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SOME FORMULAS FOR THE GENERALIZED ANALYTIC FEYNMAN INTEGRALS ON THE WEINER SPACE

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Abstract. In this paper, we analyze the analytic Feynman integrals on the Wiener space. We define a new concept of analytic Feynman integral on the Wiener space, which is called the generalized analytic Feynman integral, to explain various physical circumstances. Furthermore, we evaluate the generalized analytic Feynman integrals for several important classes of functionals. We also establish various properties of these generalized analytic Feynman integrals. We conclude the paper by giving several applications involving the Cameron-Storvick theorem and quantum mechanics.

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1. Introduction

Let $C_0[0,T]$ denote the one-parameter Wiener space, that is, the space of continuous real-valued functions x on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m denote the Wiener measure. Observe that $(C_0[0,T], \mathcal{M}, m)$ is a complete measure space, and denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)dm(x).$$

Feynman [5] has introduced an integral over a space of paths, and used his integral in a formal way in his approach to quantum mechanics. Since then the notion of Feynman integral was developed and was applied in various theories. For the procedure of analytic continuation, to define the analytic Feynman integral, we refer the reader

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to [5], [12]- [15], [18, 20]. Many mathematicians have studied the analytic Feynman integrals of functionals in several classes of functionals (see, [1] – [4], [6, 8, 10, 16, 18, 21, 22]). The differential equation

$$\frac{\partial}{\partial t}\psi(u,t) = \frac{1}{2\lambda}\Delta\psi(u,t) - V(u)\psi(u,t) \tag{1.1}$$

is called a diffusion equation with initial condition $\psi(u,0) = \varphi(u)$, where Δ is the Laplacian and V is an appropriate potential function. Many mathematicians have considered the Wiener integral of functionals of the form $F(\lambda^{-\frac{1}{2}}x + u)$, where u is a real number. It is a well-known fact that the Wiener integral of the functional

$$\exp\Big\{-\int_0^T V(\lambda^{-\frac{1}{2}}x(t)+u)dt\Big\}\varphi(\lambda^{-\frac{1}{2}}x(T)+u) \tag{1.2}$$

gives solutions of the diffusion equation (1.1) by the Feynman-Kac formula. In the case where time is replaced by imaginary time, this diffusion equation becomes the Schrödinger equation:

$$i\frac{\partial}{\partial t}\psi(u,t) = -\frac{1}{2}\Delta\psi(u,t) + V(u)\psi(u,t) \tag{1.3}$$

with initial condition $\psi(u,0) = \varphi(u)$. Hence, a solution of Schrödinger equation (1.3) is obtained via an analytic Feynman integral. In particular, the authors found the solutions of the diffusion equation (1.1) and the Schrödinger equation (1.3) for the harmonic oscillator $V(u) = \frac{k}{2}u^2$ (for a more detailed study see [8, 23]). On the other hand, it is not easy to find the solutions of the diffusion equation (1.1) and the Schrödinger equation (1.3) with respect to nonharmonic oscillator.

In this paper we consider the following functional:

$$\exp\Big\{-\int_0^T V(\lambda^{-\frac{1}{2}}x(t)+h(t))dt\Big\}\varphi(\lambda^{-\frac{1}{2}}x(T)+h(t)), \tag{1.4}$$

where $h(t)$ is a continuous function on $[0,T]$. When $h(t)$ is a constant function, then the functional F in (1.4) reduces to that of in (1.2). That is, our functional (1.4) is more general than that of in (1.2). Therefore, the results and formulas for functional (1.2) will be special cases of the results and formulas obtained in this paper for functional (1.4).

2. Preliminaries and Definitions

A subset B of $C_0[0,T]$ is said to be scale-invariant measurable if ρB is $\mathcal{M}$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set if $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.) [11]. Throughout this paper we will assume that each functional $F : C_0[0,T] \to \mathbb{C}$ that we consider is scale-invariant measurable and that for each $\rho > 0$

$$\int_{C_0[0,T]} |F(\rho x)|dm(x) < \infty.$$

For $v \in L_2[0,T]$ and $x \in C_0[0,T]$, let $\langle v, x\rangle$ denote the Paley-Wiener-Zygmund (PWZ) stochastic integral. The following assertions hold:

(1) For each $v \in L_2[0,T]$, the PWZ stochastic integral $\langle v, x\rangle$ exists for a.e. $x \in C_0[0,T]$.

(2) If $v \in L_2[0,T]$ is a function of bounded variation on $[0,T]$, then $\langle v, x\rangle$ is equal to the Riemann-Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_0[0,T]$.

(3) The PWZ stochastic integral $\langle v, x\rangle$ has the expected linearity property.

(4) The PWZ stochastic integral $\langle v, x\rangle$ is a Gaussian process with mean 0 and variance $\|v\|_2^2$.

For a more detailed study of the PWZ stochastic integral see [7]– [10].

Now we define the analytic Feynman integral of functionals on Wiener space.

