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UNIFORM CONVERGENCE OF DOUBLE VILENKIN-FOURIER SERIES

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Abstract. In this paper we study the problem of uniform convergence for the rectangular partial sums of double Fourier series on a bounded Vilenkin group of functions of partial bounded oscillation.

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1. Introduction

Let N_+ denote the set of positive integers, and $N := N_+ \cup \{0\}$. Let $m_0, m_1, ...$ be a sequence of positive integers not less then 2. Denote by $Z_{m_k} = \{0, 1, ..., m_k - 1\}$ the additive group of integers modulo m_k. Define the group G as the complete direct product of the groups Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s. If the sequence $m_0, m_1, ...$ is bounded, then G is called a bounded Vilenkin group. In this paper we consider only the bounded Vilenkin group. The elements of G can be represented by sequences $x := (x_0, x_1, ..., x_j, ...)$, $(x_j \in Z_{m_j})$. The group operation "+" in G is given by

$$x + y = ((x_0 + y_0) \, mod \, m_0, ..., (x_k + y_k) \, mod \, m_k, ...),$$

where $x := (x_0, ..., x_k, ...)$ and $y := (y_0, ..., y_k, ...) \in G$.

The inverse of operation "+" will be denoted by "−". It is easy to give a base for the neighborhoods of G :

$$I_0(x) := G, \qquad I_n(x) := \{y \in G : y_0 = x_0, ..., y_{n-1} = x_{n-1}\},$$

for some choice of $(x_j \in Z_{m_j})$, $j = 0, 1, ..., n - 1$. Let $I_n(0) = I_n$.

We denote $e_n = (0, ..., 0, 1, 0, ...) \in G$ the element of G in which the nth coordinate is 1 and the rest are zeros $(n \in N)$.

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If we define the so-called generalized number system based on m in the following way: $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in N)$, then every $n \in N$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in N_+)$ and only a finite number of n_j's differ from zero, and $G^2 = G \times G$ is the product of the group G.

Define

$$z_{\alpha}^{(n)} := (x_0, x_1, ..., x_{n-1}, 0, 0, ...) \in G;$$

where

$$\alpha := \sum_{j=0}^{n-1} \left(\frac{x_j}{M_{j+1}} \right) M_n, \quad (x_j \in Z_{m_j}), \; j = 0, 1, ..., n-1.$$

Then it is easy to show that

$$G := \bigcup_{\alpha=0}^{M_n - 1} \left(I_n + z_{\alpha}^{(n)} \right). \tag{1.1}$$

Next, on the group G we introduce an orthonormal system, which is called Vilenkin system. We first define the complex valued functions $r_k(x) : G \to \mathbb{C}$, called the generalized Rademacher functions, in the following way:

$$r_k(x) := \exp\left(\frac{2\pi i x_k}{m_k} \right), \qquad (i^2 = -1, \; x \in G, \; k \in N).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \subset N)$ on G as follows (see [1]):

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \qquad (n \in N).$$

In the special case where $m_j \equiv 2$ $(j \in N)$, the system ψ is called Walsh-Paley system.

For the system ψ the Dirichlet kernel is defined as follows:

$$D_n := \sum_{k=0}^{n-1} \psi_k \qquad (n \in \mathbb{N}_+), \quad D_0 = 0.$$

The following properties of the kernel D_n are well known (see, e.g., [20]).

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in G \backslash I_n. \end{cases} \tag{1.2}$$

and

$$\int_G D_n(t)\, d\mu(t) = 1, \quad n \in N_+. \tag{1.3}$$

Let $n = a_k M_k + n'$, with $0 < a_k < m_k$ and $0 \leq n' < M_k$, then

$$D_n(x) = \frac{1 - \psi_{M_k}^{a_k}(x)}{1 - \psi_{M_k}(x)} D_{M_k}(x) + \psi_{M_k}^{a_k}(x) D_{n'}(x). \tag{1.4}$$

The next property of the kernel D_n can be found in [19].

$$\left|D_k\left(z_\alpha^{(n)}\right)\right| < (p+1)\,M_n/\alpha \tag{1.5}$$

for all k, n, and α $(0 < \alpha < M_n)$, where $p = \sup_j m_j$.

