tag{2.8}$$

We know that $Q_2 \leq \limsup_{j\to\infty}(S_1^j + S_2^j)$, where

$$S_1^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f(\varphi(z))||u''(z)|, S_2^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f(r_j\varphi(z))||u''(z)|.$$

Using the fact that $\|f\|_{B_p} \leq 1$ and Lemma 2.2, we obtain

$$\begin{aligned} S_1^j &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f(\varphi(z))||u''(z)| \\ &\lesssim \frac{1}{3} \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |u''(z)| \Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}} \\ &\lesssim \sup_{|a|>r_N} \|uC_\varphi(f_a - g_a + \frac{1}{3}h_a)\|_{\mathcal{Z}^\alpha} \\ &\lesssim \sup_{|a|>r_N} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha} + \sup_{|a|>r_N} \|uC_\varphi g_a\|_{\mathcal{Z}^\alpha} + \sup_{|a|>r_N} \|uC_\varphi h_a\|_{\mathcal{Z}^\alpha}. \end{aligned}$$

Taking limit as $N \to \infty$, we obtain

$$\begin{aligned} &\limsup_{j\to\infty} S_1^j \lesssim \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\alpha |u''(z)| \Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}} = E \\ \lesssim\ & \limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha} + \limsup_{|a|\to 1} \|uC_\varphi g_a\|_{\mathcal{Z}^\alpha} + \limsup_{|a|\to 1} \|uC_\varphi h_a\|_{\mathcal{Z}^\alpha} \\ =\ & A + B + C. \end{aligned}$$

Similarly, $\limsup_{j\to\infty} S_2^j \lesssim E \lesssim A + B + C$, and hence, we get

$$Q_2 \lesssim E \lesssim A + B + C \lesssim \max\{A, B, C\}. \tag{2.9}$$

We have $Q_4 \leq \limsup_{j\to\infty}(S_3^j + S_4^j)$, where

$$S_3^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f'(\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)|$$

and

$$S_4^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha r_j |f'(r_j\varphi(z))||2u'(z)\varphi'(z) + u(z)\varphi''(z)|.$$

Using the fact that $\|f\|_{B_p} \le 1$ and $B_p \subset \mathcal{B}$, we can write

$$\begin{aligned}
S_3^j &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f'(\varphi(z))||2u'(z)\varphi'(z)+u(z)\varphi''(z)| \\
&\lesssim \|f\|_{B_p} \sup_{|\varphi(z)|>r_N} \frac{(1-|z|^2)^\alpha |2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \frac{(1-|z|^2)^\alpha |2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi(x_a-2y_a)\|_{\mathcal{Z}^\alpha} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi x_a\|_{\mathcal{Z}^\alpha} + \sup_{|a|>r_N} \|uC_\varphi y_a\|_{\mathcal{Z}^\alpha}.
\end{aligned}$$

Taking limit as $N \to \infty$, we obtain

$$\begin{aligned}
\limsup_{j\to\infty} S_3^j &\lesssim \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2} = F \\
&\lesssim \limsup_{|a|\to 1} \|uC_\varphi x_a\|_{\mathcal{Z}^\alpha} + \limsup_{|a|\to 1} \|uC_\varphi y_a\|_{\mathcal{Z}^\alpha} = P+Q.
\end{aligned}$$

Similarly, $\limsup_{j\to\infty} S_4^j \lesssim F \lesssim P+Q$, and hence, we get

$$Q_4 \lesssim F \lesssim P+Q. \tag{2.10}$$

We have $Q_6 \le \limsup_{j\to\infty}(S_5^j + S_6^j)$, where

$$S_5^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f''(\varphi(z))||\varphi'(z)|^2|u(z)|$$

and

$$S_6^j = \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha r_j^2 |f''(r_j\varphi(z))||\varphi'(z)|^2|u(z)|.$$

