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ON THE EQUATION $f^n(z) + g^n(z) = e^{\alpha z+\beta}$

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Abstract. In this paper we describe the meromorphic solutions of the equations $f^n(z) + (f')^n(z) = e^{\alpha z+\beta}$ and $f^n(z) + f^n(z+c) = e^{\alpha z+\beta}$ $(c \neq 0)$ over the complex plane C for integers $n \geq 1$.

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1. Introduction

This paper is devoted to the description of meromorphic solutions for the following functional equation:

$$f^n(z) + g^n(z) = e^{\alpha z+\beta}, \tag{1.1}$$

where $g(z) = f'(z)$ or $g(z) = f(z+c)$ for $\alpha, \beta, c(\neq 0) \in \mathbb{C}$, when $n \geq 1$.

In particular, when $\alpha = \beta = 0$, then (1.1) is reduced to the following well known Fermat-type functional equation, initialed by Gross [8, 9] and Baker [1]:

$$f^n(z) + g^n(z) = 1. \tag{1.2}$$

Below, we summarize all the possible meromorphic solutions of equation (1.2) (see Theorem 2.3 in Han [10]).

Proposition 1.1. *The following assertions hold.*

(A) *For $n = 2$, the only nonconstant meromorphic solutions of equation (1.2) are the functions $f = \frac{2\omega}{1+\omega^2}$ and $g = \frac{1-\omega^2}{1+\omega^2}$ for a nonconstant meromorphic function ω.*

(B) *For $n = 3$, the only nonconstant meromorphic solutions of equation (1.2) are the functions $f = \frac{1}{2\wp(h)}\left(1 + \frac{\sqrt{3}}{3}\wp'(h)\right)$ and $g = \frac{\eta}{2\wp(h)}\left(1 - \frac{\sqrt{3}}{3}\wp'(h)\right)$ for a nonconstant entire function h and a cubic root η of unity, where $\wp$ denotes the Weierstrass $\wp$-function.*

(C) *For $n \geq 4$, there is no nonconstant meromorphic solution for equation (1.2).*

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In view of the transformation $\omega = \tan\left(\frac{h}{2}\right)$, where h is an entire function, we see that in the case **(A)** (for $n = 2$) the functions $f = \frac{2\omega}{1+\omega^2} = \sin(h)$ and $g = \frac{1-\omega^2}{1+\omega^2} = \cos(h)$ are the only entire solutions of equation (1.2). Moreover, the Weierstrass elliptic $\wp$-function $\wp(z)$ with periods ω_1 and ω_2 is defined to be

$$\wp(z;\omega_1,\omega_2) := \frac{1}{z^2} + \sum_{\mu,\nu\in\mathbf{Z};\mu^2+\nu^2\neq 0} \left\{ \frac{1}{(z+\mu\omega_1+\nu\omega_2)^2} - \frac{1}{(\mu\omega_1+\nu\omega_2)^2} \right\},$$

which is an even function and, with appropriately chosen ω_1 and ω_2, satisfies the equation:

$$(\wp')^2 = 4\wp^3 - 1. \tag{1.3}$$

For meromorphic solutions of partial differential equations similar to equation (1.1), we refer the reader to Li [11, 12], Chang and Li [4], Han [10], and the references therein.

In what follows, we assume the familiarity with the basics of Nevanlinna's theory of meromorphic functions in $\mathbf{C}$ (see [15]), such as the first and second main theorems, and the standard notation, such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$, and the counting functions $N(r, f)$ (counting multiplicity) and $\bar{N}(r, f)$ (ignoring multiplicity). By $S(r, f)$ we denote a quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, except possibly on a set of finite logarithmic measure, which is not necessarily the same at each occurrence.

2. Main results

We first consider meromorphic solutions of equation $f^n + (f')^n = \gamma^n$ for $n \geq 4$ and $\gamma \not\equiv 0$. According to Proposition 1.1, both $\frac{f}{\gamma}$ and $\frac{f'}{\gamma}$ are constant. Assume $f = c_1\gamma$ and $f' = c_2\gamma$ to see that $c_1\gamma' = c_2\gamma$ with $c_1^n + c_2^n = 1$. If $c_1 = 0$, then $f \equiv 0$ and hence $\gamma \equiv 0$. So, $c_1 \neq 0$. If $c_2 = 0$, then f is a constant and so is γ. When $c_1c_2 \neq 0$, then γ cannot have zeros and poles, and hence $\gamma^n(z) = e^{\alpha z+\beta}$ with $\alpha = n\frac{c_2}{c_1}$. This is another reason why in our study we are focusing on the function $e^{\alpha z+\beta}$.