The rectangular partial sums of the double Vilenkin-Fourier series are defined as follows:

$$S_{n,m}(f;x,y) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\widehat{f}(i,j)\,\psi_i(x)\,\psi_j(y),$$

where the number

$$\widehat{f}(i,j) = \int_{G^2} f(x,y)\,\overline{\psi}_i(x)\,\overline{\psi}_j(y)\,d\mu(x,y)$$

is called the (i,j)-th Vilenkin-Fourier coefficient of function f.

By $C(G^2)$ we denote the space of continuous functions on G^2 with the supremum norm:

$$\|f\|_C := \sup_{x,y\in G}|f(x,y)| \qquad (f\in C(G^2)).$$

The partial moduli of continuity of a function $f\in C(G^2)$ are defined by

$$\omega_1\left(f;\frac{1}{M_k}\right) := \sup_{x,y\in G,t\in I_k}|f(x-t,y)-f(x,y)|$$

and

$$\omega_2\left(f;\frac{1}{M_l}\right) := \sup_{x,y\in G,t\in I_l}|f(x,y-t)-f(x,y)|.$$

We also will use the notion of mixed modulus of continuity of a function $f\in C(G^2)$, defined as follows:

$$\omega_{1,2}\left(f;\frac{1}{M_k}\times\frac{1}{M_l}\right)$$

$$:= \sup_{(x,y)\in G^2}\ \sup_{(s,t)\in I_k\times I_l}|f(x-s,y-t)-f(x-s,y)-f(x,y-t)+f(x,y)|.$$

It is well known that there is a wide analogy between harmonic analysis on the bounded Vilenkin groups and the classical Fourier analysis. However, in the trigonometric case there is a class of functions such that their Fourier series are always convergent, and the convergence is uniform if the additional assumption of continuity of a function is made. An example of such class is the class of functions of bounded variation (BV) (see Jordan [15]).

The contributions of Wiener [21], Mercinkiewicz [17], Waterman [22], Chanturia [4], Kita and Yoneda [16], Akhobadze [2], Goginava [6] and their collaborators have shown that many of the results concerning the class of functions of bounded variation (BV) can be extended to more general classes. For Vilenkin system in one-dimensional case, the class of bounded fluctuation (BF) and the class of generalized bounded fluctuation (GBF) were introduced by Onneweer and Waterman [19].

In two-dimensional case, the class BV of functions of bounded variation was introduced and studied by Hardy [14]. An analogous result for double Walsh-Fourier series was obtained by Moricz [18]. Goginava [5] has proved that in Hardy's theorem there is no need to require the boundedness of mixed variation. In particular, in [5] it was proved that if f is a continuous function and has bounded partial variation, then its double trigonometric Fourier series converges uniformly on $[0, 2\pi]^2$ in the Pringsheim sense. An analogous result for double Walsh-Fourier series was established in [7]. Different classes of generalized bounded variation for functions of two-variables were studied by Golubov [12], [13], Akhobadze [3], and Goginava and Sahakian [8]-[11]. In the present paper, we partially develop the above mentioned analogy for two-dimensional bounded Vilenkin groups, concerned with uniform convergence of Fourier series.

To state the main results of this paper, we first need to introduce the classes of functions of two variables of bounded variation and of partial bounded variation. Define

$$O_1(f; M_k, y) := \sum_{\alpha=0}^{M_k-1} \omega_1\left(f; I_k + z_\alpha^{(k)}, y\right),$$

$$O_2(f; M_l, x) := \sum_{\beta=0}^{M_l-1} \omega_2\left(f; x, I_l + z_\beta^{(l)}\right),$$

and

$$O_{1,2}(f; M_k, M_l) := \sum_{\alpha=0}^{M_k-1} \sum_{\beta=0}^{M_l-1} \omega_{1,2}\left(f; \left(I_k + z_\alpha^{(k)}\right) \times \left(I_l + z_\beta^{(l)}\right)\right),$$

where

$$\omega_1\left(f, I_k + z_\alpha^{(k)}, y\right) := \sup_{x,x' \in I_k + z_\alpha^{(k)}} \left|f(x, y) - f(x', y)\right|,$$

$$\omega_2\left(f, x, I_l + z_\beta^{(l)}\right) := \sup_{y,y' \in I_l + z_\beta^{(l)}} \left|f(x, y) - f(x, y')\right|,$$

and

$$\omega_{1,2}\left(f, \left(I_k + z_\alpha^{(k)}\right) \times \left(I_l + z_\beta^{(l)}\right)\right)$$

$$:= \sup_{x,x' \in I_k + z_\alpha^{(k)}, y,y' \in I_l + z_\beta^{(l)}} \left|f(x, y) - f(x', y) - f(x, y') + f(x', y')\right|.$$