Using the fact that $\|f\|_{B_p} \le 1$ and $B_p \subset \mathcal{B}$, we can write

$$\begin{aligned}
S_5^j &= \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |f''(\varphi(z))||\varphi'(z)|^2|u(z)| \\
&\lesssim \frac{1}{r_N}\|f\|_{B_p} \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\alpha |\varphi'(z)|^2|u(z)| \frac{|\varphi(z)|}{(1-|\varphi(z)|^2)^2} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \frac{2(1-|z|^2)^\alpha |\varphi'(z)|^2|u(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^2} \lesssim \sup_{|a|>r_N} \|uC_\varphi y_a\|_{\mathcal{Z}^\alpha}.
\end{aligned}$$

Taking limit as $N \to \infty$, we obtain

$$\limsup_{j\to\infty} S_5^j \lesssim \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\varphi'(z)|^2|u(z)|}{(1-|\varphi(z)|^2)^2} = G \lesssim Q.$$

Similarly, $\limsup_{j\to\infty} S_6^j \lesssim G \lesssim Q$, and hence, we have

$$Q_6 \lesssim G \lesssim Q. \tag{2.11}$$

Hence, by (2.2)-(2.11), we get

$$\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{Z}^\alpha} = \limsup_{j\to\infty} \sup_{\|f\|_{B_p}\le 1} \|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{Z}^\alpha}$$

$$(2.12)\quad = \limsup_{j\to\infty} \sup_{\|f\|_{B_p}\le 1} \|u\cdot(f - f_{r_j})\circ\varphi\|_* \lesssim E + F + G \lesssim A + B + C + P + Q.$$

Therefore, by (2.1) and (2.12), we obtain

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \lesssim E + F + G \lesssim \max\{E, F, G\}$$

and

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \lesssim A + B + C + P + Q \lesssim \max\{A, B, C, P, Q\}.$$

This completes the proof. Theorem 2.1 is proved. □

Theorem 2.2. *Let* $1 < p < \infty, 0 < \alpha < \infty, u \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$ *with* $\|\varphi\|_\infty = 1$ *be such that* $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ *is bounded. Then*

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \approx \max\left\{ \limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha}, \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}^\alpha} \right\}.$$

Proof. The lower estimate. For each nonnegative integer n, let $p_n(z) = z^n$. Then $p_n \in B_p$ and the sequence $\{p_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Thus, by Lemma 2.1, for any compact operator $K : B_p \to \mathcal{Z}^\alpha$, we have $\lim_{n\to\infty} \|Kp_n\|_{\mathcal{Z}^\alpha} = 0$. Hence

$$\|uC_\varphi - K\|_{B_p\to\mathcal{Z}^\alpha} \gtrsim \limsup_{n\to\infty} \|(uC_\varphi - K)p_n\|_{\mathcal{Z}^\alpha} \ge \limsup_{n\to\infty} \|uC_\varphi p_n\|_{\mathcal{Z}^\alpha}.$$

From the definition of essential norm, we get

$$(2.13)\qquad \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}^\alpha} \le \|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha}.$$

By Theorem 2.1 and (2.13), we get the desired lower estimate.

The upper estimate. For $a \in \mathbb{D}$, we define

$$\lambda_a(z) = \frac{1 - |a|^2}{1 - \bar{a}z}, \qquad \mu_a(z) = \frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^2}, \qquad z \in \mathbb{D}.$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. As shown in [4], $f_{\varphi(z_j)}, \lambda_{\varphi(z_j)}$ and $\mu_{\varphi(z_j)}$ belong to B_p and converge to zero uniformly on compact subsets of $\mathbb{D}$. Moreover, we have

$$f_{\varphi(z_j)}(\varphi(z_j)) = (M_{\varphi(z_j)})^{1-\frac{1}{p}}, \quad f'_{\varphi(z_j)}(\varphi(z_j)) = (M_{\varphi(z_j)})^{-\frac{1}{p}} \frac{\overline{\varphi(z_j)}}{1 - |\varphi(z_j)|^2},$$

$$f''_{\varphi(z_j)}(\varphi(z_j)) = (M_{\varphi(z_j)})^{-\frac{1}{p}} \frac{(\overline{\varphi(z_j)})^2}{(1 - |\varphi(z_j)|^2)^2},$$