Next, for $f^3 + (f')^3 = e^{\alpha z+\beta}$, the function f must be entire, and thus both $\frac{f}{\gamma}$ and $\frac{f'}{\gamma}$ are constant, so that the same conclusion holds as above. Now, for $f^2 + (f')^2 = e^{\alpha z+\beta}$, the function f must again be entire and, by Proposition 1.1, we have $f(z) = e^{\frac{\alpha z+\beta}{2}}\sin(h(z))$ and $f'(z) = e^{\frac{\alpha z+\beta}{2}}\cos(h(z))$, so that $\frac{\alpha}{2}\tan(h) \equiv 1 - h'$. Since $T(r, h') = O(T(r, h)) + S(r, h)$ as $r \to \infty$ and $\lim_{r\to\infty}\frac{T(r,\tan(h))}{T(r,h)} = +\infty$ (see Clunie [6, Theorem 2 (i)] that extends Pólya's result from [16]), we see that either $\alpha = 0$ and $h' = 1$, or h is a constant.

Summarizing the above discussions we can state the following result.

Theorem 2.1. *The meromorphic solutions f of the following differential equation:*

$$f^n(z) + (f')^n(z) = e^{\alpha z+\beta} \tag{2.1}$$

must be entire functions, and the following assertions hold.
(A) *For $n = 1$, the general solutions of (2.1) are $f(z) = \frac{e^{\alpha z+\beta}}{\alpha+1} + ae^{-z}$ for $\alpha \neq -1$ and $f(z) = ze^{-z+\beta} + ae^{-z}$.*
(B) *For $n = 2$, either $\alpha = 0$ and the general solutions of (2.1) are $f(z) = e^{\frac{\beta}{2}}\sin(z+b)$, or $f(z) = de^{\frac{\alpha z+\beta}{2}}$.*
(C) *For $n \geq 3$, the general solutions of (2.1) are $f(z) = de^{\frac{\alpha z+\beta}{n}}$.*
Here, $\alpha, \beta, a, b, d \in \mathbf{C}$ with $d^n\left(1 + \left(\frac{\alpha}{n}\right)^n\right) = 1$ for $n \geq 1$.

Note when $n \geq 2$, equation (2.1) may have no meromorphic solution for $\alpha = ne^{\frac{(2k+1)\pi i}{n}}$, $k = 0, 1, \ldots, n-1$. Also, for some related interesting results, we refer the reader to Li and Yang [18], and Li [13].

Now, consider the meromorphic solutions $f(z)$ of the following difference equation with $c \neq 0$:

$$f^n(z) + f^n(z+c) = e^{\alpha z+\beta}. \tag{2.2}$$

When $n \geq 1$, take $f(z) = c_1 e^{\frac{\alpha z+\beta}{n}}$ and $f(z+c) = c_2 e^{\frac{\alpha z+\beta}{n}}$ to see that $c_1 e^{\frac{\alpha c}{n}} = c_2$ with $c_1^n + c_2^n = 1$, inspired by the case (C) of Proposition 1.1. Note that $c_1 c_2 \neq 0$ and $f(z+c) = e^{\frac{\alpha c}{n}} f(z)$. As a result, all the *trivial* meromorphic solutions of (2.2) are the functions $f(z) = de^{\frac{\alpha z+\beta}{n}}$ with $d^n(1 + e^{\alpha c}) = 1$ for $n \geq 1$.

Next, we discuss the existence of nontrivial meromorphic solutions for (2.2) when $n = 3$. It should be noted that a similar yet simpler approach has been applied in Han and Lü [14].

Theorem 2.2. *There is no meromorphic solution of finite order for the difference equation:*

$$f^3(z) + f^3(z+c) = e^{\alpha z+\beta}. \tag{2.3}$$

Here, the order of f is defined to be $\rho(f) := \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r}$.