Definition 2.1. *Let $\mathbb{C}$ denote the set of complex numbers, $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$, and let $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } Re(\lambda) \geq 0\}$. Let $F : C_0[0,T] \to \mathbb{C}$ be a measurable functional such that for each $\lambda > 0$ the Wiener integral*

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2}x)dm(x)$$

exists. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$J^*(\lambda) = \int_{C_0[0,T]}^{\text{anw}_\lambda} F(x)dm(x).$$

Let $q \neq 0$ be a real number and let F be a functional such that $J^*(\lambda)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F

with parameter q, and write

$$\int_{C_0[0,T]}^{\mathrm{an}f_q} F(x)dm(x) = \lim_{\lambda \to -iq} \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)dm(x),$$

where $\lambda \to -iq$ through values in $\mathbb{C}_+$.

The following theorem provides a well-known integration formula which we will use several times in this paper.

Theorem 2.1. *Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal set of functions in L^2, and let $f : \mathbb{R}^n \to \mathbb{C}$ be Borel measurable. Let $|\vec{v}| = \sqrt{v_1^2 + \cdots + v_n^2}$, and let*

$$F(x) = f(\langle \alpha_1, x\rangle, \cdots, \langle \alpha_n, x\rangle) \equiv f(\langle \vec{\alpha}, x\rangle).$$

Then

$$\int_{C_0[0,T]} F(x)dm(x) = \int_{C_0[0,T]} f(\langle \vec{\alpha}, x\rangle)dm(x) \tag{2.1}$$
$$\doteq \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{v}) \exp\left\{ -\frac{|\vec{v}|^2}{2}\right\} d\vec{v}$$

in the sense that if either side of (2.1) exists, then both sides exist and the equality holds.

3. An analogue of the analytic Feynman integral

Now we explain the importance of the functionals given by equation (1.4). For a constant k, when the potential function is given by $V(u) = \frac{k}{2}u^2$, then the equation (1.1) is called a diffusion equation for harmonic oscillator with potential V. For $\xi \in \mathbb{R}$, the function

$$V_1(u) \equiv V(u+\xi) = \frac{k}{2}(u+\xi)^2$$

is the translation of V, and so, the equation (1.1) is called a diffusion equation for harmonic oscillator with potential V_1. However, for an appropriate function $h(t)$ on $[0, T]$, the function

$$V_2(u) \equiv V(u + h(u)) = \frac{k}{2}(u + h(u))^2$$

might be a nonharmonic oscillator.

Example 3.1. Let $h(u) = u^2$ defined on $[0, T]$. Then

$$V_3(u) = \frac{k}{2}(u^2 + 2u^3 + u^4).$$

In this case, the equation (1.1) is called a diffusion equation for a nonharmonic oscillator with potential V_3 because it contains the "u^3"term. The above facts show that in certain physical circumstances the status of the harmonic oscillator can be exchanged by the status of the nonharmonic oscillator, which can be explained by studying the Wiener integral of the functional given by (1.4).

Example 3.2. For $\gamma \in \mathbb{R}$ let $h(u) = -u + \sqrt{u^2(u^2-\gamma^2)}$ defined on $[0,T]$. Then

$$V_4(u) = \frac{k}{2}u^2(u^2-\gamma^2).$$

In this case, the equation (1.1) is called a diffusion equation for double-well potential V_4. Thus, the functionals considered in this paper are more useful in applications than the functionals considered in the earlier papers [1] – [4], [6, 8, 10, 12, 23].

Now we are ready to state the definition of a generalized analytic Feynman integral.

Definition 3.1. *Let $h \in C_0[0,T]$ be given, and let $F : C_0[0,T] \longrightarrow \mathbb{C}$ be such that the function space integral*

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)dm(x)$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the modified analytic function space integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$J^*(\lambda) = \int_{C_0[0,T]}^{an_\lambda^h} F(x)dm(x).$$

Let $q \neq 0$ be a real number and let F be a functional such that the integral $\int_{C_0[0,T]}^{an_\lambda^h} F(x)dm(x)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it a modified generalized analytic Feynman integral of F with parameter q and we write

$$\int_{C_0[0,T]}^{anf_q^h} F(x)dm(x) = \lim_{\lambda \to -iq} \int_{C_0[0,T]}^{an_\lambda^h} F(x)dm(x),$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$.

Remark 3.1. If $h(t) \equiv 0$ on $[0,T]$, then we can write

$$\int_{C_0[0,T]}^{an_\lambda^h} F(x)dm(x) = \int_{C_0[0,T]}^{anw_\lambda} F(x)dm(x)$$

and

$$\int_{C_0[0,T]}^{\text{anf}_q^h} F(x)dm(x) = \int_{C_0[0,T]}^{\text{anf}_q} F(x)dm(x).$$

4. Examples involving generalized analytic Feynman integrals

In this section we establish the existence of the generalized analytic Feynman integrals for several classes of functionals. Let $M(L_2[0,T])$ be the class of all complex valued countably additive Borel measures f on $L_2[0,T]$.