$$\lambda_{\varphi(z_j)}(\varphi(z_j)) = 1, \quad \lambda'_{\varphi(z_j)}(\varphi(z_j)) = \frac{\overline{\varphi(z_j)}}{1-|\varphi(z_j)|^2}, \quad \lambda''_{\varphi(z_j)}(\varphi(z_j)) = \frac{2(\overline{\varphi(z_j)})^2}{(1-|\varphi(z_j)|^2)^2},$$

$$\mu_{\varphi(z_j)}(\varphi(z_j)) = 1, \quad \mu'_{\varphi(z_j)}(\varphi(z_j)) = \frac{2\overline{\varphi(z_j)}}{1-|\varphi(z_j)|^2}, \quad \mu''_{\varphi(z_j)}(\varphi(z_j)) = \frac{6(\overline{\varphi(z_j)})^2}{(1-|\varphi(z_j)|^2)^2}.$$

Here $M_{\varphi(z_j)} = \log \frac{e}{1-|\varphi(z_j)|^2}$. Next, we can write

$$\begin{aligned} \|uC_\varphi f_{\varphi(z_j)}\|_{\mathcal{Z}^\alpha} &\geq \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(uC_\varphi f_{\varphi(z_j)})''(z)| \\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\Big|u''(z)f_{\varphi(z_j)}(\varphi(z)) + u(z)(\varphi'(z))^2 f''_{\varphi(z_j)}(\varphi(z)) \\ &\quad +(2u'(z)\varphi'(z)+u(z)\varphi''(z))f'_{\varphi(z_j)}(\varphi(z))\Big| \\ &\geq (1-|z_j|^2)^\alpha|u''(z_j)|(M_{\varphi(z_j)})^{1-\frac{1}{p}} \\ &\quad -\frac{(1-|z_j|^2)^\alpha|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1-|\varphi(z_j)|^2}(M_{\varphi(z_j)})^{-\frac{1}{p}} \\ &\quad -\frac{(1-|z_j|^2)^\alpha|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2}(M_{\varphi(z_j)})^{-\frac{1}{p}}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \|uC_\varphi \lambda_{\varphi(z_j)}\|_{\mathcal{Z}^\alpha} &\geq \frac{(1-|z_j|^2)^\alpha|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1-|\varphi(z_j)|^2} \\ &\quad -2\frac{(1-|z_j|^2)^\alpha|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2} - (1-|z_j|^2)^\alpha|u''(z_j)| \end{aligned} \tag{2.15}$$

$$\begin{aligned} \|uC_\varphi \mu_{\varphi(z_j)}\|_{\mathcal{Z}^\alpha} &\geq 6\frac{(1-|z_j|^2)^\alpha|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2} - (1-|z_j|^2)^\alpha|u''(z_j)| \\ &\quad -2\frac{(1-|z_j|^2)^\alpha|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1-|\varphi(z_j)|^2}. \end{aligned} \tag{2.16}$$

Taking limit as $j \to \infty$ on both sides of (2.14), we get

$$\begin{aligned} &\limsup_{j\to\infty}\|uC_\varphi f_{\varphi(z_j)}\|_{\mathcal{Z}^\alpha} \\ &+\limsup_{j\to\infty}\frac{(1-|z_j|^2)^\alpha|u(z_j)||\varphi'(z_j)|^2|\varphi(z_j)|^2}{(1-|\varphi(z_j)|^2)^2}(M_{\varphi(z_j)})^{-\frac{1}{p}} \\ &+\limsup_{j\to\infty}\frac{(1-|z_j|^2)^\alpha|2u'(z_j)\varphi'(z_j)+u(z_j)\varphi''(z_j)||\varphi(z_j)|}{1-|\varphi(z_j)|^2}(M_{\varphi(z_j)})^{-\frac{1}{p}} \\ \geq\ &\limsup_{j\to\infty}(1-|z_j|^2)^\alpha|u''(z_j)|\big(M_{\varphi(z_j)}\big)^{1-\frac{1}{p}} \\ \geq\ &\limsup_{j\to\infty}(1-|z_j|^2)^\alpha|u''(z_j)|, \end{aligned}$$

which implies that

$$\begin{aligned}
&\limsup_{|\varphi(z)|\to 1}\|uC_\varphi f_{\varphi(z)}\|_{\mathcal{Z}^\alpha}+\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{-\frac{1}{p}}\\
&+\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{-\frac{1}{p}}\\
\geq\ &\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}}\\
\geq\ &\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|.
\end{aligned}$$