3. Proof of Theorem 2.2

By Proposition 1.1, one has

$$f(z) = \frac{1}{2}\frac{\left\{1 + \frac{\sqrt{3}}{3}\mathfrak{p}'(h(z))\right\}}{\mathfrak{p}(h(z))}e^{\frac{\alpha z+\beta}{3}} \text{ and } f(z+c) = \frac{\eta}{2}\frac{\left\{1 - \frac{\sqrt{3}}{3}\mathfrak{p}'(h(z))\right\}}{\mathfrak{p}(h(z))}e^{\frac{\alpha z+\beta}{3}}. \tag{3.1}$$

A routine computation leads to

$$\text{(3.2)} \qquad \frac{\eta\left\{1-\frac{\sqrt{3}}{3}\mathfrak{p}'(h(z))\right\}}{\mathfrak{p}(h(z))} = \frac{\left\{1+\frac{\sqrt{3}}{3}\mathfrak{p}'(h(z+c))\right\}}{\mathfrak{p}(h(z+c))} e^{\frac{\alpha c}{3}}.$$

Assuming $\rho(f) < \infty$, from (1.3) and the first equality in (3.1), we obtain

$$\text{(3.3)} \qquad \frac{3f^2(z)\mathfrak{p}^2(h(z))}{e^{\frac{2}{3}(\alpha z+\beta)}} - \frac{3f(z)\mathfrak{p}(h(z))}{e^{\frac{1}{3}(\alpha z+\beta)}} + 1 = \mathfrak{p}^3(h(z)).$$

Recall the estimate (2.7) from Bank and Langley [2], stating that

$$\text{(3.4)} \qquad T(r, \mathfrak{p}) = \frac{\pi}{A} r^2 (1 + o(1)) \quad \text{and} \quad \rho(\mathfrak{p}) = 2,$$

where A is the area of the parallelogram $\mathfrak{S}$ with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$.

Therefore, taking into account that $T(r, e^{\alpha z}) = \frac{|\alpha|}{\pi} r (1 + o(1))$, we can combine (3.3) and (3.4) to obtain

$$\text{(3.5)} \qquad T(r, \mathfrak{p}(h)) \le 2T(r, f) + \frac{2}{3}T(r, e^{\alpha z}) + O(1),$$

and hence $\rho(\mathfrak{p}(h)) < \infty$ as well. By Corollary 1.2 of Edrei and Fuchs [7] (see also Theorem 1 of Bergweiler [3] for a different and elegant proof), h must be a polynomial.

A side note here is that $T(r, \mathfrak{p}(h)) = O(r^{2l})$ for some positive integer $l \ge 1$.

Notice that when $\mathfrak{p}(z_0) = 0$, then by (1.3) we have $(\mathfrak{p}')^2(z_0) = -1$. Now, we denote by $\{z_j\}_{j=1}^{\infty}$ all the zeros of $\mathfrak{p}(z)$ that satisfy $|z_j| \to \infty$ as $j \to \infty$, and assume that $h(a_{j,k}) = z_j$ for $k = 1, 2, \ldots, \deg(h)$. Then, we have $(\mathfrak{p}')^2(h(a_{j,k})) = (\mathfrak{p}')^2(z_j) = -1$.

Suppose there is a subsequence of $\{a_{j,k}\}_{j=1}^{\infty}$ with respect to j such that $\mathfrak{p}(h(a_{j,k} + c)) = 0$. Denote this subsequence still by $\{a_{j,k}\}_{j=1}^{\infty}$ and, without loss of generality, fix the index k below. So, we have $(\mathfrak{p}')^2(h(a_{j,k} + c)) = -1$. Differentiate (3.2) and use substitution to obtain

$$\eta\{1-\frac{\sqrt{3}}{3}\mathfrak{p}'(h(a_{j,k}))\}\mathfrak{p}'(h(a_{j,k}+c))h'(a_{j,k}+c) = \{1+\frac{\sqrt{3}}{3}\mathfrak{p}'(h(a_{j,k}+c))\}\mathfrak{p}'(h(a_{j,k}))h'(a_{j,k})\, e^{\frac{\alpha c}{3}},$$

from which we observe that one and only one of the following situations can appear:

$$\begin{cases} \eta\left\{1-i\frac{\sqrt{3}}{3}\right\} h'(a_{j,k}+c) = \left\{1+i\frac{\sqrt{3}}{3}\right\} h'(a_{j,k})\, e^{\frac{\alpha c}{3}}, \\ \eta\, h'(a_{j,k}+c) = -\, h'(a_{j,k})\, e^{\frac{\alpha c}{3}}, \\ \eta\left\{1+i\frac{\sqrt{3}}{3}\right\} h'(a_{j,k}+c) = \left\{1-i\frac{\sqrt{3}}{3}\right\} h'(a_{j,k})\, e^{\frac{\alpha c}{3}}. \end{cases}$$

Taking into account that $h(z)$ and $h(z + c)$ are polynomials of the same leading coefficient, and there are infinitely many $a_{j,k}$'s with $|a_{j,k}| \to \infty$ as $j \to \infty$, we conclude that

$$
(3.6)\qquad \begin{cases} \eta\left\{1-i\frac{\sqrt{3}}{3}\right\}h'(z+c)=\left\{1+i\frac{\sqrt{3}}{3}\right\}h'(z)e^{\frac{\alpha c}{3}}, \\ \eta h'(z+c)=-h'(z)e^{\frac{\alpha c}{3}}, \\ \eta\left\{1+i\frac{\sqrt{3}}{3}\right\}h'(z+c)=\left\{1-i\frac{\sqrt{3}}{3}\right\}h'(z)e^{\frac{\alpha c}{3}}. \end{cases}
$$

This is possible only if α and c satisfy $e^{\frac{\alpha c}{3}}=-1,\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$ because $\eta=1,-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$.