4.1. The Banach algebra $\mathcal{S}$. Let $\mathcal{S}$ be the class of functionals of the form:

$$F(x) = \int_{L_2[0,T]} \exp\{i\langle v, x\rangle\} df(v) \tag{4.1}$$

for s-a.e. $x \in C_0[0,T]$ for some $f \in M(L_2[0,T])$. One can show that $\mathcal{S}$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{L_2[0,T]} |df(v)|.$$

Example 4.1. Let $h(t) = \int_0^t z_h(s)ds$ for some $z_h \in L_2[0,T]$ and let $F \in \mathcal{S}$ be given by equation (4.1). Then for all $\lambda > 0$, we have

$$\begin{aligned} &\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)dm(x) \\ &= \int_{C_0[0,T]} \int_{L_2[0,T]} \exp\{i\langle v, \lambda^{-\frac{1}{2}}x + h\rangle\} df(v)dm(x) \\ &= \int_{L_2[0,T]} \exp\Big\{-\frac{1}{2\lambda}\|v\|_2 + i(v, z_h)_2\Big\} df(v). \end{aligned} \tag{4.2}$$

But the expression above can be extended to the open right-hand plane $\lambda = p - iq$ with $p > 0$. Then letting $p \to 0$ we obtain that

$$\int_{C_0[0,T]}^{\text{anf}_q^h} F(x)dm(x) = \int_{L_2[0,T]} \exp\Big\{-\frac{i}{2q}\|v\|_2 + i(v, z_h)_2\Big\} df(v) \tag{4.3}$$

and that

$$\left|\int_{C_0[0,T]}^{\text{anf}_q^h} F(x)dm(x)\right| \leq \|f\| < +\infty$$

for all $q \in \mathbb{R} - \{0\}$.

4.2. The class $\mathcal{A}_n^{(p)}$. Let $\mathcal{A}_n^{(p)}$ be the class of all functionals of the form:

$$F(x) = f(\langle\alpha_1, x\rangle, \cdots, \langle\alpha_n, x\rangle) \equiv f(\langle\vec{\alpha}, x\rangle),$$

where $f \in L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$ and $\{\alpha_1, \cdots, \alpha_n\}$ is an orthonormal set in $L_2[0,T]$.

Example 4.2. Let $h(t) = \int_0^t z_h(s)ds$ for some $z_h \in L_2[0,T]$ and let $F \in \mathcal{A}_n^{(p)}$. Then for all $q \in \mathbb{R} - \{0\}$, the generalized analytic Feynman integral of F exists and is given by formula

$$\int_{C_0[0,T]}^{\mathrm{anf}_q^h} F(x)dm(x) = \left(\frac{-iq}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{\frac{iq}{2}\sum_{j=1}^{n}(u_j - (\alpha_j, z_h)_2)^2\right\} d\vec{u}. \tag{4.4}$$

Furthermore, we have

$$\left|\int_{C_0[0,T]}^{\mathrm{anf}_q^h} F(x)dm(x)\right| \leq \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} |f(\vec{u})|d\vec{u} \leq \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \|f\|_1 < +\infty.$$

4.3. The class of Fourier-type functionals. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of infinitely differentiable functions $f(\vec{u})$ together with all their derivatives each of which decays at infinity faster than any polynomial of $|\vec{u}|^{-1}$. Let $\widehat{f}$ be the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$, that is,

$$\widehat{f}(\vec{\xi}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\{i\vec{u}\cdot\vec{\xi}\} d\vec{u}, \tag{4.5}$$

where $\vec{u}$ and $\vec{\xi}$ are in $\mathbb{R}^n$ and $\vec{u}\cdot\vec{\xi} = u_1\xi_1 + \cdots + u_n\xi_n$.

Note that the Fourier transform is an isomorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In addition, $\Delta^k f$ and $\widehat{\Delta^k f}$ are elements of $\mathcal{S}(\mathbb{R}^n)$ for all $k = 1, 2, \cdots$, where Δ denotes the Laplacian.

Next, following [9], we introduce the Fourier-type functionals. Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal set of functions in L^2. For $f \in \mathcal{S}(\mathbb{R}^n)$, we set

$$\Delta^k F(x) = (\Delta^k f)(\langle\vec{\alpha}, x\rangle), \quad k = 0, 1, \cdots \tag{4.6}$$

and

$$\widehat{\Delta^k F}(x) = \widehat{\Delta^k f}(\langle\vec{\alpha}, x\rangle), \quad k = 0, 1, \cdots. \tag{4.7}$$

The functionals in (4.6) and (4.7) are called Fourier-type functionals defined on the Wiener space $C_0[0,T]$.

Example 4.3. Let $\widehat{\Delta^k F}$ be as in (4.7), and let $h(t) = \int_0^t z_h(s)ds$ for some $z_h \in L_2[0,T]$. Then it is not hard to show that for all $q \neq 0$, the generalized analytic Feynman integral of $\widehat{\Delta^k F}$ exists and is given by the formula:

$$(4.8) \quad \int_{C_0[0,T]}^{\mathrm{an}f_q^h} \widehat{\Delta^k F}(x)dm(x) \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} (\Delta^k f)(\vec{v}) \exp\Big\{ -\frac{i|\vec{v}|^2}{2q} + i\vec{v}\cdot(\vec{\alpha}, z_h)_2 \Big\} d\vec{v}$$

for each $k = 0, 1, \cdots$, and hence

$$\Big| \int_{C_0[0,T]}^{\mathrm{an}f_q^h} \widehat{\Delta^k F}(x)dm(x) \Big| \leq \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} |(\Delta^k f)(\vec{v})| d\vec{v} < +\infty.$$

5. Properties of generalized analytic Feynman integrals

The following lemma is useful in establishing various relationships among generalized analytic Feynman integrals.