Similarly, by (2.15) and (2.16), we get

$$\begin{aligned}
&\limsup_{|\varphi(z)|\to 1}\|uC_\varphi \lambda_{\varphi(z)}\|_{\mathcal{Z}^\alpha}+\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|\\
\geq\ &\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}\\
&-\limsup_{|\varphi(z)|\to 1}2\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2},
\end{aligned}$$

$$\begin{aligned}
&\limsup_{|\varphi(z)|\to 1}\|uC_\varphi \mu_{\varphi(z)}\|_{\mathcal{Z}^\alpha}+\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|\\
\geq\ &\limsup_{|\varphi(z)|\to 1}6\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\\
&-\limsup_{|\varphi(z)|\to 1}2\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}.
\end{aligned}$$

By the boundedness of $uC_\varphi : B_p \to \mathcal{Z}^\alpha$, we see that

$$\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{1-|\varphi(z)|^2}\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{-\frac{1}{p}}=0$$

and

$$\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{-\frac{1}{p}}=0.$$

Thus, we can write

$$\begin{aligned}
E=\ &\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|u''(z)|\Big(\log\frac{e}{1-|\varphi(z)|^2}\Big)^{1-\frac{1}{p}}\\
\leq\ &\limsup_{|\varphi(z)|\to 1}\|uC_\varphi f_{\varphi(z)}\|_{\mathcal{Z}^\alpha}\leq\limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{Z}^\alpha},
\end{aligned}$$

$$G = \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}$$
$$\le \frac{3}{2}\limsup_{|\varphi(z)|\to 1} \|uC_\varphi f_{\varphi(z)}\|_{\mathcal{Z}^\alpha} + \limsup_{|\varphi(z)|\to 1} \|uC_\varphi \lambda_{\varphi(z)}\|_{\mathcal{Z}^\alpha} + \frac{1}{2}\limsup_{|\varphi(z)|\to 1} \|uC_\varphi \mu_{\varphi(z)}\|_{\mathcal{Z}^\alpha}$$
$$\le \frac{3}{2}\limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha} + \limsup_{|a|\to 1} \|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha} + \frac{1}{2}\limsup_{|a|\to 1} \|uC_\varphi \mu_a\|_{\mathcal{Z}^\alpha},$$

$$F = \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{1-|\varphi(z)|^2}$$
$$\le 4\limsup_{|\varphi(z)|\to 1} \|uC_\varphi f_{\varphi(z)}\|_{\mathcal{Z}^\alpha} + 3\limsup_{|\varphi(z)|\to 1} \|uC_\varphi \lambda_{\varphi(z)}\|_{\mathcal{Z}^\alpha} + \limsup_{|\varphi(z)|\to 1} \|uC_\varphi \mu_{\varphi(z)}\|_{\mathcal{Z}^\alpha}$$
$$\le 4\limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha} + 3\limsup_{|a|\to 1} \|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha} + \limsup_{|a|\to 1} \|uC_\varphi \mu_a\|_{\mathcal{Z}^\alpha}.$$

By Theorem 2.1 and the last three inequalities we obtain

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \approx \max\Big\{E, F, G\Big\} \tag{2.17}$$
$$\le \max\Big\{\limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha}, \limsup_{|a|\to 1} \|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha}, \limsup_{|a|\to 1} \|uC_\varphi \mu_a\|_{\mathcal{Z}^\alpha}\Big\}.$$

Finally, we prove that

$$\max\{\limsup_{|a|\to 1} \|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha},\quad \limsup_{|a|\to 1} \|uC_\varphi \mu_a\|_{\mathcal{Z}^\alpha}\} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}^\alpha}.$$