Definition 1.1. *We say that a function f is of Bounded Oscillation, and write $f \in BO\left(G^2\right)$, if it satisfies the following conditions:*

$$\sup_k O_1\left(f; M_k, 0\right) < \infty, \quad \sup_l O_2\left(f; M_l, 0\right) < \infty, \quad \sup_{k,l} O_{1,2}\left(f; M_k, M_l\right) < \infty. \tag{1.6}$$

We note that if $f \in BO\left(G^2\right)$, then $\sup_{y\in G}\sup_k O_1\left(f; M_k, y\right) < \infty$.

Indeed, in view of (1.6) and (??), we can write

$$\sup_{y\in G}\sup_k \sum_{\alpha=0}^{M_k-1} \omega_1\left(f, I_k + z_\alpha^{(k)}, y\right)$$

$$\leq \sup_{y\in G}\sup_k \sum_{\alpha=0}^{M_k-1} \sup_{x,x'\in I_k+z_\alpha^{(k)}} \left|f\left(x,y\right) - f\left(x,0\right) - f\left(x',y\right) + f\left(x',0\right)\right|$$

$$+\sup_k \sum_{\alpha=0}^{M_k-1} \sup_{x,x'\in I_k+z_\alpha^{(k)}} \left|f\left(x,0\right) - f\left(x',0\right)\right| < \infty.$$

Analogously, we can show that $\sup_{x\in G}\sup_l O_2\left(f; M_l, x\right) < \infty$.

Definition 1.2. *We say a bounded, measurable function f is of Partial Bounded Oscillation, and write $f \in PBO\left(G^2\right)$, if the following conditions hold:*

$$\sup_{y\in G}\sup_k O_1\left(f; M_k, y\right) < \infty, \quad \sup_{x\in G}\sup_l O_2\left(f; M_l, x\right) < \infty. \tag{1.7}$$

Define

$$\Delta_k^{(1)} f\left(x,y\right) := f\left(x - e_k, y\right) - f\left(x,y\right), \quad \Delta_l^{(2)} f\left(x,y\right) := f\left(x, y - e_l\right) - f\left(x,y\right), \tag{1.8}$$

and

$$\Delta_{k,l}^{(1,2)} f\left(x,y\right) := f\left(x - e_k, y - e_l\right) - f\left(x - e_k, y\right) - f\left(x, y - e_l\right) + f\left(x,y\right).$$

It is easy to see that

$$\left|\Delta_{k,l}^{(1,2)} f\left(x,y\right)\right| \leq \left|\Delta_k^{(1)} f\left(x,y\right)\right| + \left|\Delta_k^{(1)} f\left(x, y - e_l\right)\right|$$

and

$$\left|\Delta_{k,l}^{(1,2)} f\left(x,y\right)\right| \leq \left|\Delta_l^{(2)} f\left(x,y\right)\right| + \left|\Delta_l^{(2)} f\left(x - e_k, y\right)\right|.$$

2. Main results

In this section we state the main results of this paper.

Theorem 2.1. *Let $f \in C\left(G^2\right)$, and let the following conditions hold:*

(2.1)
$$\lim_{k\to\infty}\sum_{\alpha=1}^{M_k-1}\frac{1}{\alpha}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|=0,$$

(2.2)
$$\lim_{l\to\infty}\sum_{\beta=1}^{M_l-1}\frac{1}{\beta}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|=0,$$

(2.3)
$$\lim_{l,k\to\infty}\sum_{\alpha=1}^{M_k-1}\sum_{\beta=1}^{M_l-1}\frac{1}{\alpha}\frac{1}{\beta}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|=0$$

uniformly with respect to $(x,y)\in G^2$. Then the double Vilenkin-Fourier series of function f converges uniformly on G^2.