Lemma 5.1. *(1)* (Translation theorem). *Let F be a Wiener integrable functional, and let $x_0(t) = \int_0^t z_0(s)ds$ for some $z_0 \in L_2[0,T]$. Then*

$$(5.1) \quad \int_{C_0[0,T]} F(x + x_0)dm(x) = \exp\Big\{ -\frac{1}{2}\|z_0\|_2^2 \Big\} \int_{C_0[0,T]} F(x)\exp\{(z_0, x)\}dm(x).$$

(2) (Fubini theorem for Wiener integrals). *Let F be a Wiener integrable functional on $C_0[0,T]$. Then for all non-zero real numbers p_1 and p_2,*

$$\begin{aligned}
&\int_{C_0[0,T]} \Big(\int_{C_0[0,T]} F(p_1x_1 + p_2x_2)dm(x_1) \Big) dm(x_2) \\
(5.2) \quad &= \int_{C_0[0,T]} F(\sqrt{p_1^2 + p_2^2}x)dm(x) \\
&= \int_{C_0[0,T]} \Big(\int_{C_0[0,T]} F(p_1x_1 + p_2x_2)dm(x_2) \Big) dm(x_1).
\end{aligned}$$

In Theorem 5.1 below, we list several relationships in a table format.

Theorem 5.1. *Let F be as in Lemma 5.1. Let $h_j(t) = \int_0^t z_j(s)ds$ for some $z_j \in L_2[0,T], j = 1,2,3$, and let $H_q(x) = F(x)\exp\{(-iq)\langle z_3, x\rangle\}$ for $q \in \mathbb{R} - \{0\}$. Then for all non-zero real numbers q_1 and q_2 with $q_1 + q_2 \neq 0$, we have the following relationships:*

1. Commutative:

$$\int_{C_0[0,T]}^{\mathrm{an}f_{q_2}^{h_2}} \Big(\int_{C_0[0,T]}^{\mathrm{an}f_{q_1}^{h_1}} F(x+y)dm(x) \Big) dm(y) \doteq \int_{C_0[0,T]}^{\mathrm{an}f_{q_1}^{h_1}} \Big(\int_{C_0[0,T]}^{\mathrm{an}f_{q_2}^{h_2}} F(x+y)dm(y) \Big) dm(x).$$

2. Fubini theorem: $\int_{C_0[0,T]}^{anf_{q_2}^{h_2}}\left(\int_{C_0[0,T]}^{anf_{q_1}^{h_1}}F(x+y)dm(x)\right)dm(y) \doteq \int_{C_0[0,T]}^{anf_{\frac{q_1q_2}{q_1+q_2}}^{h_1+h_2}} F(z)dm(z)$.

3. Translation theorem: $\int_{C_0[0,T]}^{anf_q^{h_3}} F(x)dm(x) \doteq \exp\left\{\frac{iq}{2}\|z_3\|_2^2\right\}\int_{C_0[0,T]}^{anf_q^{h_3}} H_q(x)dm(x)$.

4. Integration formula: $\int_{C_0[0,T]}^{anf_{-q}^{-h_j}}\int_{C_0[0,T]}^{anf_q^{h_j}} F(x+y)dm(x)dm(y) = 0$.

Proof of Relationship 1:

First, using the symmetric property, for all $\lambda, \beta > 0$, we have

$$\int_{C_0^2[0,T]} F(\lambda^{-\frac{1}{2}}x+\beta^{-\frac{1}{2}}y+h_1+h_2)d(m\times m)(x,y)$$
$$\doteq \int_{C_0^2[0,T]} F(\beta^{-\frac{1}{2}}y+\lambda^{-\frac{1}{2}}x+h_2+h_1)d(m\times m)(y,x).$$

It can be analytically continued in λ and β for (λ,β), and so we have for all $(\lambda,\beta) \in \mathbb{C}_+ \times \mathbb{C}_+$,

$$\int_{C_0[0,T]}^{an_\beta^{h_2}}\left(\int_{C_0[0,T]}^{an_\lambda^{h_1}} F(x+y)dm(x)\right)dm(y)$$
$$\doteq \int_{C_0[0,T]}^{an_\lambda^{h_1}}\left(\int_{C_0[0,T]}^{an_\beta^{h_2}} F(x+y)d\mu(y)\right)d\mu(x). \tag{5.3}$$

Next, let E be a subset of $\tilde{\mathbb{C}}_+ \times \tilde{\mathbb{C}}_+$ containing the point $(-iq_1, -iq_2)$ and be such that $(\lambda,\beta) \in E$ implies that $\lambda+\beta \neq 0$. Then the function

$$\mathcal{H}(\lambda,\beta) \equiv \int_{C_0[0,T]}^{an_\beta^{h_2}}\left(\int_{C_0[0,T]}^{an_\lambda^{h_1}} F(y+z)dm(y)\right)dm(z)$$

is continuous on E and is uniformly continuous on E provided that E is compact. By the continuity of $\mathcal{H}$ and equation (5.3), the Relationship 1 follows.