Let $a \in \mathbb{D}$. For any fixed positive integer $n \ge 1$, it follows from triangle inequality, the fact that $\sup_{0\le k<\infty} \|u\varphi^k\|_{\mathcal{Z}^\alpha} < \infty$ and

$$\lambda_a(z) = (1-|a|^2)\sum_{k=0}^\infty \bar{a}^k z^k, \quad z \in \mathbb{D},$$

that we have

$$\begin{aligned}
\|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha} &\le (1-|a|^2)\sum_{k=0}^\infty |a|^k \|u\varphi^k\|_{\mathcal{Z}^\alpha} \\
&= (1-|a|^2)\sum_{k=0}^{n-1} |a|^k \|u\varphi^k\|_{\mathcal{Z}^\alpha} + (1-|a|^2)\sum_{k=n}^{\infty} |a|^k \|u\varphi^k\|_{\mathcal{Z}^\alpha} \\
&\le n(1-|a|^2)\sup_{0\le k\le n-1} \|u\varphi^k\|_{\mathcal{Z}^\alpha} + (1-|a|^2)\sum_{k=n}^{\infty} |a|^k \sup_{j\ge n} \|u\varphi^j\|_{\mathcal{Z}^\alpha} \\
&\lesssim n(1-|a|^2) + 2\sup_{k\ge n} \|u\varphi^k\|_{\mathcal{Z}^\alpha}.
\end{aligned}$$

Letting $|a| \to 1$ in the above inequality, we get

$$\limsup_{|a|\to 1} \|uC_\varphi \lambda_a\|_{\mathcal{Z}^\alpha} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{Z}^\alpha}. \tag{2.18}$$

Let $a \in \mathbb{D}$. Note that (see [28])

$$\frac{1}{(1-|a|)^\beta} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\beta)}{k!\Gamma(\beta)}|a|^k \quad \text{and} \quad \frac{\Gamma(k+\beta)}{k!} \approx k^{\beta-1}, \qquad k \to \infty.$$

for any fixed positive integer $n \geq 1$. Hence, using the triangle inequality, the fact that $u \in \mathcal{Z}^\alpha$, $\sup_{0 \leq k < \infty} \|u\varphi^k\|_{\mathcal{Z}^\alpha} < \infty$, and

$$\mu_a(z) = (1-|a|^2)^2 \sum_{k=0}^{\infty} \frac{\Gamma(k+2)}{\Gamma(2)k!} \bar{a}^k z^k, \quad z \in \mathbb{D},$$

we obtain

$$\|uC_\varphi \mu_a\|_{\mathcal{Z}^\alpha} \leq (1-|a|^2)^2 \sum_{k=0}^{\infty} \frac{\Gamma(k+2)}{\Gamma(2)k!} |a|^k \|u\varphi^k\|_{\mathcal{Z}^\alpha}$$

$$\begin{aligned}
&\lesssim (1-|a|^2)^2 \|u\|_{\mathcal{Z}^\alpha} + (1-|a|^2)^2 \sum_{k=1}^{\infty} k|a|^k \|u\varphi^k\|_{\mathcal{Z}^\alpha} \\
&= (1-|a|^2)^2 \|u\|_{\mathcal{Z}^\alpha} + (1-|a|^2)^2 \sum_{k=0}^{n-1} k|a|^k \|u\varphi^k\|_{\mathcal{Z}^\alpha} + (1-|a|^2)^2 \sum_{k=n}^{\infty} k|a|^k \|u\varphi^k\|_{\mathcal{Z}^\alpha} \\
&\leq (1-|a|^2)^2 \|u\|_{\mathcal{Z}^\alpha} + n(n-1)(1-|a|^2)^2 \sup_{0 \leq k \leq n-1} \|u\varphi^k\|_{\mathcal{Z}^\alpha} \\
&\quad + (1-|a|^2)^2 \sum_{k=n}^{\infty} k|a|^k \sup_{j \geq n} \|u\varphi^j\|_{\mathcal{Z}^\alpha} \\
&\lesssim (1-|a|^2)^2 \|u\|_{\mathcal{Z}^\alpha} + n(n-1)(1-|a|^2)^2 \sup_{0 \leq k \leq n-1} \|u\varphi^k\|_{\mathcal{Z}^\alpha} + 4 \sup_{k \geq n} \|u\varphi^k\|_{\mathcal{Z}^\alpha}.
\end{aligned}$$