Theorem 2.2. *Let f be a continuous function on G^2 and $f\in PBO\left(G^2\right)$. Then the Fourier series of f converges uniformly on G^2.*

Corollary 2.1. *Let f be a continuous function on G^2 and $f\in BO\left(G^2\right)$. Then the Fourier series of f converges uniformly on G^2.*

3. Proof of Main Results

In this section we prove the main results of this paper, stated in Section 2.

Proof of Theorem 2.1. Let

$$n=\sum_{i=0}^{k}a_iM_i,\ \text{with}\ a_k\neq 0\ \text{and}\ \ 0\le a_i<m_i\ \text{for}\ \ 0\le i\le k,\ \text{and}\ n'=n-a_kM_k$$

and

$$m=\sum_{j=0}^{l}b_jM_j,\ \text{with}\ b_l\neq 0\ \text{and}\ \ 0\le b_j<m_j\ \text{for}\ 0\le j\le l,\ \text{and}\ \ m'=m-b_lM_l.$$

Then, in view of (1.3) and (1.4), we can write

(3.1)
$$S_{n,m}(f;x,y)-f(x,y)$$
$$=\int_{G^2}\left(f(x-s,y-t)-f(x,y)\right)D_n(s)\,D_m(t)\,d\mu(s)\,d\mu(t)$$
$$=\int_{G^2}\left(f(x-s,y-t)-f(x,y)\right)\left(1+\psi_{M_k}(s)+\ldots+\psi_{M_k}^{a_k-1}(s)\right)D_{M_k}(s)$$
$$\times\left(1+\psi_{M_l}(t)+\ldots+\psi_{M_l}^{b_l-1}(t)\right)D_{M_l}(t)\,d\mu(s)\,d\mu(t)$$

$$+\int_{G^2}(f(x-s,y-t)-f(x,y))\left(1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right)$$
$$\times D_{M_l}(t)\,\psi_{M_k}^{a_k}(s)\,D_{n'}(s)\,d\mu(s)\,d\mu(t)$$
$$+\int_{G^2}(f(x-s,y-t)-f(x,y))\left(1+\psi_{M_k}(s)+...+\psi_{M_k}^{a_k-1}(s)\right)$$
$$\times D_{M_k}(s)\,\psi_{M_l}^{b_l}(t)\,D_{m'}(t)\,d\mu(s)\,d\mu(t)$$
$$+\int_{G^2}(f(x-s,y-t)-f(x,y))\,\psi_{M_k}^{a_k}(s)\,D_{n'}(s)\,\psi_{M_l}^{b_l}(t)\,D_{m'}(t)\,d\mu(s)\,d\mu(t)$$
$$=: A_1+A_2+A_3+A_4.$$

From (1.2) we obtain

$$|A_1|\le M_kM_l\int_{I_k}\int_{I_l}|f(x-s,y-t)-f(x,y)| \tag{3.2}$$
$$\times\left|1+\psi_{M_k}(s)+...+\psi_{M_k}^{a_k-1}(s)\right|\left|1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right|d\mu(s)\,d\mu(t)$$
$$\le M_kM_l\frac{1}{M_k}\frac{1}{M_l}\left(\omega_1\left(f;\frac{1}{M_k}\right)+\omega_2\left(f;\frac{1}{M_l}\right)\right)a_k\,b_l$$
$$\le p^2\left(\omega_1\left(f;\frac{1}{M_k}\right)+\omega_2\left(f;\frac{1}{M_l}\right)\right)=o(1),$$

as $l,k\to\infty$ uniformly with respect to $(x,y)\in G^2$.

We observe that if $t\in I_k,\ 0\le\alpha<M_k$, then

$$D_{n'}\left(z_\alpha^{(k)}+t\right)=D_{n'}\left(z_\alpha^{(k)}\right).$$

Hence, in view of (1.1), we can write

$$A_2=\sum_{\alpha=0}^{M_k-1}\int_{I_k+z_\alpha^{(k)}}\int_G f(x-s,y-t)\left(1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right) \tag{3.3}$$
$$\times D_{M_l}(t)\,\psi_{M_k}^{a_k}(s)\,D_{n'}(s)\,d\mu(s)\,d\mu(t)$$
$$=\int_{I_k}\int_G\sum_{\alpha=0}^{M_k-1}f\left(x-z_\alpha^{(k)}-s,y-t\right)\left(1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right)$$
$$\times D_{M_l}(t)\,D_{n'}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}(s)\,d\mu(s,t)$$
$$=\int_{I_k}\int_G f(x-s,y-t)\left(1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right)$$
$$\times D_{M_l}(t)\,D_{n'}(0)\,\psi_{M_k}^{a_k}(0)\,\psi_{M_k}^{a_k}(s)\,d\mu(s,t)$$
$$+\int_{I_k}\int_G\sum_{\alpha=1}^{M_k-1}f\left(x-z_\alpha^{(k)}-s,y-t\right)\left(1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right)D_{M_l}(t)$$