When this is true, one has uniformly by (3.6) that $h(z)=az+b$ for $ac\neq 0$. Since $\mathfrak{p}(z)$ has two distinct zeros in $\mathfrak{S}$, and thus, in each associated lattice, we observe that all the zeros $\{z_j\}_{j=1}^{\infty}$ of $\mathfrak{p}(z)$ are transferred to each other through (a multiple of) ac. For simplicity, we can consider two cases where either $ac=\omega_1,\omega_2,\omega_1+\omega_2$, or $ac\neq\omega_1,\omega_2,\omega_1+\omega_2$ and $ac\in\mathfrak{S}$. The former cannot occur in view of (3.2) and the periodicity of $\mathfrak{p}(z)$ and $\mathfrak{p}'(z)$, and the latter cannot occur either because $\mathfrak{p}(z)$ has a unique double pole in each lattice. We substitute $z_\infty=-\frac{b}{a}$ into (3.2) to get a contradiction:

$$
\infty=\frac{\eta\left\{1-\frac{\sqrt{3}}{3}\mathfrak{p}'(0)\right\}}{\mathfrak{p}(0)}=\frac{\left\{1+\frac{\sqrt{3}}{3}\mathfrak{p}'(ac)\right\}}{\mathfrak{p}(ac)}e^{\frac{\alpha c}{3}}<\infty.
$$

Thus, $\mathfrak{p}(h(a_{j,k}+c))=0$ may occur only for finitely many $a_{j,k}$'s. Without loss of generality, assume that $\mathfrak{p}(h(a_{j,k}+c))\neq 0$ for each $k=1,2,\ldots,\deg(h)$ and all $j>J$, with J being a sufficiently large positive integer. Since $\mathfrak{p}(h(a_{j,k}))=0$ and $(\mathfrak{p}')^2(h(a_{j,k}))=-1$, by (3.2) we have $\mathfrak{p}(h(a_{j,k}+c))=\infty$ for $j>J$. Observing that $O(\log r)=S(r,\mathfrak{p}(h))$, we can write

$$
(3.7)\qquad \begin{aligned} N\left(r,\frac{1}{\mathfrak{p}(h(z))}\right)&\leq\bar{N}\left(r,\frac{1}{\mathfrak{p}(h(z))}\right)+2N\left(r,\frac{1}{h'(z)}\right)\\ &\leq\bar{N}(r,\mathfrak{p}(h(z+c)))+2T(r,h')+O(\log r)\leq\bar{N}(r,\mathfrak{p}(h(z+c)))+S(r,\mathfrak{p}(h)). \end{aligned}
$$

Now we use the first equality in (3.1) and estimate (3.4) to obtain

$$
(3.8)\qquad T(r,f)\leq T(r,\mathfrak{p}(h))+T(r,\mathfrak{p}'(h))+\frac{1}{3}T(r,e^{\alpha z})+O(1)\leq O(T(r,\mathfrak{p}(h))).
$$

Hence, in view of (3.5) and the side note after it, we have $\rho(f)=\rho(\mathfrak{p}(h))$ and $S(r,f)=S(r,\mathfrak{p}(h))$. Thus, $T(r,e^{\alpha z})=S(r,f)$. Since all the zeros of the functions $f-e^{\frac{\alpha z+\beta}{3}}$, $f-\eta e^{\frac{\alpha z+\beta}{3}}$ and $f-\eta^2e^{\frac{\alpha z+\beta}{3}}$ $(\eta\neq 1)$ are of multiplicities at least 3, from (2.3) and Yamanoi's second main theorem (see [17]), we obtain

$$
\begin{aligned} 2T(r,f)&\leq\sum_{m=1}^{3}\bar{N}\left(r,\frac{1}{f-\eta^m e^{\frac{\alpha z+\beta}{3}}}\right)+\bar{N}(r,f)+S(r,f)\\ &\leq\frac{1}{3}\sum_{m=1}^{3}N\left(r,\frac{1}{f-\eta^m e^{\frac{\alpha z+\beta}{3}}}\right)+N(r,f)+S(r,f)\leq 2T(r,f)+S(r,\mathfrak{p}(h)). \end{aligned}
$$

Therefore, we have $T(r, f) = N(r, f) + S(r, \mathfrak{p}(h))$, and hence $m(r, f) = S(r, \mathfrak{p}(h))$.