Letting $|a| \to 1$ in the above inequality, we get

$$\limsup_{|a| \to 1} \|uC_\varphi \mu_a\|_{\mathcal{Z}^\alpha} \lesssim \limsup_{n \to \infty} \|u\varphi^n\|_{\mathcal{Z}^\alpha}. \tag{2.19}$$

Therefore, by (2.17) - (2.19) we obtain the desired upper estimate:

$$\|uC_\varphi\|_{e, B_p \to \mathcal{Z}^\alpha} \lesssim \max\left\{ \limsup_{|a| \to 1} \|uC_\varphi f_a\|_{\mathcal{Z}^\alpha}, \limsup_{n \to \infty} \|u\varphi^n\|_{\mathcal{Z}^\alpha} \right\}.$$

The proof is complete. □

3. A NEW CHARACTERIZATION OF OPERATOR uC_φ:

In this section, we give a new characterization for the boundedness, compactness and essential norm of the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$. For this purpose, we first state some definitions and lemmas.

Let $v : \mathbb{D} \to R_+$ be a continuous, strictly positive and bounded function. Here we call v a weight function. The weighted space, denoted by H_v^∞, is a space which consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

Observe that H_v^∞ is a Banach space with the norm $\|\cdot\|_v$. A weight v is called radial if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. The associated weight $\widetilde{v}$ of v is defined by

$$\widetilde{v} = \frac{1}{\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \le 1\}}, z \in \mathbb{D}.$$

When $v = v_\alpha(z) = (1-|z|^2)^\alpha (0 < \alpha < \infty)$, then it is easy to check that $\widetilde{v}_\alpha(z) = v_\alpha(z)$. In this case, instead of H_v^∞ we use the notation $H_{v_\alpha}^\infty$, that is,

$$H_{v_\alpha}^\infty = \{f \in H(\mathbb{D}) : \|f\|_{v_\alpha} = \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^\alpha < \infty\}.$$

When $v = v_{\log,p}(z) = \left((\log \frac{e}{1-|z|^2})^{1-\frac{1}{p}}\right)^{-1}$, then it is also easy to see that $\widetilde{v}_{\log,p} = v_{\log,p}$. Indeed, if

$$v(z) = \left(\max\{|g(w)|; |w| = |z|\}\right)^{-1}$$

is a weight for some $g \in H(\mathbb{D})$, then $\widetilde{v}(z) = v(z)$. Hence the statement follows with $g(z) = (\log \frac{e}{1-|z|^2})^{1-\frac{1}{p}}$.

Lemma 3.1 ([11]). *For $\alpha > 0$, we have $\lim_{k\to\infty} k^\alpha \|z^{k-1}\|_{v_\alpha} = (\frac{2\alpha}{e})^\alpha$.*

Also, we have the following result.

Lemma 3.2. *For $1 < p < \infty$, we have $\lim_{k\to\infty} (\log k)^{1-\frac{1}{p}} \|z^k\|_{v_{\log,p}} \approx 1$.*

Lemma 3.3 ([21]). *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then the following statements hold.*

(a) The weighted composition operator $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)| < \infty.$$

(b) Suppose $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded. Then

$$\|uC_\varphi\|_{e,H_v^\infty \to H_w^\infty} = \lim_{s\to 1^-} \sup_{|\varphi(z)|>s} \frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)|.$$

Lemma 3.4 ([10]). *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then the following statements hold.*

(a) The weighted composition operator $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if

$$\sup_{k\ge 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty,$$

with the norm comparable to the above supremum.

(b) Suppose $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded. Then $\|uC_\varphi\|_{e,H_v^\infty \to H_w^\infty} = \limsup_{k\to\infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}$.

Now we are in position to state and prove our main results in this section.