Proof of Relationship 2:

Using equation (5.2), it follows that for $\lambda > 0$ and $\beta > 0$,

$$\int_{C_0^2[0,T]} F(\lambda^{-\frac{1}{2}}x+\beta^{-\frac{1}{2}}y+h_1+h_2)d(m\times m)(x,y)$$
$$\doteq \int_{C_0[0,T]} F(\sqrt{\lambda^{-1}+\beta^{-1}}z+h_1+h_2)dm(z).$$

This last expression is defined for $\lambda > 0$ and $\beta > 0$. For $\beta > 0$ it can be analytically continued in $\lambda \in \mathbb{C}_+$. Also, for $\lambda > 0$ it can be analytically continued in $\beta \in \mathbb{C}_+$. Therefore $\lambda \in \mathbb{C}_+, \beta \in \mathbb{C}_+$ implies that $\frac{\lambda\beta}{\lambda+\beta} \in \mathbb{C}_+$, and hence it can be analytically continued into $\mathbb{C}_+$ to equal the generalized analytic Wiener integral:

$$\int_{C_0[0,T]}^{an_\gamma^{h_1+h_2}} F(z)dm(z), \tag{5.4}$$

where $\gamma = \frac{\lambda\beta}{\lambda+\beta}$. Next, note that for all $q_1, q_2 \in \mathbb{R} - \{0\}$ with $q_1 + q_2 \neq 0$, if $\lambda \to -iq_1$ and $\beta \to -iq_2$, then $\frac{\lambda\beta}{\lambda+\beta} \to -i\frac{q_1 q_2}{q_1+q_2}$. Now, using this fact and equation (5.4), we can write

$$\int_{C_0[0,T]}^{anf_{q_2}^{h_2}} \left(\int_{C_{a,b}[0,T]}^{anf_{q_1}^{h_1}} F(x+y)dm(x) \right) dm(y)$$

$$\doteq \lim_{\beta \to -iq_2} \lim_{\lambda \to -iq_1} \int_{C_0[0,T]}^{an_\gamma^{h_1+h_2}} F(z)dm(z)$$

$$\doteq \lim_{\frac{\lambda\beta}{\lambda+\beta} \to -i\frac{q_1q_2}{q_1+q_2}} \int_{C_0[0,T]}^{an_\gamma^{h_1+h_2}} F(z)dm(z)$$

$$\doteq \int_{C_0[0,T]}^{anf_{\frac{q_1q_2}{q_1+q_2}}^{h_1+h_2}} F(z)dm(z),$$

which completes the proof of Relationship 2.

Proof of Relationship 3:

Using equation (5.1) with $G_\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$ (instead of F) and $x_0(t) = \lambda^{\frac{1}{2}}h_3(t)$, we can write

$$\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h_3)dm(x) \doteq \int_{C_0[0,T]} G_\lambda(x + x_0)dm(x)$$

$$\doteq \exp\left\{-\frac{\lambda}{2}\|z_3\|_2^2\right\} \int_{C_0[0,T]} G_\lambda(x)\exp\{\lambda^{\frac{1}{2}}\langle z_3, x\rangle\}$$

$$\doteq \exp\left\{-\frac{\lambda}{2}\|z_3\|_2^2\right\} \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x)\exp\{\lambda^{\frac{1}{2}}\langle z_3, \rangle\}$$

$$\doteq \exp\left\{-\frac{\lambda}{2}\|z_3\|_2^2\right\} \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x)\exp\{\lambda\langle z_3, \lambda^{-\frac{1}{2}}x\rangle\}.$$

It can be analytically continued in $\lambda \in \mathbb{C}_+$, and hence we have established Relationship 3 as $\lambda \to -iq$.

Proof of Relationship 4:

In view of equation (5.2), it follows that for all nonzero real numbers γ and β,

$$\int_{C_0[0,T]} \int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + \beta^{-\frac{1}{2}}y + h_j - h_j)dm(x_1)dm(x_2)$$

$$= \int_{C_0[0,T]} F(\sqrt{\lambda^{-1} + \beta^{-1}}x)dm(x).$$

Let $\lambda \to -iq$ and $\beta \to -i(-q) = iq$. Then $\lambda^{-1} + \beta^{-1} \to 0$, and hence Relationship 4 follows.

6. Application to the Cameron-Storvick theorem

In our first application, we establish the generalized Cameron-Storvick theorem for the generalized analytic Feynman integral. To do this, we need to define the concept of first variation of functionals on $C_0[0,T]$.

Definition 6.1. *Let F be a functional defined on $C_0[0,T]$. Then the first variation of F is defined by the formula:*

$$\delta F(x|u) = \frac{\partial}{\partial k} F(x+kw)\bigg|_{k=0}, \quad x, w \in C_0[0,T], \tag{6.1}$$

if it exists.