Next, applying the lemma of logarithmic derivative and again using the first equality in (3.1), we get

$$m\left(r, \frac{1}{\mathfrak{p}(h)}\right) \le m(r, f) + m\left(r, \frac{\mathfrak{p}'(h)h'}{\mathfrak{p}(h)}\right) + S(r, \mathfrak{p}(h)) = S(r, \mathfrak{p}(h)). \tag{3.9}$$

Finally, combining (3.7) and (3.9), and applying Theorem 2.1 from Chiang and Feng [5], we obtain

$$\begin{aligned} T(r, \mathfrak{p}(h)) + O(1) &= T\left(r, \frac{1}{\mathfrak{p}(h)}\right) = m\left(r, \frac{1}{\mathfrak{p}(h(z))}\right) + N\left(r, \frac{1}{\mathfrak{p}(h(z))}\right) \\ &\le \bar{N}(r, \mathfrak{p}(h(z+c))) + S(r, \mathfrak{p}(h)) \le \frac{1}{2}N(r, \mathfrak{p}(h(z+c))) + S(r, \mathfrak{p}(h)) \\ &\le \frac{1}{2}T(r, \mathfrak{p}(h(z+c))) + S(r, \mathfrak{p}(h)) \le \frac{1}{2}T(r, \mathfrak{p}(h)) + S(r, \mathfrak{p}(h)), \end{aligned} \tag{3.10}$$

which is a contradiction, and the result follows. □

4. Examples

Example 4.1. Let $f(z)$ be given by (3.1) through $h(z) = e^z$. Then $\rho(f) = \infty$, and for $c = \pi i$ and each α with $e^{\alpha c} = 1$, we have $f^3(z) + f^3(z+c) = e^{\alpha z+\beta}$ for all $\beta \in \mathbf{C}$.

Example 4.2. Let $f_1(z) = e^{\frac{\alpha z+\beta}{2}} \sin(z)$ and $f_2(z) = e^{\frac{\alpha z+\beta}{2}} \sin(e^{4iz}+z)$. Then $\rho(f_1) \le 1$ and $\rho(f_2) = \infty$. For $c = \frac{\pi}{2}$ and each α with $e^{\alpha c} = 1$, we have $f_j^2(z)+f_j^2(z+c) = e^{\alpha z+\beta}$ for $j = 1, 2$ and all $\beta \in \mathbf{C}$.

Example 4.3. Let $f_1(z) = e^z + \frac{e^{\alpha z+\beta}}{2}$ and $f_2(z) = e^{e^{2z}+z} + \frac{e^{\alpha z+\beta}}{2}$. Then $\rho(f_1) \le 1$ and $\rho(f_2) = \infty$. For $c = i\pi$ and each α with $e^{\alpha c} = 1$, we have $f_j(z)+f_j(z+c) = e^{\alpha z+\beta}$ for $j = 1, 2$ and all $\beta \in \mathbf{C}$.

In contrast to Theorem 2.1, even though the existence of finite or infinite order solutions of equation $f^2(z) + f^2(z+c) = e^{\alpha z+\beta}$ may be described for special α and c, we could not characterize systematically all the possible solutions of this difference equation. The same concern occurs for the existence of infinite order solutions of equation $f^3(z) + f^3(z+c) = e^{\alpha z+\beta}$ in a systematic manner.

Finally, we briefly consider the equation $f(z) + f(z+c) = e^{\alpha z+\beta}$. Recall (2.2) to choose $f(z) = de^{\alpha z+\beta}$ and $f(z+c) = e^{\alpha c}f(z)$. When $d(1+e^{\alpha c}) = 1$, we are done. The general solutions may be of the form $f(z) = \delta(z) + de^{\alpha z+\beta}$ for a meromorphic function $\delta(z)$ with $\delta(z+c) = -\delta(z)$ and $d(1+e^{\alpha c}) = 1$. In addition, the general solutions may be of the form $f(z) = \delta(z) - \frac{z}{c}e^{\alpha z+\beta}$ for $e^{\alpha c} = -1$. When $n \ge 2$, then equation (2.2) may have no solution if $e^{\alpha c} = -1$.

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