$$\times D_{n'}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}(s)\,d\mu(s,t)=A_{21}+A_{22}.$$

It is clear that

$$\psi_{M_k}^{-a_k}(e_k)\,\psi_{M_k}^{a_k}(s)=e^{-\frac{2\pi i}{m_k}a_k}e^{\frac{2\pi i s_k}{m_k}a_k}=e^{\frac{2\pi i(s_k-1)}{m_k}a_k}=\psi_{M_k}^{a_k}(s-e_k)$$

and

$$0<c_1\leq\left|1-\psi_{M_k}^{-a_k}(e_k)\right|\leq 2.$$

We have

$$\psi_{M_k}^{-a_k}(e_k)\,A_{21}=\psi_{M_k}^{-a_k}(e_k)\int_{I_k}\int_G f(x-s,y-t)$$

$$\times\left(1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right)D_{M_l}(t)\,D_{n'}(0)\,\psi_{M_k}^{a_k}(s)\,d\mu(s,t)$$

$$=\int_{I_k}\int_G f(x-s,y-t)\left(1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right)$$

$$D_{M_l}(t)\,D_{n'}(0)\,\psi_{M_k}^{a_k}(s-e_k)\,d\mu(s)\,d\mu(t)$$

$$=\int_{I_k}\int_G f(x-s-e_k,y-t)\left(1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right)$$

$$\times D_{M_l}(t)\,D_{n'}(0)\,\psi_{M_k}^{a_k}(s)\,d\mu(s,t).$$

Hence, we have

$$\left|A_{21}-\psi_{M_k}^{-a_k}(e_k)\,A_{21}\right|\leq\int_{I_k}\int_G\left|\Delta_k^{(1)}f(x-s,y-t)\right.$$

$$\left.\times\left(1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right)D_{M_l}(t)\,D_{n'}(0)\,\psi_{M_k}^{a_k}(s)\right|d\mu(s,t)$$

$$\leq M_kM_l\frac{c}{M_k}\frac{1}{M_l}\omega_1\left(f,\frac{1}{M_k}\right)b_l\leq cp\omega_1\left(f,\frac{1}{M_k}\right). \tag{3.4}$$

Analogously, in view of (1.5) and (2.1), we can write

$$\left|A_{22}-\psi_{M_k}^{-a_k}(e_k)\,A_{22}\right|$$

$$\leq(p+1)\,M_k\int_{I_k}\int_G\left|\left(1+\psi_{M_l}(t)+...+\psi_{M_l}^{b_l-1}(t)\right)D_{M_l}(t)\right|$$

$$\times\sum_{\alpha=1}^{M_k-1}\frac{1}{\alpha}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)}-s,y\right)\psi_{M_k}^{a_k}(s)\right|d\mu(s,t)$$

$$\leq c\,(p+1)\,M_kM_l\frac{c}{M_k}\frac{1}{M_l}o(1)\,b_l=o(1). \tag{3.5}$$

Combining (3.4) and (3.5) we get

$$A_2=o(1) \tag{3.6}$$

as $k \to \infty$ uniformly with respect to $(x, y) \in G^2$. Analogously, by (2.2) we have

$$A_3 = o(1) \tag{3.7}$$

as $l \to \infty$ uniformly with respect to $(x, y) \in G^2$.