Theorem 3.1. *Let $1 < p < \infty$, $0 < \alpha < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the weighted composition operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded if and only if $u \in \mathcal{Z}^\alpha$,*

$$\sup_{j\geq 1} j\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha} < \infty,\ \sup_{j\geq 1} j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha} < \infty,$$

$$\sup_{j\geq 1}(\log j)^{1-\frac{1}{p}}\|u''\varphi^j\|_{v_\alpha} < \infty.$$

Proof. Observe first that by Theorem 3 of [5], the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded if and only if

$$M_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|u''(z)|\Big(\log\frac{e}{1-|a|^2}\Big)^{1-\frac{1}{p}} < \infty,$$

$$M_2 = \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{1-|\varphi(z)|^2} < \infty$$

and

$$M_3 = \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} < \infty.$$

By Lemma 3.3, the condition $M_2 < \infty$ and the boundedness of the weighted composition operator $(2u'\varphi' + u\varphi'')C_\varphi : H_{v_1}^\infty \to H_{v_\alpha}^\infty$ are equivalent. By Lemma 3.4, this is equivalent to the following:

$$\sup_{j\geq 1}\frac{\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_1}} < \infty.$$

By Lemma 3.3, the condition $M_3 < \infty$ and the boundedness of the operator $u\varphi'^2C_\varphi : H_{v_2}^\infty \to H_{v_\alpha}^\infty$ are equivalent. By Lemma 3.4, this is equivalent to the following:

$$\sup_{j\geq 1}\frac{\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_2}} < \infty.$$

Next, by Lemma 3.3, the condition $M_1 < \infty$ and the boundedness of the operator $u''C_\varphi : H_{v_{\log,p}}^\infty \to H_{v_\alpha}^\infty$ are equivalent. By Lemma 3.4, this is equivalent to the following:

$$\sup_{j\geq 1}\frac{\|u''\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_{\log,p}}} < \infty.$$

Finally, in view of Lemmas 3.1 and 3.2, we conclude that that the operator $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded if and only if

$$\sup_{j\geq 1} j\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha} \approx \sup_{j\geq 1}\frac{j\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha}}{j\|z^{j-1}\|_{v_1}} < \infty,$$

$$\sup_{j\geq 1} j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha} \approx \sup_{j\geq 1}\frac{j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha}}{j^2\|z^{j-1}\|_{v_2}} < \infty$$

and

$$\begin{aligned}&\max\left\{\|u\|_{\mathcal{Z}^\alpha}, \sup_{j\geq 1}(\log j)^{1-\frac{1}{p}}\|u''\varphi^j\|_{v_\alpha}\right\}\\ = \; &\max\left\{\|u\|_{\mathcal{Z}^\alpha}, \sup_{j\geq 2}(\log(j-1))^{1-\frac{1}{p}}\|u''\varphi^{j-1}\|_{v_\alpha}\right\} \approx \sup_{j\geq 1}\frac{\|u''\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_{\log,p}}} < \infty,\end{aligned}$$

and the result follows. Theorem 3.1 is proved. □

Theorem 3.2. *Let $1 < p < \infty$, $0 < \alpha < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ with $\|\varphi\|_\infty = 1$ be such that $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is bounded. Then*

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \approx \max\{N_1, N_2, N_3\},$$

where

$$N_1 = \limsup_{j\to\infty} j\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha},$$

and

$$N_2 = \limsup_{j\to\infty} j^2\|u(\varphi')^2\varphi^{j-1}\|_{v_\alpha} \qquad \textit{and} \qquad N_3 = \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|u''\varphi^j\|_{v_\alpha}.$$

Proof. From the proof of Theorem 3.1 we see that the boundedness of $uC_\varphi : B_p \to \mathcal{Z}^\alpha$ is equivalent to the boundedness of the operators $(2u'\varphi' + u\varphi'')C_\varphi : H^\infty_{v_1} \to H^\infty_{v_\alpha}$, $u''C_\varphi : H^\infty_{v_{\log,p}} \to H^\infty_{v_\alpha}$ and $u\varphi'^2C_\varphi : H^\infty_{v_2} \to H^\infty_{v_\alpha}$.