Now we are ready to state the generalized Cameron-Storvick theorem for the generalized analytic Feynman integrals.

Theorem 6.1. (Generalized Cameron-Storvick theorem). *Let F be an m-integrable functional on $C_0[0,T]$ such that*

$$\sup_{|k| \le \eta} |\delta F(x+h|w)|$$

is an m-integrable functional on $C_0[0,T]$, and let $w(t) = \int_0^t z_w(s)ds$ for some $z_w \in L_2[0,T]$. Then

$$\begin{aligned} \int_{C_0[0,T]}^{\text{anf}_q^h} \delta F(x|w)dm(x) &\doteq \int_{C_0[0,T]}^{\text{anf}_q^h} F(x)dm(x) \\ &+ iq(z_w, h)_2 \int_{C_0[0,T]}^{\text{anf}_q^h} F(x)dm(x) - iq \int_{C_0[0,T]}^{\text{anf}_q^h} \langle z_w, x\rangle F(x)dm(x). \end{aligned} \tag{6.2}$$

Proof of Theorem 6.1: First, let $F_h(x) = F(x+h)$ and $G_\lambda(x) = F_h(\lambda^{-\frac{1}{2}}x)$. Then for $\lambda > 0$, we obtain that

$$\begin{aligned}\int_{C_0[0,T]} \delta F(\lambda^{-\frac{1}{2}}x + h|w)dm(x) &= \frac{\partial}{\partial k}\left[\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h + kw)dm(x)\right]\Big|_{k=0} \\ &= \frac{\partial}{\partial k}\left[\int_{C_0[0,T]} F_h(\lambda^{-\frac{1}{2}}x + kw)dm(x)\right]\Big|_{k=0} \\ &= \frac{\partial}{\partial k}\left[\int_{C_0[0,T]} F_h(\lambda^{-\frac{1}{2}}(x + \lambda^{\frac{1}{2}}kw))dm(x)\right]\Big|_{k=0} \\ &= \frac{\partial}{\partial k}\left[\int_{C_0[0,T]} G_\lambda(x + x_0)dm(x)\right]\Big|_{k=0},\end{aligned}$$

where $x_0(t) = \lambda^{\frac{1}{2}}kw(t) = \int_0^t \lambda^{\frac{1}{2}}kz_w(s)ds$. Now applying the translation theorem for functional G_λ, we get

$$\begin{aligned}&\int_{C_0[0,T]} \delta F(\lambda^{-\frac{1}{2}}x + h|w)dm(x) \\ &= \frac{\partial}{\partial k}\left[\exp\left\{-\frac{\lambda k^2}{2}\|z_w\|_2^2\right\}\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)\exp\{\lambda^{\frac{1}{2}}k(z_w, x)\}dm(x)\right]\Big|_{k=0} \\ &= \frac{\partial}{\partial k}\left[\exp\left\{-\frac{\lambda k^2}{2}\|z_w\|_2^2\right\}\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)\right. \\ &\qquad \left.\cdot \exp\{\lambda k\langle z_w, \lambda^{-\frac{1}{2}}x + h\rangle - \lambda k\langle z_w, h\rangle\}dm(x)\right]\Big|_{k=0}. \end{aligned} \tag{6.3}$$

The last expression in (6.3) can be decomposed into three terms

$$\begin{aligned}\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)dm(x) - \lambda(z_w, h)_2\int_{C_0[0,T]} F(\lambda^{-\frac{1}{2}}x + h)dm(x) \\ + \lambda\int_{C_0[0,T]} \langle z_w, \lambda^{-\frac{1}{2}}x + h\rangle F(\lambda^{-\frac{1}{2}}x + h)dm(x).\end{aligned}$$

It can be analytically continued in $\lambda \in \mathbb{C}_+$, and hence, we have

$$\int_{C_0[0,T]}^{\mathrm{an}_\lambda^h} \delta F(x|w)dm(x) \doteq \int_{C_0[0,T]}^{\mathrm{an}_\lambda^h} F(x)dm(x)$$

$$-\lambda(z_w, h)_2\int_{C_0[0,T]}^{\mathrm{an}_\lambda^h} F(x)dm(x) + \lambda\int_{C_0[0,T]}^{\mathrm{an}_\lambda^h} \langle z_w, x\rangle F(x)dm(x).$$

Passing to the limit as $\lambda \to -iq$, we obtain the desired equation.

From Theorem 6.1 we have the following corollary, which is known as ordinary Cameron-Storvick theorem for the analytic Feynman integral.

Corollary 6.1. *Let $h(t) \equiv 0$ on $[0,T]$ and $(z_w, h)_2 = 0$. Then for the analytic Feynman integral we have*

$$\int_{C_0[0,T]}^{\operatorname{anf}_q} \delta F(x|w)dm(x) \doteq \int_{C_0[0,T]}^{\operatorname{anf}_q} F(x)dm(x) - iq\int_{C_0[0,T]}^{\operatorname{anf}_q} \langle z_w, x\rangle F(x)dm(x).$$

We conclude this section by giving two relationships concerning generalized analytic Feynman integrals. From Theorems 5.1 and 6.1 we have the following relationships, which we state without any conditions.