For A_4 we can write

$$A_4 = \sum_{\alpha=0}^{M_k-1} \sum_{\beta=0}^{M_l-1} \int_{I_l+z_\beta^{(l)}} \int_{I_k+z_\alpha^{(k)}} (f(x-s, y-t) - f(x, y)) \tag{3.8}$$

$$\times \psi_{M_k}^{a_k}(s) D_{n'}(s) \psi_{M_l}^{b_l}(t) D_{m'}(t) d\mu(s) d\mu(t)$$

$$= \sum_{\alpha=0}^{M_k-1} \sum_{\beta=0}^{M_l-1} \int_{I_l} \int_{I_k} f\left(x - z_\alpha^{(k)} - s, y - z_\beta^{(l)} - t\right)$$

$$\times D_{n'}\left(z_\alpha^{(k)}\right) \psi_{M_k}^{a_k}\left(z_\alpha^{(k)}\right) \psi_{M_k}^{a_k}(s) D_{m'}\left(z_\beta^{(l)}\right) \psi_{M_l}^{b_l}\left(z_\beta^{(l)}\right) \psi_{M_l}^{b_l}(t) d\mu(s,t)$$

$$= \int_{I_l} \int_{I_k} f(x-s, y-t) D_{n'}(0) \psi_{M_k}^{a_k}(s) D_{m'}(0) \psi_{M_l}^{b_l}(t) d\mu(s,t)$$

$$+ \int_{I_l} \int_{I_k} \sum_{\alpha=1}^{M_k-1} f\left(x - z_\alpha^{(k)} - s, y - t\right) D_{m'}(0) \psi_{M_l}^{b_l}(t) D_{n'}\left(z_\alpha^{(k)}\right) \psi_{M_k}^{a_k}(s) d\mu(s,t)$$

$$+ \int_{I_l} \int_{I_k} \sum_{\beta=1}^{M_l-1} f\left(x - s, y - z_\beta^{(l)} - t\right) D_{n'}(0) \psi_{M_k}^{a_k}(s) D_{m'}\left(z_\beta^{(l)}\right) \psi_{M_l}^{b_l}(t) d\mu(s,t)$$

$$+ \int_{I_l} \int_{I_k} \sum_{\alpha=1}^{M_k-1} \sum_{\beta=1}^{M_l-1} f\left(x - z_\alpha^{(k)} - s, y - z_\beta^{(l)} - t\right) D_{n'}\left(z_\alpha^{(k)}\right) \psi_{M_k}^{a_k}(s)$$

$$\times D_{m'}\left(z_\beta^{(l)}\right) \psi_{M_l}^{b_l}(t) d\mu(s) d\mu(t) = A_{41} + A_{42} + A_{43} + A_{44}.$$

We have

$$\left|A_{41} - \psi_{M_k}^{-a_k}(e_k) A_{41}\right|$$

$$\leq \int_{I_l} \int_{I_k} \left|\Delta_k^{(1)} f(x-s, y-t) D_{m'}(0) \psi_{M_l}^{b_l}(t) D_{n'}(0) \psi_{M_k}^{a_k}(s)\right| d\mu(s,t)$$

$$\leq M_k M_l \frac{c}{M_k} \frac{1}{M_l} \omega_1\left(f, \frac{1}{M_k}\right) = o(1) \tag{3.9}$$

as $k, l \to \infty$ uniformly with respect to $(x, y) \in G^2$.

Analogously, we get

$$\left|A_{42} - \psi_{M_k}^{-a_k}(e_k) A_{42}\right| \leq (p+1) M_k$$

$$\times \int_{I_l}\int_{I_k} \left| D_{m'}(0)\,\psi_{M_l}^{b_l}(t)\right| \sum_{\alpha=1}^{M_k-1} \frac{1}{\alpha}\left|\Delta_k^{(1)} f\left(x - z_\alpha^{(k)} - s, y - t\right)\psi_{M_k}^{a_k}(s)\right| d\mu(s,t)$$

(3.10) $$\leq (p+1)\,M_k M_l \frac{1}{M_k M_l} o(1) = o(1) \text{ as } k, l \to \infty.$$

(3.11) $$A_{43} = o(1)$$

as $k, l \to \infty$ uniformly with respect to $(x, y) \in G^2$.