The upper estimate. In view of Lemmas 3.1 - 3.4 we can write

$$\begin{aligned}\|(2u'\varphi' + u\varphi'')C_\varphi\|_{e,H^\infty_{v_1}\to H^\infty_{v_\alpha}} &= \limsup_{j\to\infty}\frac{\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_1}}\\ = \limsup_{j\to\infty}\frac{j\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha}}{j\|z^{j-1}\|_{v_1}} &\approx \limsup_{j\to\infty} j\|(2u'\varphi' + u\varphi'')\varphi^{j-1}\|_{v_\alpha},\end{aligned}$$

$$\begin{aligned}\|u\varphi'^2C_\varphi\|_{e,H^\infty_{v_2}\to H^\infty_{v_\alpha}} &= \limsup_{j\to\infty}\frac{\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_2}} = \limsup_{j\to\infty}\frac{j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha}}{j^2\|z^{j-1}\|_{v_2}}\\ &\approx \limsup_{j\to\infty} j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha},\end{aligned}$$

$$\begin{aligned}\|u''C_\varphi\|_{e,H^\infty_{v_{\log,p}}\to H^\infty_{v_\alpha}} &= \limsup_{j\to\infty}\frac{\|u''\varphi^{j-1}\|_{v_\alpha}}{\|z^{j-1}\|_{v_{\log,p}}} = \limsup_{j\to\infty}\frac{(\log(j-1))^{1-\frac{1}{p}}\|u''\varphi^{j-1}\|_{v_\alpha}}{(\log(j-1))^{1-\frac{1}{p}}\|z^{j-1}\|_{v_{\log,p}}}\\ \approx \limsup_{j\to\infty}(\log(j-1))^{1-\frac{1}{p}}\|u''\varphi^{j-1}\|_{v_\alpha} &= \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|u''\varphi^j\|_{v_\alpha}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} &\lesssim \|(2u'\varphi'+u\varphi'')C_\varphi\|_{e,H^\infty_{v_1}\to H^\infty_{v_\alpha}} + \|u''C_\varphi\|_{e,H^\infty_{v_{\log,p}}\to H^\infty_{v_\alpha}}\\ &+\|u\varphi'^2C_\varphi\|_{e,H^\infty_{v_2}\to H^\infty_{v_\alpha}} \lesssim \max\{N_1,N_2,N_3\}.\end{aligned}$$

The lower estimate. From Theorem 2.1, Lemmas 3.1 – 3.4, and the above proof, we have

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \gtrsim F = \|(2u'\varphi'+u\varphi'')C_\varphi\|_{e,H^\infty_{v_1}\to H^\infty_{v_\alpha}} \approx \limsup_{j\to\infty} j\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_{v_\alpha},$$

$$\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \gtrsim G = \|u\varphi'^2C_\varphi\|_{e,H^\infty_{v_2}\to H^\infty_{v_\alpha}} \approx \limsup_{j\to\infty} j^2\|u\varphi'^2\varphi^{j-1}\|_{v_\alpha},$$

$$\begin{aligned}\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} &\gtrsim E = \|u''C_\varphi\|_{e,H^\infty_{v_{\log,p}}\to H^\infty_{v_\alpha}}\\ \approx \limsup_{j\to\infty}(\log(j-1))^{1-\frac{1}{p}}\|u''\varphi^{j-1}\|_{v_\alpha} &= \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|u''\varphi^{j}\|_{v_\alpha}.\end{aligned}$$

Therefore, $\|uC_\varphi\|_{e,B_p\to\mathcal{Z}^\alpha} \gtrsim \max\{N_1,N_2,N_3\}$, as desired. Theorem 3.2 is proved. □

From Theorem 3.2, we immediately get the following result.

Theorem 3.3. *Let $1<p<\infty$, $0<\alpha<\infty$, $u\in H(\mathbb{D})$ and $\varphi\in S(\mathbb{D})$ be such that $uC_\varphi : B_p\to\mathcal{Z}^\alpha$ is bounded. Then the operator $uC_\varphi : B_p\to\mathcal{Z}^\alpha$ is compact if and only if*

$$\limsup_{j\to\infty} j\|(2u'\varphi'+u\varphi'')\varphi^{j-1}\|_{v_\alpha}=0, \qquad \limsup_{j\to\infty} j^2\|u(\varphi')^2\varphi^{j-1}\|_{v_\alpha}=0$$

and

$$\limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|u''\varphi^{j}\|_{v_\alpha}=0.$$

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