Relationship R1: (Cameron-Storvick theorem, and Relationships 1 and 2 from Theorem 5.1).

$$\int_{C_0[0,T]}^{\operatorname{anf}_{q_2}^{h_2}} \left(\int_{C_0[0,T]}^{\operatorname{anf}_{q_1}^{h_1}} \delta F(x+y|w)dm(x)\right)dm(y) \doteq \int_{C_0[0,T]}^{\operatorname{anf}_{\frac{q_1q_2}{q_1+q_2}}^{h_1+h_2}} \delta F(z|w)dm(z)$$

$$\doteq \int_{C_0[0,T]}^{\operatorname{anf}_{\frac{q_1q_2}{q_1+q_2}}^{h_1+h_2}} F(z)dm(z) + i\frac{q_1q_2}{q_1+q_2}(z_w, h_1+h_2)_2 \int_{C_0[0,T]}^{\operatorname{anf}_{\frac{q_1q_2}{q_1+q_2}}^{h_1+h_2}} F(z)dm(z)$$

$$- i\frac{q_1q_2}{q_1+q_2}\int_{C_0[0,T]}^{\operatorname{anf}_{\frac{q_1q_2}{q_1+q_2}}^{h_1+h_2}} \langle z_w, z\rangle F(z)dm(z).$$

To state the next relationship, we first give some observations. Let F and G be functionals on $C_0[0,T]$, and let H_q be as in Theorem 5.1. Then for all $x, w \in C_0[0,T]$ we have $\delta(FG)(x|w) = \delta F(x|w)G(x) + F(x)\delta G(x|w)$, provided that it exists. Also, note that $\delta H(x|w) = (-iq)\langle z_3, x\rangle H(x)$, where $H(x) = \exp\{(-iq)\langle z_3, x\rangle\}$. Hence we have

$$(6.4)\quad \delta(H_q)(x|w) = \delta F(x|w)\exp\{(-iq)\langle z_3, x\rangle\} - iq\langle z_3, x\rangle F(x)\exp\{(-iq)\langle z_3, x\rangle\}$$

provided that they exist.

Relationship R2: (Relationship 3 from Theorem 5.1 and equation (6.4)).

$$\int_{C_0[0,T]}^{\operatorname{anf}_q^{h_3}} \delta F(x|w)dm(x) \doteq \exp\left\{\frac{iq}{2}\|z_3\|_2^2\right\}\int_{C_0[0,T]}^{\operatorname{anf}_q^{h_3}} \delta F(x|w)\exp\{(-iq)\langle z_3, x\rangle\}dm(x)$$

$$\doteq \exp\left\{\frac{iq}{2}\|z_3\|_2^2\right\}\left[\int_{C_0[0,T]}^{\operatorname{anf}_q^{h_3}} \delta H_q(x|w)dm(x)\right.$$

$$+ iq \int_{C_0[0,T]}^{anf_q^{h_3}} \langle z_3, x\rangle F(x) \exp\{(-iq)\langle z_3, x\rangle\} dm(x)\Big]$$

$$\doteq \exp\Big\{\frac{iq}{2}\|z_3\|_2^2\Big\}\Big[\int_{C_0[0,T]}^{anf_q^{h_3}} H_q(x)dm(x) + iq(z_w, h_3)_2 \int_{C_0[0,T]}^{anf_q^{h_3}} H_q(x)dm(x)$$

$$- iq \int_{C_0[0,T]}^{anf_q^{h_3}} \langle z_w, x\rangle H_q(x)dm(x) + iq \int_{C_0[0,T]}^{anf_q^{h_3}} \langle z_3, x\rangle F(x) \exp\{(-iq)\langle z_3, x\rangle\} dm(x)\Big].$$

7. Application to quantum mechanics

The equation (1.3) with $V(u) = a^2u^2, a \in \mathbb{R} - \{0\}$ is called diffusion equation for harmonic oscillator:

$$\frac{\partial}{\partial t}\psi(u,t) = \frac{1}{2\lambda}\Delta\psi(u,t) - a^2u^2\psi(u,t) \tag{7.1}$$

with the initial condition $\psi(u,0) = \varphi(u)$. Hence the solution of the diffusion equation for harmonic oscillator is given by

$$\int_{C_0[0,T]} \varphi(\lambda^{-1/2}x(T)) \exp\Big\{ - \frac{a^2}{\lambda}\int_0^T x^2(s)ds\Big\} dm(x).$$

Also, when time is replaced by imaginary time, the equation (7.1) becomes the Schrödinger equation for harmonic oscillator:

$$i\frac{\partial}{\partial t}\psi(u,t) = -\frac{1}{2}\Delta\psi(u,t) + a^2u^2\psi(u,t) \tag{7.2}$$

with the initial condition $\psi(u,0) = \varphi(u)$. In [8, 16], the authors have described an approach for finding solutions for the diffusion equation for the harmonic oscillator (7.1) and the Schrödinger equation for harmonic oscillator (7.2) as follows.