For A_{44} we can write

$$\psi_{M_k}^{-a_k}(e_k)\,A_{44} = \int_{I_l}\int_{I_k} \sum_{\alpha=1}^{M_k-1}\sum_{\beta=1}^{M_l-1} f\left(x - z_\alpha^{(k)} - s - e_k, y - z_\beta^{(l)} - t\right)$$

$$\times D_{n'}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}(s)\,D_{m'}\left(z_\beta^{(l)}\right)\psi_{M_l}^{b_l}(t)\,d\mu(s,t),$$

$$\psi_{M_l}^{-b_l}(e_l)\,A_{44} = \int_{I_l}\int_{I_k} \sum_{\alpha=1}^{M_k-1}\sum_{\beta=1}^{M_l-1} f\left(x - z_\alpha^{(k)} - s, y - z_\beta^{(l)} - t - e_l\right)$$

$$\times D_{n'}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}(s)\,D_{m'}\left(z_\beta^{(l)}\right)\psi_{M_l}^{b_l}(t)\,d\mu(s)\,d\mu(t),$$

$$\psi_{M_k}^{-a_k}(e_k)\,\psi_{M_l}^{-b_l}(e_l)\,A_{44} = \int_{I_l}\int_{I_k} \sum_{\alpha=1}^{M_k-1}\sum_{\beta=1}^{M_l-1} f\left(x - z_\alpha^{(k)} - s - e_k, y - z_\beta^{(l)} - t - e_l\right)$$

$$\times D_{n'}\left(z_\alpha^{(k)}\right)\psi_{M_k}^{a_k}(s)\,D_{m'}\left(z_\beta^{(l)}\right)\psi_{M_l}^{b_l}(t)\,d\mu(s,t).$$

So, using (2.3), we obtain

(3.12) $$\left|A_{44} - \psi_{M_k}^{-a_k}(e_k)\,A_{44} - \psi_{M_l}^{-b_l}(e_l)\,A_{44} + \psi_{M_k}^{-a_k}(e_k)\,\psi_{M_l}^{-b_l}(e_l)\,A_{44}\right|$$

$$= \left|1 - \psi_{M_k}^{-a_k}(e_k)\right|\left|1 - \psi_{M_l}^{-b_l}(e_l)\right|\left|A_{44}\right|$$

$$\leq (p+1)^2\,M_k M_l \int_{I_l}\int_{I_k} \sum_{\alpha=1}^{M_k-1}\sum_{\beta=1}^{M_l-1} \frac{1}{\alpha}\frac{1}{\beta}$$

$$\times \left|\Delta_{k,l}^{(1,2)} f\left(x - z_\alpha^{(k)} - s, y - z_\beta^{(l)} - t\right)\psi_{M_k}^{a_k}(s)\,\psi_{M_l}^{b_l}(t)\right| d\mu(s)\,d\mu(t)$$

$$\leq (p+1)^2\,M_k M_l \frac{1}{M_k M_l} o(1) = o(1)$$

as $k, l \to \infty$ uniformly with respect to $(x, y) \in G^2$. From (3.8)-(3.12) we get

(3.13) $$A_4 = o(1)$$

as $k, l \to \infty$ uniformly with respect to $(x, y) \in G^2$.

Combining (3.1), (3.2), (3.6) and (3.13) we complete the proof of the theorem. □

Proof of Theorem 2.2. In view of Theorem 2.1, it is enough to prove that the conditions (2.1)-(2.3) are fulfilled. Let $\theta(M_k)$ and $\eta(M_l)$ be sequences of natural numbers tending to infinity and depending on M_k and M_l, respectively. Using (1.7) we can write

$$\sum_{\alpha=1}^{M_k-1}\frac{1}{\alpha}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|$$

$$=\sum_{\alpha=1}^{\theta(M_k)}\frac{1}{\alpha}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|+\sum_{\alpha=\theta(M_k)+1}^{M_k-1}\frac{1}{\alpha}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|$$

$$\leq\omega_1\left(f,\frac{1}{M_k}\right)\log\theta(M_k)+\frac{c}{\theta(M_k)+1}. \tag{3.14}$$

Next, we can choose $\theta(M_k)$ so that both terms on the last relation tend to 0 as $k\to\infty$ uniformly with respect to $(x,y)\in G^2$, and (2.1) follows.