(1) Note that there is a function f_m in $S(\mathbb{R}^m)$ so that $\widehat{f_m}(\vec{\xi}) = \exp\Big\{-a^2\sum_{j=1}^m \beta_j\xi_j^2\Big\}$.

In fact, f_m is given by the inverse Fourier transform of $\exp\Big\{-a^2\sum_{j=1}^m \beta_j\xi_j^2\Big\}$.

Now, let $V_m(x) = f_m(\langle\vec{\alpha}, x\rangle)$. Then V_m is a Fourier-type functional, and so, $\widehat{V_m}$ is also a Fourier-type functional. Furthermore, we have

$$\widehat{V_m}(x) = \exp\Big\{-a^2\sum_{j=1}^m \beta_j\langle\alpha_j, x\rangle^2\Big\} \tag{7.3}$$

and

$$\lim_{m\to\infty}\widehat{V_m}(x) = \exp\Big\{ - a^2\int_0^T x^2(s)ds\Big\},$$

for a.e. $x \in C_0[0,T]$, where $\beta_m = \left(\frac{T}{(m-\frac{1}{2})\pi}\right)^2$. Also, we have $|\widehat{V_m}(x)| \le 1$ for all $m = 1, 2, \cdots$, and

$$\lim_{m\to\infty} \varphi(x(T))\widehat{V_m}(x) = \varphi(x(T)) \exp\left\{ -a^2 \int_0^T x^2(s)ds \right\}$$

for a.e. $x \in C_0[0,T]$.

(2) The solution of the diffusion equation for harmonic oscillator (7.1) is the limit of Wiener integrals for Fourier-type functionals. Assume that φ is a bounded function. Then the limit of Wiener integrals for the Fourier-type functionals

$$\lim_{m\to\infty} \int_{C_0[0,T]} \varphi(\lambda^{-\frac{1}{2}}x(T))\widehat{V_m}(\lambda^{-\frac{1}{2}}x)dm(x)$$

is a solution of the diffusion equation for harmonic oscillator (7.1). Furthermore, the solution of the Schrödinger equation for harmonic oscillator (7.2) is the limit of analytic Feynman integrals for the Fourier-type functionals,

$$\lim_{m\to\infty} \int_{C_0[0,T]}^{anf_1} \varphi(x(T))\widehat{V_m}(x)dm(x)$$

is a solution of the Schrödinger equation for harmonic oscillator (7.2).

(3) In particular, we can choose the following initial condition:

$$\psi(u,0) = \varphi(u) = \begin{cases} A, & |u| \le L/2 \\ 0, & |u| > L/2, \end{cases}$$

where A is a real constant. In view of the Schrödinger equation this condition corresponds to a pulse wave packet with constant amplitude A in the given range of $|u| \le L/2$ (see [17, 19]). Then the solution of the diffusion equation for harmonic oscillator with the wave packet is:

$$A \lim_{m\to\infty} \prod_{j=1}^{m} \left(\frac{(j-\frac{1}{2})^2\pi^2\lambda}{2a^2T + (j-\frac{1}{2})^2\pi^2\lambda} \right) = A\,\mathrm{sech}\left(\sqrt{\frac{2a^2T}{\lambda}} \right).$$

Furthermore, the solution of the Schrödinger equation for harmonic oscillator with the wave packet is:

$$\lim_{m\to\infty} \int_{C_0[0,T]}^{anf_1} \varphi(x(T))\widehat{V_m}(x)dm(x) = A \lim_{m\to\infty} \prod_{j=1}^{m} \left(-\frac{(j-\frac{1}{2})^2\pi^2 i}{2a^2T - (j-\frac{1}{2})^2\pi^2 i} \right)$$
$$= A\mathrm{sech}\left(\sqrt{\frac{2a^2T}{-i}} \right) = A\sec\left(\sqrt{-i2a^2T} \right).$$

It was not easy to obtain the solutions for the diffusion equation and the Schrödinger equation for nonharmonic oscillators. However, we would like to obtain the solutions of these equations, by using the generalized analytic Feynman integral introduced in Section 3. Given the potential function $V(u) = a^2u^2, a \in \mathbb{R} - \{0\}$, if we take $h(u)$ so that $V(u + h(u))$ is the potential function for the nonharmonic oscillator, then we can conclude that the solution of the diffusion equation for nonharmonic oscillator is the limit of Wiener integrals for Fourier-type functionals. That is, the limit of the Wiener integrals for the Fourier-type functionals:

$$\lim_{m\to\infty} \int_{C_0[0,T]} \varphi(\lambda^{-\frac{1}{2}}x(T) + h(T))\widehat{V_m}(\lambda^{-\frac{1}{2}}x + h)dm(x)$$

is a solution of the diffusion equation for nonharmonic oscillator, and the solution of the Schrödinger equation for nonharmonic oscillator is the limit of analytic Feynman integrals for the Fourier-type functionals. Furthermore,

$$\lim_{m\to\infty} \int_{C_0[0,T]}^{\mathrm{anf}_1^h} \varphi(x(T))\widehat{V_m}(x)dm(x)$$

is a solution of the Schrödinger equation for the nonharmonic oscillator (7.2).

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