Analogously, by using (??), we get

$$\lim_{l\to\infty}\sum_{\beta=1}^{M_l-1}\frac{1}{\beta}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|=0 \tag{3.15}$$

as $l\to\infty$ uniformly with respect to $(x,y)\in G^2$, and (2.2) follows. To verify (2.3), we write

$$B:=\sum_{\alpha=1}^{M_k-1}\sum_{\beta=1}^{M_l-1}\frac{1}{\alpha}\frac{1}{\beta}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|$$

$$=\sum_{s=0}^{k-1}\sum_{r=0}^{l-1}\sum_{\alpha=M_s}^{M_{s+1}-1}\sum_{\beta=M_r}^{M_{r+1}-1}\frac{1}{\alpha}\frac{1}{\beta}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|$$

$$\leq\sum_{s=0}^{k-1}\sum_{r=0}^{l-1}\frac{1}{M_s}\frac{1}{M_r}\left(\sum_{\alpha=M_s}^{M_{s+1}-1}\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{\alpha=M_s}^{M_{s+1}-1}\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|\right)^{\frac{1}{2}}$$

From (1.8) and (??) we get

$$\sum_{\alpha=M_s}^{M_{s+1}-1}\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|$$

$$\leq 2pM_r\sup_y\sum_{\alpha=M_s}^{M_{s+1}-1}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|$$

and

$$\sum_{\alpha=M_s}^{M_{s+1}-1}\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|$$
$$\leq 2pM_s\sup_x\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|.$$

Hence, we can write

$$B\leq 2p\sum_{s=0}^{k-1}\sum_{r=0}^{l-1}\frac{1}{(M_s)^{\frac12}}\frac{1}{(M_r)^{\frac12}}$$
$$\times\sup_x\left(\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|\right)^{\frac12}\sup_y\left(\sum_{\alpha=M_s}^{M_{s+1}-1}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|\right)^{\frac12}$$
$$=2p\sum_{s=0}^{k-1}\frac{1}{(M_s)^{\frac12}}\sup_y\left(\sum_{\alpha=M_s}^{M_{s+1}-1}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|\right)^{\frac12}$$
$$\times\sum_{r=0}^{l-1}\frac{1}{(M_r)^{\frac12}}\sup_x\left(\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|\right)^{\frac12}$$
$$=2p\left(\sum_{s=0}^{\theta(k)-1}+\sum_{s=\theta(k)}^{k-1}\right)\left(\frac{1}{(M_s)^{\frac12}}\sup_y\left(\sum_{\alpha=M_s}^{M_{s+1}-1}\left|\Delta_k^{(1)}f\left(x-z_\alpha^{(k)},y\right)\right|\right)^{\frac12}\right)$$
$$\times\left(\sum_{r=0}^{\eta(l)-1}+\sum_{r=\eta(l)}^{l-1}\right)\left(\frac{1}{(M_r)^{\frac12}}\sup_x\left(\sum_{\beta=M_r}^{M_{r+1}-1}\left|\Delta_l^{(2)}f\left(x,y-z_\beta^{(l)}\right)\right|\right)^{\frac12}\right)$$
$$\leq 2p^2\left(\sqrt[2]{\omega_1\left(f;\frac{1}{M_k}\right)}\theta(k)+\frac{c}{\left(M_{\theta(k)}\right)^{\frac12}}\right)$$
$$\times\left(\sqrt[2]{\omega_2\left(f;\frac{1}{M_l}\right)}\eta(l)+\frac{c}{\left(M_{\eta(l)}\right)^{\frac12}}\right)$$

since we can choose $\theta(k)$ and $\eta(l)$ such that $\theta(k),\eta(l)\to\infty$ as $k,l\to\infty$,

$$\sqrt[2]{\omega_1\left(f;\frac{1}{M_k}\right)}\theta(k)\to 0 \text{ as } k,l\to\infty \quad\text{and}\quad \sqrt[2]{\omega_2\left(f;\frac{1}{M_l}\right)}\eta(l)\to 0 \text{ as } k,l\to\infty.$$

Therefore, we have

$$\lim_{l,k\to\infty}\sum_{\alpha=1}^{M_k-1}\sum_{\beta=1}^{M_l-1}\frac{1}{\alpha}\frac{1}{\beta}\left|\Delta_{k,l}^{(1,2)}f\left(x-z_\alpha^{(k)},y-z_\beta^{(l)}\right)\right|=0, \tag{3.16}$$

as $l,k\to\infty$ uniformly with respect to $(x,y)\in G^2$, and (2.3) follows.

Combining (3.14)-(3.16) we complete the proof of the theorem. □

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